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PROPERTIES OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH THE DZIOK-SRIVASTAVA OPERATOR

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Abstract: In the present paper, we study the class $W_p(\mathcal{H}(b_j+1);A,B)$ of multivalent analytic functions with respect to the parameters $b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ ($\mathbb{Z}_0^- = 0,-1,-2,\cdots;\ j=1,2,\cdots,s$), which is defined by the Dziok-Srivastava operator $\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_s)$. By using the methods of differential subordination and the properties of convolution, we obtain the characterization properties and inclusion results for this class, which generalize some previous known results.

Keywords: analytic functions; subordination; Hadamard product (or convolution); Dziok-Srivastava operator; starlike functions; convex functions

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1,2,\cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z|<1\}$. Also, let $\mathcal{A}_1 = \mathcal{A}$.

Let $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}.$$

Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g*f)(z).$$

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For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0)=0 \ \text{ and } \ |\omega(z)|<1 \ (z\in\mathbb{U}),$$

such that

$$f(z)=g(\omega(z)) \ (z\in\mathbb{U}).$$

We denote this subordination by $f(z)\prec g(z)$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details, [3, 12]; see also [19, 20]):

$$f(z)\prec g(z) \ (z\in\mathbb{U}) \Longleftrightarrow f(0)=g(0) \ \text{ and } \ f(\mathbb{U})\subset g(\mathbb{U}).$$

A function $f\in\mathcal{A}$ is said to be the class $\mathcal{K}$ of convex functions in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)>-1 \quad (z\in\mathbb{U}). \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be close-to-convex of order α $(0\le\alpha\le1)$ in $\mathbb{U}$ if there exists a convex univalent function $h\in\mathcal{A}$ and a real β such that

$$\operatorname{Re}\left(\frac{f'(z)}{e^{i\beta}h'(z)}\right)>\alpha \quad (z\in\mathbb{U}). \tag{1.3}$$

Janowski [11] introduced the class

$$\mathcal{S}^*(a,b)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+az}{1+bz} \ \ (z\in\mathbb{U};-1\le b<a\le1)\right\}. \tag{1.4}$$

For $a=1,b=-1$, we have the class of starlike functions $\mathcal{S}^*=\mathcal{S}^*(1,-1)$.

For parameters $a_i\in\mathbb{C}$ $(i=1,2,\cdots,q)$ and $b_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$ $(\mathbb{Z}_0^-=0,-1,-2,\cdots;\ j=1,2,\cdots,s)$, the generalized hypergeometric function ${}_qF_s(a_1,\cdots,a_q;b_1,\cdots,b_s;z)$ is defined by

$$\begin{aligned}&{}_qF_s(a_1,\cdots,a_q;b_1,\cdots,b_s;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_q)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!}\\&(q\le s+1;\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ z\in\mathbb{U}),\end{aligned}$$

where $(\lambda)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(\lambda)_k=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}=\begin{cases}1 & (k=0;\ \lambda\in\mathbb{C}\setminus\{0\}),\\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N};\ \lambda\in\mathbb{C}).\end{cases}$$

Dziok and Srivastava in [7] (see also [8, 9]) considered a linear operator

$$\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_s):\mathcal{A}_p\longrightarrow\mathcal{A}_p,$$

defined by the Hadamard product

$$\begin{aligned}\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_s)f(z) &= [z^p\cdot {}_qF_s(a_1,\cdots,a_q;b_1,\cdots,b_s;z)]*f(z)\\ &= z^p+\sum_{k=1}^{\infty}\frac{(a_1)_k\cdots(a_q)_k}{(b_1)_k\cdots(b_s)_k}\frac{a_{k+p}}{k!}z^{k+p},\end{aligned} \tag{1.5}$$

where $f\in\mathcal{A}_p$ is given by (1.1).

It follows from (1.5) that for all $j\in\{1,2,\cdots,s\}$,

$$b_j\mathcal{H}(b_j)f(z)=z\,[\mathcal{H}(b_j+1)f(z)]'+(b_j-p)\mathcal{H}(b_j+1)f(z), \tag{1.6}$$

where, for convenice

$$\mathcal{H}(b_j)f(z)=\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_j,\cdots,b_s)f(z)$$

and

$$\mathcal{H}(b_j+1)f(z)=\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_j+1,\cdots,b_s)f(z).$$

The Dziok-Srivastava operator $\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_s)$ includes various linear operators, which were considered in earlier works, such as (for example) the linear operators introduced by Hohlov [10], Carlson and Shaffer [2], Ruschewyh [13] and Srivastava and Owa [18].

In particular, we mention here the Bernardi integral operator $\mathcal{J}_\nu:\mathcal{A}\longrightarrow\mathcal{A}$, defined by (see [1])

$$\mathcal{J}_\nu[f(z)]=\frac{\nu+1}{z^\nu}\int_0^z t^{\nu-1}f(t)dt\ \ (\nu\in\mathbb{C}). \tag{1.7}$$

Note that for $f(z)=z+a_2z^2+\cdots$, we have

$$\mathcal{J}_\nu[f(z)]=\sum_{k=1}^{\infty}\frac{\nu+1}{\nu+k}a_kz^k. \tag{1.8}$$

Therefore the Bernardi operator and the Dziok-Srivastava operator are connected in the following way

$$\mathcal{J}_\nu[f(z)]=\mathcal{H}(1+\nu,1;\nu+2)f(z).$$

Definition 1.1 Let us suppose

$$-1\le B\le 0,\ \ A\in\mathbb{C}\ \text{ and }\ |A|<1. \tag{1.9}$$

We denote by $W_p(\mathcal{H}(b_j+1);A,B)$ the class of functions $f\in\mathcal{A}_p$ of form (1.1) which satisfy the following condition

$$b_j\frac{\mathcal{H}(b_j)f(z)}{\mathcal{H}(b_j+1)f(z)}+p-b_j\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}). \tag{1.10}$$

By using (1.6), condition (1.10) becomes

$$\frac{z[z^{1-p}\mathcal{H}(b_j+1)f(z)]'}{z^{1-p}\mathcal{H}(b_j+1)f(z)}=\frac{z[\mathcal{H}(b_j+1)f(z)]'}{\mathcal{H}(b_j+1)f(z)}-p+1\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}). \tag{1.11}$$

From (1.9), we see that

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.12}$$

Thus we have

$$f \in W_p(\mathcal{H}(b_j+1); A, B) \Longrightarrow z^{1-p}\mathcal{H}(b_j+1)f(z) \in \mathcal{S}^*.$$

Moreover, for $-1 \le B < A \le 1$, this means that $z^{1-p}\mathcal{H}(b_j+1)f(z)$ belongs to the class $\mathcal{S}^*(A,B)$ defined by (1.4).

Many interesting subclasses of analytic functions associated with the Dziok-Srivastava operator $\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_s)$ were investigated recently (for example) by Dziok [4, 5], Dziok and Sokol [6], Sokol [16, 17] and others. They obtained various properties and characterizations for these subclasses with respect to the parameters $a_i \in \mathbb{C}$ $(i=1,2,\cdots,q)$. However, in this paper, we aim to investigate some characterizations and inclusion relationships for the class $W_p(\mathcal{H}(b_j+1); A, B)$, which are in connection with the parameters $b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(\mathbb{Z}_0^- = 0,-1,-2,\cdots;\ j=1,2,\cdots,s)$.

2 Main Results

First, we begin by proving the following two characterization theorems.

Theorem 2.1 If $f \in \mathcal{A}_p$ and $j \in \{1,2,\cdots,s\}$, then

$$z[z^{1-p}\mathcal{H}(b_j+1)f(z)]'' = b_j[z^{1-p}\mathcal{H}(b_j)f(z)]' - b_j[z^{1-p}\mathcal{H}(b_j+1)f(z)]'. \tag{2.1}$$

Proof From (1.6), we easily get

$$z[\mathcal{H}(b_j+1)f(z)]' + (1-p)\mathcal{H}(b_j+1)f(z) = b_j\mathcal{H}(b_j)f(z) + (1-b_j)\mathcal{H}(b_j+1)f(z). \tag{2.2}$$

Multiplying both sides of (2.2) by z^{1-p}, equality (2.2) becomes

$$z[z^{1-p}\mathcal{H}(b_j+1)f(z)]' = b_j[z^{1-p}\mathcal{H}(b_j)f(z)] + (1-b_j)[z^{1-p}\mathcal{H}(b_j+1)f(z)]. \tag{2.3}$$

Then differentiating (2.3), we immediately obtain (2.1).

Theorem 2.2 If $f \in \mathcal{A}_p$ and $z^{1-p}\mathcal{H}(b_j+1)f(z)$ is convex univalent function, then $z^{1-p}\mathcal{H}(b_j)f(z)$ is close-to-convex of order $\mathrm{Re}\left(\frac{b_j-1}{|b_j|}\right)$ with respect to $z^{1-p}\mathcal{H}(b_j+1)f(z)$, where $j \in \{1,2,\cdots,s\}$.

Proof From (2.1), we conclude that

$$\frac{[z^{1-p}\mathcal{H}(b_j)f(z)]'}{[z^{1-p}\mathcal{H}(b_j+1)f(z)]'} = \frac{z[z^{1-p}\mathcal{H}(b_j+1)f(z)]''}{b_j[z^{1-p}\mathcal{H}(b_j+1)f(z)]'} + 1. \tag{2.4}$$

Hence, from (1.2) and (2.4), we have

$$\begin{aligned} \mathrm{Re}\left\{\frac{b_j}{|b_j|}\cdot\frac{[z^{1-p}\mathcal{H}(b_j)f(z)]'}{[z^{1-p}\mathcal{H}(b_j+1)f(z)]'}\right\} &= \mathrm{Re}\left\{\frac{z}{|b_j|}\cdot\frac{[z^{1-p}\mathcal{H}(b_j+1)f(z)]''}{[z^{1-p}\mathcal{H}(b_j+1)f(z)]'} + \frac{b_j}{|b_j|}\right\} \\ &> \mathrm{Re}\left(\frac{b_j-1}{|b_j|}\right) \end{aligned}$$

and using (1.3), we obtain the asserted result.

In order to obtain inclusion properties, we first recall the following lemma.

Lemma 2.1 (see [12]) Let $\nu, A \in \mathbb{C}$ and $B \in [-1, 0]$ satisfy either

$$\mathrm{Re}[1 + AB + \nu(1 + B^2)] \geq |A + B + B(\nu + \overline{\nu})| \ \text{ for } \ B \in (-1, 0], \tag{2.5}$$

or

$$1 + A > 0, \quad \mathrm{Re}[1 - A + 2\nu] \geq 0 \ \text{ for } \ B = -1. \tag{2.6}$$

If $f \in \mathcal{A}$ and $F(z) = \mathcal{J}_\nu[f(z)]$ is given by (1.7), then $F \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \Longrightarrow \frac{zF'(z)}{F(z)} \prec \frac{1 + Az}{1 + Bz}.$$

Theorem 2.3 If $f \in \mathcal{A}_p$ and $j \in \{1, 2, \cdots, s\}$, then

$$z^{1-p}\mathcal{H}(b_j + 1)f(z) = \mathcal{J}_{b_j - 1}[z^{1-p}\mathcal{H}(b_j)f(z)], \tag{2.7}$$

where $\mathcal{J}_{b_j-1}$ is the Bernardi operator (1.7) with $\nu = b_j - 1$.

Proof From (1.5), we have

$$\begin{aligned}
\mathcal{H}(b_j + 1)f(z) &= z^p + \sum_{k=1}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_j + 1)_k \cdots (b_s)_k} \frac{a_{k+p}}{k!} z^{k+p} \\
&= z^p + \sum_{k=1}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_j)_k \left(\frac{b_j + k}{b_j}\right) \cdots (b_s)_k} \frac{a_{k+p}}{k!} z^{k+p} \\
&= \mathcal{H}(b_j)f(z) * \left[z^p + \sum_{k=1}^{\infty} \frac{b_j}{b_j + k} z^{k+p}\right] \\
&= z^{p-1}\left\{z^{1-p}[\mathcal{H}(b_j)f(z)] * \left[z + \sum_{k=1}^{\infty} \frac{b_j}{b_j + k} z^{k+1}\right]\right\} \\
&= z^{p-1}\left\{z^{1-p}[\mathcal{H}(b_j)f(z)] * \left[\sum_{k=1}^{\infty} \frac{(b_j - 1) + 1}{(b_j - 1) + k} z^k\right]\right\}.
\end{aligned}$$

Hence, by (1.8) with $\nu = b_j - 1$, we obtain

$$\mathcal{H}(b_j + 1)f(z) = z^{p-1}\mathcal{J}_{b_j-1}[z^{1-p}\mathcal{H}(b_j)f(z)],$$

which implies that (2.7) holds.

Theorem 2.4 Let $m \in \mathbb{N}$ and $j \in \{1, 2, \cdots, s\}$. If $A \in \mathbb{C}$ and $B \in [-1, 0]$ satisfy (2.5) or (2.6) with $\nu = b_j - 1$, then

$$W_p(\mathcal{H}(b_j); A, B) \subseteq W_p(\mathcal{H}(b_j + m); A, B). \tag{2.8}$$

Proof Clearly, it is sufficient to prove (2.8) only for $m = 1$. Let $f \in W_p(\mathcal{H}(b_j); A, B)$, then from (1.11) we have

$$\frac{z[z^{1-p}\mathcal{H}(b_j)f(z)]'}{z^{1-p}\mathcal{H}(b_j)f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{2.9}$$

By applying Lemma 2.1 and Theorem 2.3 to (2.9), we get

$$\frac{z[z^{1-p}\mathcal{H}(b_j+1)f(z)]'}{z^{1-p}\mathcal{H}(b_j+1)f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

which means that $f \in W_p(\mathcal{H}(b_j+1); A, B)$.

It is natural to ask about the inclusion relation (2.8) when m is not positive integer. Next, we will give a partial answer to the question by using a different method. We need the following lemma.

Lemma 2.2 (see [15]) Let $f \in \mathcal{K}$ and $g \in \mathcal{S}^*$. Then, for every analytic function h in $\mathbb{U}$,

$$\frac{(f * hg)(\mathbb{U})}{(f * g)(\mathbb{U})} \subset \overline{\mathrm{co}}[h(\mathbb{U})],$$

where $\overline{co}[h(\mathbb{U})]$ denotes the closed convex hull of $h(\mathbb{U})$.

Theorem 2.5 If $f \in W_p(\mathcal{H}(b_j); A, B)$, $H(z) = z^{1-p}\mathcal{H}(b_j)f(z) \in \mathcal{S}^*$ and $G(z) = \sum\limits_{k=0}^{\infty} \frac{(b_j)_k}{(\widetilde{b_j})_k} z^{k+1} \in \mathcal{K}$, then $f \in W_p(\mathcal{H}(\widetilde{b_j}); A, B)$ and $z^{1-p}\mathcal{H}(\widetilde{b_j})f(z) \in \mathcal{S}^*$.

Proof Let $f \in W_p(\mathcal{H}(b_j); A, B)$. Then by the definition of the class $W_p(\mathcal{H}(b_j); A, B)$, we have

$$\frac{z[z^{1-p}\mathcal{H}(b_j)f(z)]'}{z^{1-p}\mathcal{H}(b_j)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} = \phi[\omega(z)] \quad (z \in \mathbb{U}), \tag{2.10}$$

where ϕ is convex univalent mapping of $\mathbb{U}$ and $|\omega(z)| < 1$ in $\mathbb{U}$ with $\omega(0) = 0 = \phi(0) - 1$. Also, we have $\mathrm{Re}[\phi(z)] > 0$ because of $H(z) \in \mathcal{S}^*$. Using (2.10) and the properties of convolution, we get

$$\begin{aligned} \frac{z[z^{1-p}\mathcal{H}(\widetilde{b_j})f(z)]'}{z^{1-p}\mathcal{H}(\widetilde{b_j})f(z)} &= \frac{z\left[\sum\limits_{k=0}^{\infty} \frac{(b_j)_k}{(\widetilde{b_j})_k} z^{k+1} * z^{1-p}\mathcal{H}(b_j)f(z)\right]'}{\sum\limits_{k=0}^{\infty} \frac{(b_j)_k}{(\widetilde{b_j})_k} z^{k+1} * z^{1-p}\mathcal{H}(b_j)f(z)} \\ &= \frac{G(z) * zH'(z)}{G(z) * H(z)} = \frac{G(z) * \phi[\omega(z)]H(z)}{G(z) * H(z)}. \end{aligned} \tag{2.11}$$

Since $H(z) \in \mathcal{S}^*$, $G(z) \in \mathcal{K}$ and ϕ is convex univalent, then by applying Lemma 2.2 to (2.11), we conclude that (2.11) is subordinate to ϕ in $\mathbb{U}$. Thus, by (1.11), we obtain that $z^{1-p}\mathcal{H}(\widetilde{b_j})f(z) \in \mathcal{S}^*(A, B) \subseteq \mathcal{S}^*$ and so $f \in W_p(\mathcal{H}(\widetilde{b_j}); A, B)$.

3 Some Corollaries

Lemma 3.1 (see [14]) If either $0 < a \le c$ and $c \ge 2$ when a, c are real, or $\mathrm{Re}[a+c] \ge 3$, $\mathrm{Re}[a] \le \mathrm{Re}[c]$ and $\mathrm{Im}[a] = \mathrm{Im}[c]$ when a, c are complex, then the function

$$f(z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (z \in \mathbb{U})$$

belongs to the class $\mathcal{K}$ of convex functions.

Corollary 3.1 If $b_j, \widetilde{b_j}$ are real such that $0 < b_j \le \widetilde{b_j}$ and $\widetilde{b_j} \ge 2$ or $b_j, \widetilde{b_j}$ are complex $(b_j, \widetilde{b_j} \ne 0, -1, -2, \cdots)$ such that $\mathrm{Re}[b_j + \widetilde{b_j}] \ge 3$, $\mathrm{Re}[b_j] \le \mathrm{Re}[\widetilde{b_j}]$ and $\mathrm{Im}[b_j] = \mathrm{Im}[\widetilde{b_j}]$, then $W_p(\mathcal{H}(b_j); A, B) \subseteq W_p(\mathcal{H}(\widetilde{b_j}); A, B)$.

Proof Since A, B satisfy (1.12), so if $f \in W_p(\mathcal{H}(b_j); A, B)$, then $H(z) = z^{1-p}\mathcal{H}(b_j)f(z) \in \mathcal{S}^*$. By Lemma 3.1, the function

$$G(z) = \sum_{k=0}^{\infty} \frac{(b_j)_k}{(\widetilde{b_j})_k} z^{k+1} \quad (z \in \mathbb{U})$$

belongs to the class $\mathcal{K}$ of convex functions. Therefore, in view of Theorem 2.5, we obtain that $f \in W_p(\mathcal{H}(\widetilde{b_j}); A, B)$.

Lemma 3.2 (see [12]) If a, b, c are real and satisfy $-2 \le a < 0$, $b \ne 0$, $b \ge -1$ and $c > M(a, b)$, where

$$M(a, b) = \max\{2 + |a + b|, 1 - ab\},$$

then the Gaussian hypergeometric function

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$$

is convex in $\mathbb{U}$.

Corollary 3.2 Let $b_j \in (-1, 0) \cup (0, 1)$ and $j \in \{1, 2, \cdots, s\}$. If $\widetilde{b_j} > 3 + |b_j|$, then

$$\sum_{k=0}^{\infty} \frac{(b_j)_k}{(\widetilde{b_j})_k} z^{k+1} \in \mathcal{K}.$$

Proof If we choose $b = 1,\ a = b_j - 1,\ c = \widetilde{b_j} - 1$ in Lemma 3.2, then we obtain that

$$F(z) = \sum_{k=0}^{\infty} \frac{(b_j - 1)_k}{(\widetilde{b_j} - 1)_k} z^k$$

is convex in $\mathbb{U}$ for $b_j \ne 0, -1, -2, \cdots$; $-2 \le b_j - 1 < 0$ and $\widetilde{b_j} - 1 > M(a, b) = 2 + |b_j|$. It is clear that $G(z) = \frac{\widetilde{b_j}-1}{b_j-1}[F(z) - 1] \in \mathcal{K}$. After some calculations we have that

$$G(z) = \sum_{k=0}^{\infty} \frac{(b_j)_k}{(\widetilde{b_j})_k} z^{k+1}$$

and this completes the proof.

Corollary 3.3 Let $b_j \in (-1, 0) \cup (0, 1)$ and $j \in \{1, 2, \cdots, s\}$. If $\widetilde{b_j} > 3 + |b_j|$, then

$$W_p(\mathcal{H}(b_j); A, B) \subseteq W_p(\mathcal{H}(\widetilde{b_j}); A, B).$$

Proof The proof follows as the proof of Corollary 3.1 by using Corollary 3.2.

Corollary 3.4 Let $m \in \mathbb{N}$ and $j \in \{1, 2, \cdots, s\}$. If $\mathrm{Re}(b_j) > 1$, then

$$W_p(\mathcal{H}(b_j); A, B) \subseteq W_p(\mathcal{H}(b_j + m); A, B).$$

Proof Obviously, it is sufficient to prove this corollary only for $m = 1$. If $f \in W_p(\mathcal{H}(b_j); A, B)$, then $H(z) = z^{1-p}\mathcal{H}(b_j)f(z) \in \mathcal{S}^*(A, B) \subseteq \mathcal{S}^*$. Let us denote

$$\frac{zH'(z)}{H(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} = \phi[\omega(z)] \quad (z \in \mathbb{U}),$$

where ϕ is convex univalent and $|\omega(z)| < 1$ in $\mathbb{U}$ with $\omega(0) = 0 = \phi(0) - 1$ and $\mathrm{Re}[\phi(z)] > 0$. If $\mathrm{Re}(b_j) > 1$, then

$$G(z) = \sum_{k=1}^{\infty} \frac{(b_j - 1) + 1}{(b_j - 1) + k} z^k \quad (z \in \mathbb{U})$$

belongs to the class $\mathcal{K}$ of convex functions (see [14]). Therefore, by (2.7), we have

$$\begin{aligned} \frac{z[z^{1-p}\mathcal{H}(b_j + 1)f(z)]'}{z^{1-p}\mathcal{H}(b_j + 1)f(z)} &= \frac{[G(z) * zH(z)]'}{G(z) * H(z)} = \frac{G(z) * zH'(z)}{G(z) * H(z)} \\ &= \frac{G(z) * \phi[\omega(z)]H(z)}{G(z) * H(z)} \in \overline{\mathrm{co}}\phi(\mathbb{U}). \end{aligned}$$

Analogous to the proof of Theorem 2.5, we obtain that $f \in W_p(\mathcal{H}(b_j + 1); A, B)$.

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由 Dziok-Srivastava 算子定义的多叶解析函数类的性质

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摘 要: 本文研究了由 Dziok-Srivastava 算子 $\mathcal{H}(a_1,\cdots,a_q;b_1,\cdots,b_s)$ 定义的关于参数 $b_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$ ($\mathbb{Z}_0^-=0,-1,-2,\cdots;\ j=1,2,\cdots,s$) 的多叶解析函数类 $W_p(\mathcal{H}(b_j+1);A,B)$. 利用微分从属的方法和卷积的性质, 获得了该类函数的特征性质和包含结果, 推广了一些已知结果.

关键词: 解析函数; 从属; 卷积; Dziok-Srivastava 算子; 星象函数; 凸象函数

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