CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO $k$-SYMMETRIC POINTS

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Abstract: In this article, we introduce two new subclasses $R^b_m(b, k, \lambda)$ and $K^\alpha_m(\alpha, b, k, \lambda, \delta)$ of analytic functions with respect to $k$-symmetric points. By using the principle of subordination, we obtain the integral representations, coefficient inequalities, covering theorems and arc-length estimates for these function classes, which would provide extensions of those given in earlier works.

Keywords: analytic functions; $k$-symmetric points; subordination; arc-length

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk $U := \{z : |z| < 1\}$.

Let $S, S^*(\gamma), C(\gamma), K(\gamma)$ be the subclasses of $\mathcal{A}$ whose members are univalent, starlike of order $\gamma$, convex of order $\gamma$, and close-to-convex of order $\gamma$, respectively, where $0 \leq \gamma < 1$.

Let $f$ and $g$ be analytic in $U$. Then $f$ is said to be subordinate to $g$, written $f \prec g$, if there exists an analytic function $\omega(z)$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. Indeed, it is known that

$$f(z) \prec g(z) (z \in U) \implies f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence:

$$f(z) \prec g(z) (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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Sakaguchi [13] introduced a class $S^*_s$ of starlike functions with respect to symmetric points which satisfy the inequality
\[ \text{Re} \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0 \quad (z \in U). \]

Since then, many authors discussed this class and its subclasses. Also, a function $f(z) \in A$ is in the class $C_s$ if and only if $zf'(z) \in S^*_s$.

Let $P_m(\gamma)$ be the class of functions $p$ analytic in $U$ satisfying the conditions $p(0) = 1$ and
\[ \int_0^{2\pi} \left| \text{Re}(p(z)) - \gamma \right| d\theta \leq m\pi \quad (0 \leq \gamma < 1; m \geq 2; z = re^{i\theta}). \]

This class was introduced in [11]. We note that $P_m(0) \equiv P_m$ is introduced in [12] and $P_2(\gamma) \equiv P(\gamma)$ is the class of functions with positive real part of order $\gamma$. With $m = 2, \gamma = 0$, we have the class $P$ of functions with positive real part.

The classes $V_m(\gamma)$ of functions of bounded boundary rotation of order $\gamma$ and $R_m(\gamma)$ of functions of bounded radius rotation of order $\gamma$ are closely related with $P_m(\gamma)$. A function $f \in A$ is said to be in the class $V_m(\gamma)$ if and only if $zf'(z) \in P_m(\gamma)$ ($z \in U$).

Moreover, we know that
\[ f \in R_m(\gamma) \iff \frac{zf'(z)}{f(z)} \in P_m(\gamma) \quad (z \in U). \]

Motivated essentially by the above work, we introduce and study the following classes $R^*_m(b, k, \lambda)$ and $K^*_m(\alpha, b, k, \lambda, \delta)$ with respect to $k$-symmetric points.

**Definition 1.1** Suppose that $b \in \mathbb{C} \setminus \{0\}$, $0 \leq \lambda \leq 1$, $m \geq 2$ and $k$ is a fixed positive integer. A function $f \in A$ is said to be in the class $R^*_m(b, k, \lambda)$ if and only if
\[ 1 + \frac{1}{b} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f'_k(z)} - 1 \right\} \in P_m \quad (z \in U), \]
where $f_k(z)$ is defined by
\[ f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} e^{-\nu} f(z^{\nu}z) \quad (z \in U; \; \varepsilon = \exp \left( \frac{2\pi i}{k} \right)). \]

**Remark 1.1** For some recent investigations on analytic functions involving $k$-symmetric points, one can refer to [6, 14–17].

**Definition 1.2** Let $\alpha > 0$, $m \geq 2$ and $0 \leq \delta < 1$. A function $f \in A$ is said to be in the class $K^*_m(\alpha, b, k, \lambda, \delta)$ if and only if
\[ \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{g_k(z)} \right)^\alpha \in P_m(\delta) \quad (z \in U). \]
for some $g \in R_2^b(h, k, \lambda)$.

**Remark 1.2** For special choices of $\alpha, b, k, \lambda, m$ and $\delta$, several related function classes have been studied extensively, see for example [1, 2, 8–10].

In the present paper, we aim at proving some basic properties of the classes $R_m^s(b, k, \lambda)$ and $K_m^s(\alpha, b, k, \lambda, \delta)$. Such results as integral representations, coefficient inequalities, covering theorems and arc-length estimates are derived. The results presented here would provide extensions of those given in earlier works.

2 Preliminary Results

In order to prove our main results, we need the following lemmas.

**Lemma 2.1** (see [3]) Let $h$ be convex in $U$ with $\text{Re}(\beta h(z) + \gamma) > 0$. If $q$ is analytic in $U$ with $q(0) = h(0)$, then

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \implies q(z) \prec h(z).$$

(2.1)

**Lemma 2.2** (see [5]) If $f \in S^*(\alpha)$, $0 \leq \alpha < 1$ and $|z| = r < 1$, then

$$\frac{r}{(1 + r)^{2(1 - \alpha)}} \leq |f(z)| \leq \frac{r}{(1 - r)^{2(1 - \alpha)}}.$$ 

(2.2)

**Lemma 2.3** (see [7]) Let $p \in P_m(\gamma)$ and $|z| = r < 1$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + |m^2(1 - \gamma)^2 - 1|}{1 - r^2}.$$ 

(2.3)

**Lemma 2.4** (see [4]) Let $q$ be univalent in $U$. Then there exists a point $\xi$ with $|\xi| = r$ such that for all $z$, $|z| = r$,

$$|z - \xi||q(z)| \leq \frac{2r^2}{1 - r^2}.$$ 

(2.4)

3 Some Properties of the Classes $R_m^s(b, k, \lambda)$ and $K_m^s(\alpha, b, k, \lambda, \delta)$

We begin by stating the following result which involved the connections between $R_m^s(b, k, \lambda)$ and $R_m(1 - b)$.

**Theorem 3.1** Let $f \in R_m^s(b, k, \lambda)$. Then

$$(1 - \lambda)f_k(z) + \lambda zf'_k(z) \in R_m(1 - b) \quad (z \in U).$$

(3.1)

**Proof** Let

$$F(z) = (1 - \lambda)f(z) + \lambda zf'(z)$$

(3.2)

and

$$F_k(z) = (1 - \lambda)f_k(z) + \lambda zf'_k(z).$$

(3.3)
Then condition (1.2) can be written as
\[
1 + \frac{1}{b} \left( \frac{zF'(z)}{F_k(z)} - 1 \right) = p(z) \tag{3.4}
\]
for some \( p \in P_m \). Substituting \( \varepsilon^\mu z (\mu = 0, 1, 2, \cdots, k - 1) \) in (3.4) gives
\[
1 + \frac{1}{b} \left( \frac{\varepsilon^\mu z f'(\varepsilon^\mu z) + \lambda(\varepsilon^\mu z)^2 f''(\varepsilon^\mu z)}{(1 - \lambda)f_k(\varepsilon^\mu z) + \lambda\varepsilon^\mu z f_k'(\varepsilon^\mu z)} - 1 \right) = p(\varepsilon^\mu z). \tag{3.5}
\]
We note that \( f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z), f_k'(\varepsilon^\mu z) = f_k'(z) \) and \( \varepsilon^\mu f_k''(\varepsilon^\mu z) = f_k''(z) \). Thus taking \( \mu = 0, 1, 2, \cdots, k - 1 \) in (3.5), respectively, and summing the resulting equations, we get
\[
1 + \frac{1}{b} \left( \frac{zF'_k(z)}{F_k(z)} - 1 \right) = \frac{1}{k} \sum_{\mu=0}^{k-1} p(\varepsilon^\mu z). \tag{3.6}
\]
Since \( P_m \) is a convex set, it is clear that
\[
1 + \frac{1}{b} \left( \frac{zF'_k(z)}{F_k(z)} - 1 \right) \in P_m, \tag{3.7}
\]
which implies that
\[
\frac{zF'_k(z)}{F_k(z)} \in P_m(1 - b) \tag{3.8}
\]
and hence \( F_k(z) \in R_m(1 - b) \).

Next, we give the integral representations of functions belonging to the class \( R_m^*(b, k, \lambda) \).

**Theorem 3.2** Let \( f \in R_m^*(b, k, \lambda) \) with \( 0 < \lambda \leq 1 \). Then
\[
f_k(z) = \frac{1}{\lambda} z^{1-\frac{1}{k}} \int_0^z \exp \left( \frac{b}{k} \sum_{\mu=0}^{k-1} \varepsilon^\mu \frac{p(\xi) - 1}{\xi} d\xi \right) u^{\frac{1}{k}-1} du \tag{3.9}
\]
for some \( p \in P_m \).

**Proof** Suppose that \( f \in R_m^*(b, k, \lambda) \). From (1.2), we get
\[
\frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f_k(z) + \lambda z f_k'(z)} = b(p(z) - 1) + 1 \tag{3.10}
\]
for some \( p \in P_m \). Substituting \( z \) by \( \varepsilon^\mu z (\mu = 0, 1, 2, \cdots, k - 1) \) in (3.10), we have
\[
\frac{\varepsilon^\mu z f'(\varepsilon^\mu z) + \lambda(\varepsilon^\mu z)^2 f''(\varepsilon^\mu z)}{(1 - \lambda)f_k(\varepsilon^\mu z) + \lambda\varepsilon^\mu z f_k'(\varepsilon^\mu z)} = b(p(\varepsilon^\mu z) - 1) + 1. \tag{3.11}
\]
By observing that \( f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z) \) and \( f_k'(\varepsilon^\mu z) = f_k'(z) \), we know that (3.11) can be written as
\[
\frac{zf'(\varepsilon^\mu z) + \lambda \varepsilon^\mu z^2 f''(\varepsilon^\mu z)}{(1 - \lambda)f_k(z) + \lambda \varepsilon z f_k'(z)} = b(p(\varepsilon^\mu z) - 1) + 1. \tag{3.12}
\]
Taking $\mu = 0, 1, 2, \cdots, k-1$ in (3.12), respectively, and summing the resulting equations, we obtain

\[
\frac{zf_k''(z) + \lambda z f_k''(z)}{(1-\lambda)f_k(z) + \lambda z f_k''(z)} = \frac{1}{k} \sum_{\mu=0}^{k-1} \left( b(p(\epsilon^\mu z) - 1) + 1 \right) ,
\]

(3.13)

which follows that

\[
\frac{(1-\lambda)f_k'(z) + \lambda (zf_k'(z))'}{(1-\lambda)f_k(z) + \lambda z f_k'(z)} - \frac{1}{z} = b \sum_{\mu=0}^{k-1} \frac{p(\epsilon^\mu z) - 1}{z} .
\]

(3.14)

Integrating (3.14), we get

\[
\log \left( \frac{(1-\lambda)f_k(z) + \lambda z f_k'(z)}{z} \right) = b \sum_{\mu=0}^{k-1} \int_0^z \frac{p^\mu(\xi) - 1}{\xi} d\xi
\]

(3.15)

or equivalently

\[
(1-\lambda)f_k(z) + \lambda z f_k'(z) = z \cdot \exp \left( b \sum_{\mu=0}^{k-1} \int_0^z \frac{p^\mu(\xi) - 1}{\xi} d\xi \right) .
\]

(3.16)

The assertion of Theorem 3.2 can now be derived from (3.16).

**Theorem 3.3** Let $f \in R^m_{s}(b, k, \lambda)$ with $0 < \lambda \leq 1$. Then

\[
f(z) = \frac{1}{\lambda} \int_0^z \int_0^u \exp \left( b \sum_{\mu=0}^{k-1} \int_0^{\epsilon^\mu t} \frac{p(t) - 1}{t} dt \right) \cdot (b(p(z) - 1) + 1) \ u^{k-2} du
\]

(3.17)

for some $p \in P_m$.

**Proof** Suppose that $f \in R^m_{s}(b, k, \lambda)$. From (1.2) and (3.16), we have

\[
(1-\lambda)f'(z) + \lambda z f'(z) = b \sum_{\mu=0}^{k-1} \int_0^{\epsilon^\mu z} \frac{p(t) - 1}{t} dt \cdot (b(p(z) - 1) + 1)
\]

(3.18)

Integrating (3.18) yields

\[
(1-\lambda)f(z) + \lambda z f'(z) = \int_0^z \exp \left( b \sum_{\mu=0}^{k-1} \int_0^{\epsilon^\mu t} \frac{p(t) - 1}{t} dt \right) \cdot (b(p(z) - 1) + 1) \ d\xi .
\]

(3.19)

From (3.19), we can get (3.17) easily.

In what follows, we provide some coefficient inequalities and covering theorems for functions in the class $R^m_{s}(b, k, \lambda)$.

**Theorem 3.4** Let $f \in R^m_{s}(b, k, \lambda)$ with $k \geq 2$. Then

\[
|a_2| \leq \frac{m |b|}{2(1+\lambda)} .
\]

(3.20)
Proof. Suppose that \( f \in R^n_m(b, k, \lambda) \). In view of Theorem 3.1, there exists a function \( \phi \in R_m(1 - b) \), \( \phi(z) = (1 - \lambda)f_k(z) + \lambda zf'_k(z) \) such that

\[
z f'(z) + \lambda z^2 f''(z) = \phi(z)p(z) \tag{3.21}
\]

for some \( p \in P_m(1 - b) \). Using the fact that

\[
f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^\nu f(\varepsilon^\nu z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \left( \varepsilon^\nu z + \sum_{n=2}^{\infty} a_n (\varepsilon^\nu z)^n \right) = z + \sum_{l=2}^{\infty} a_{(l-1)k+1} z^{(l-1)k+1}, \tag{3.22}
\]

we have

\[
\phi(z) = z + \sum_{l=2}^{\infty} \left[ 1 + \lambda(l-1)k \right] a_{(l-1)k+1} z^{(l-1)k+1}. \tag{3.23}
\]

Let

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{3.24}
\]

Then we find from (3.21) that

\[
z + \sum_{n=2}^{\infty} n[1 + \lambda(n-1)] a_n z^n = \left( z + \sum_{l=2}^{\infty} \left[ 1 + \lambda(l-1)k \right] a_{(l-1)k+1} z^{(l-1)k+1} \right) \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right). \tag{3.25}
\]

Comparing the coefficients \( z^2 \) in both sides of (3.25), we get \( 2(1 + \lambda)a_2 = c_1 \), which follows that

\[
|a_2| \leq \frac{|c_1|}{2(1 + \lambda)}. \tag{3.26}
\]

Since \( |c_1| \leq m|b| \) for \( p \in P_m(1 - b) \), we get the desired assertion of Theorem 3.4.

Theorem 3.5. Let \( f \in R^n_m(b, k, \lambda) \) with \( k \geq 2 \). Then the unit disk \( U \) is mapped by every univalent function \( f \) onto a domain that contains the disk \( |\omega| < r_1 \), where

\[
r_1 = \frac{2(1 + \lambda)}{4(1 + \lambda) + m|b|}. \tag{3.27}
\]

Proof. Suppose that \( f \in R^n_m(b, k, \lambda) \). Also, let \( \omega_0 \) be any complex number such that \( f(z) \neq \omega_0 \) for \( z \in U \), then \( \omega_0 \neq 0 \) and

\[
\frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left( a_2 + \frac{1}{\omega_0} \right) z^2 + \cdots \tag{3.28}
\]

for every univalent function \( f \). This leads to

\[
\left| a_2 + \frac{1}{\omega_0} \right| \leq 2 \tag{3.29}
\]

and hence

\[
|\omega_0| \geq \frac{1}{|a_2| + 2}. \tag{3.30}
\]
Using (3.30) and Theorem 3.4, we obtain the required result.

Let $L_r f(z)$ denote the length of the image of the circle $|z| = r$ under $f(z)$. We finally show some basic properties of functions in the class $K_m^*(\alpha, b, k, \lambda, \delta)$ including arc-length and coefficient problems.

**Theorem 3.6** Suppose that $f \in K_m^*(\alpha, b, k, \lambda, \delta)$ with $0 < \lambda \leq 1$ and $0 < b \leq 1$. Then

$$L_r f(z) \leq \begin{cases} C(\alpha, b, \delta, m) M(r)^{1-\alpha} \left( \frac{\delta + b}{1-r} \right)^{4\lambda k + 1} & (0 < \alpha \leq 1), \\ C(\alpha, b, \delta, m) N(r)^{1-\alpha} \left( \frac{\delta + b}{1-r} \right)^{2} & (\alpha > 1), \end{cases}$$

(3.31)

where $N(r) = \min_{|z|=r} |f(z)|$, $M(r) = \max_{|z|=r} |f(z)|$, and $C(\alpha, b, \delta, m)$ is a constant which is determined by the parameters $\alpha, b, \delta$ and $m$.

**Proof** Suppose that $f \in K_m^*(\alpha, b, k, \lambda, \delta)$. From definition (1.4), we know that

$$zf'(z) \left( f(z) \over g_k(z) \right)^{\alpha} = p(z)$$

(3.32)

for some $p \in P_m(\delta)$. It follows that

$$zf'(z) = (f(z))^{1-\alpha} (g_k(z))^\alpha p(z).$$

(3.33)

For $0 < \alpha \leq 1$, we find from (3.33) that

$$L_r f(z) = \int_0^{2\pi} |zf'(z)| \, d\theta \leq \int_0^{2\pi} |f(z)|^{1-\alpha} |g_k(z)|^\alpha |p(z)| \, d\theta \leq M^{1-\alpha}(r) \int_0^{2\pi} |g_k(z)|^\alpha |p(z)| \, d\theta,$$

(3.34)

where $M(r) = \max_{|z|=r} |f(z)|$. Since $g \in R_2^*(b, k, \lambda)$, from Theorem 3.1, we have

$$(1-\lambda)g_k(z) + \lambda zg'_k(z) = G_k(z) \in R_2(1-b) \equiv S^*(1-b).$$

(3.35)

Let $q(z) = \frac{zg_k'(z)}{g_k(z)}$. It follows from (3.35) that

$$\frac{G_k(z)}{g_k(z)} = 1 - \lambda + \lambda q(z).$$

(3.36)

Differentiate both sides of (3.36) logarithmically, we obtain

$$q(z) + \frac{zq'(z)}{q(z) + 1-\lambda} = \frac{z G_k'(z)}{G_k(z)} < \frac{1 + (2b - 1)z}{1-z}.$$  

(3.37)

By noting that

$$\text{Re} \left[ \frac{1 + (2b - 1)z}{1-z} + \frac{1-\lambda}{\lambda} \right] > 0 \quad (0 < b \leq 1; \ 0 < \lambda \leq 1),$$

an application of Lemma 2.1 to (3.37) yields

$$q(z) < \frac{1 + (2b - 1)z}{1-z},$$  

(3.38)
which implies that \( g_k(z) \in S^*(1 - b) \). By Lemma 2.2, we have
\[
\frac{r}{(1 + r)^{2b}} \leq |g_k(z)| \leq \frac{r}{(1 - r)^{2b}}.
\]
(3.39)

Using (3.39) and Lemma 2.3, we deduce from (3.34) that
\[
L_r f(z) \leq M(r)^{1-\alpha} \frac{r^\alpha}{(1-r)^{2\alpha}} \int_0^{2\pi} |p(z)| d\theta
\]
\[
\leq 2\pi M(r)^{1-\alpha} \frac{r^\alpha}{(1-r)^{2\alpha}} \left( \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{\frac{1}{2}}
\]
\[
\leq 2\pi M(r)^{1-\alpha} \frac{r^\alpha}{(1-r)^{2\alpha}} \left( 1 + \frac{m^2(1-\delta)^2 - 1}{1-r^2} \right)^{\frac{1}{2}}
\]
\[
= C(\alpha, b, \delta, m) M(r)^{1-\alpha} \left( \frac{1}{1-r} \right)^{\frac{4ab+1}{2}}.
\]
(3.40)

Similarly, for \( \alpha > 1 \), we have
\[
L_r f(z) \leq C(\alpha, b, \delta, m) N(r)^{1-\alpha} \left( \frac{1}{1-r} \right)^{\frac{4ab+1}{2}}.
\]

**Theorem 3.7** Let \( f \in K^s_m(\alpha, b, k, \lambda, \delta) \) with \( 0 < \lambda \leq 1 \) and \( 0 < b \leq 1 \). Then
\[
|a_n| \leq \begin{cases} 
C_1(\alpha, b, \delta, m) M(n)^{1-\alpha} n^{\frac{4ab+1}{2}} & (0 < \alpha \leq 1), \\
C_1(\alpha, b, \delta, m) N(n)^{1-\alpha} n^{\frac{4ab+1}{2}} & (\alpha > 1).
\end{cases}
\]
(3.41)

**Proof** Suppose that \( f \in K^s_m(\alpha, b, k, \lambda, \delta) \). For \( n \geq 1 \) and \( z = re^{i\theta} \), Cauchy’s Theorem gives that
\[
n a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z) e^{-in\theta} d\theta.
\]
(3.42)

Using Theorem 3.6 for \( 0 < \alpha \leq 1 \), we get
\[
n |a_n| \leq \frac{1}{2\pi r^n} C(\alpha, b, \delta, m) M(r)^{1-\alpha} \left( \frac{1}{1-r} \right)^{\frac{4ab+1}{2}}.
\]
(3.43)

Taking \( r = 1 - \frac{1}{n} \) in (3.43), we obtain
\[
|a_n| \leq C_1(\alpha, b, \delta, m) M(n)^{1-\alpha} n^{\frac{4ab+1}{2}}.
\]
(3.44)

Using the similar techniques, we can prove the corresponding result for \( \alpha > 1 \).

**Theorem 3.8** Let \( f \in K^s_m(\alpha, b, k, \lambda, \delta) \) with \( 0 < \lambda \leq 1 \) and \( 0 < b \leq 1 \). Then
\[
||a_{n+1} - a_n|| \leq \begin{cases} 
C_2(\alpha, b, \delta, m) M(r)^{1-\alpha} \left( \frac{1}{1-r} \right)^{\frac{1}{2}} & (0 < \alpha \leq 1), \\
C_2(\alpha, b, \delta, m) N(r)^{1-\alpha} \left( \frac{1}{1-r} \right)^{\frac{4(\alpha-1)b+1}{2}} & (\alpha > 1).
\end{cases}
\]
(3.45)
Proof. It is known that for $\xi \in U, z = re^{i\theta}$ and $n \geq 1$, one has
\begin{equation}
| (n+1)\xi a_{n+1} - na_n | \leq \int_0^{2\pi} |z - \xi| |zf'(z)| d\theta. \tag{3.46}
\end{equation}

Since $f \in K^a_{m}(\alpha, b, k, \lambda, \delta)$, we obtain
\begin{equation}
zf'(z) = (f(z))^{1-\alpha} (g_k(z))^\alpha p(z) \tag{3.47}
\end{equation}
and
\begin{equation}
\frac{r}{(1+r)^{2b}} \leq |g_k(z)| \leq \frac{r}{(1-r)^{2b}}. \tag{3.48}
\end{equation}

For $0 < \alpha \leq 1$, combining (3.47), (3.48) and (3.46), we get
\begin{equation}
| (n+1)\xi a_{n+1} - na_n | \leq M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \int_0^{2\pi} |z - \xi| |g_k(z)||p(z)| d\theta. \tag{3.49}
\end{equation}

By Lemmas 2.3 and 2.4, we deduce that
\begin{align*}
| (n+1)\xi a_{n+1} - na_n | & \leq M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \frac{2r^2}{1-r^2} \int_0^{2\pi} |p(z)| d\theta \\
& \leq 2\pi M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \frac{2r^2}{1-r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{\frac{1}{2}} \\
& \leq 2\pi M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \frac{2r^2}{1-r^2} \left( \frac{1 + [m^2(1-\delta)^2 - 1]r^2}{1 - r^2} \right)^{\frac{1}{2}}.
\end{align*}

Putting $|\xi| = r = \frac{n}{n+1}$, it follows that
\begin{equation}
|| a_{n+1} | - |a_n || \leq C_2(\alpha, b, \delta, m) M(n)^{1-\alpha} \frac{n^{\frac{1}{2}}}{n^{\frac{1}{2}}}. \tag{3.50}
\end{equation}

Similarly, we can get the required result for $\alpha > 1$.

References


与k折对称点有关的解析函数族的一些新子族

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摘要: 本文引入了两类与k折对称点有关的解析函数族的子族. 利用从属理论, 得到了这些函数族的积分表达、系数不等式、覆盖定理、弧长估计结果. 所得结果推广了一些相关文献的结论.

关键词: 解析函数; k折对称点; 从属; 弧长