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CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO k-SYMMETRIC POINTS

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Abstract: In this article, we introduce two new subclasses $R_m^s(b,k,\lambda)$ and $K_m^s(\alpha,b,k,\lambda,\delta)$ of analytic functions with respect to k-symmetric points. By using the principle of subordination, we obtain the integral representations, coefficient inequalities, covering theorems and arc-length estimates for these function classes, which would provide extensions of those given in earlier works.

Keywords: analytic functions; k-symmetric points; subordination; arc-length

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} := \{z : |z| < 1\}$.

Let $S, S^*(\gamma), C(\gamma), K(\gamma)$ be the subclasses of $\mathcal{A}$ whose members are univalent, starlike of order γ, convex of order γ, and close-to-convex of order γ, respectively, where $0 \le \gamma < 1$.

Let f and g be analytic in $\mathbb{U}$. Then f is said to be subordinate to g, written $f \prec g$, if there exists an analytic function $\omega(z)$, with $\omega(0) = 0$ and $|w(z)| < 1$ such tat $f(z) = g(\omega(z))$. Indeed, it is known that

$$f(z) \prec g(z)(z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z)(z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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Sakaguchi [13] introduced a class S_s^* of starlike functions with respect to symmetric points which satisfy the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

Since then, many authors discussed this class and its subclasses. Also, a function $f(z) \in \mathcal{A}$ is in the class C_s if and only if $zf'(z) \in S_s^*$.

Let $P_m(\gamma)$ be the class of functions p analytic in $\mathbb{U}$ satisfying the conditions $p(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\operatorname{Re}(p(z)) - \gamma}{1-\gamma}\right| d\theta \leq m\pi \qquad (0 \leq \gamma < 1;\ m \geq 2;\ z = re^{i\theta}).$$

This class was introduced in [11]. We note that $P_m(0) \equiv P_m$ is introduced in [12] and $P_2(\gamma) \equiv P(\gamma)$ is the class of functions with positive real part of order γ. With $m = 2$, $\gamma = 0$, we have the class P of functions with positive real part.

The classes $V_m(\gamma)$ of functions of bounded boundary rotation of order γ and $R_m(\gamma)$ of functions of bounded radius rotation of order γ are closely related with $P_m(\gamma)$. A function $f \in \mathcal{A}$ is said to be in the class $V_m(\gamma)$ if and only if

$$\frac{(zf'(z))'}{f'(z)} \in P_m(\gamma) \qquad (z \in \mathbb{U}).$$

Moreover, we know that

$$f \in R_m(\gamma) \Longleftrightarrow \frac{zf'(z)}{f(z)} \in P_m(\gamma) \qquad (z \in \mathbb{U}).$$

Motivated essentially by the above work, we introduce and study the following classes $R_m^s(b,k,\lambda)$ and $K_m^s(\alpha,b,k,\lambda,\delta)$ with respect to k-symmetric points.

Definition 1.1 Suppose that $b \in \mathbb{C} \setminus \{0\}$, $0 \leq \lambda \leq 1$, $m \geq 2$ and k is a fixed positive integer. A function $f \in \mathcal{A}$ is said to be in the class $R_m^s(b,k,\lambda)$ if and only if

$$1 + \frac{1}{b}\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_k(z) + \lambda z f_k'(z)} - 1\right\} \in P_m \qquad (z \in \mathbb{U}), \tag{1.2}$$

where $f_k(z)$ is defined by

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z) \qquad \left(z \in \mathbb{U};\ \varepsilon = \exp\left(\frac{2\pi i}{k}\right)\right). \tag{1.3}$$

Remark 1.1 For some recent investigations on analytic functions involving k-symmetric points, one can refer to [6, 14–17].

Definition 1.2 Let $\alpha > 0$, $m \geq 2$ and $0 \leq \delta < 1$. A function $f \in \mathcal{A}$ is said to be in the class $K_m^s(\alpha,b,k,\lambda,\delta)$ if and only if

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g_k(z)}\right)^{\alpha} \in P_m(\delta) \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some $g \in R_2^s(b,k,\lambda)$.

Remark 1.2 For special choices of α, b, k, λ, m and δ, several related function classes have been studied extensively, see for example [1, 2, 8–10].

In the present paper, we aim at proving some basic properties of the classes $R_m^s(b,k,\lambda)$ and $K_m^s(\alpha,b,k,\lambda,\delta)$. Such results as integral representations, coefficient inequalities, covering theorems and arc-length estimates are derived. The results presented here would provide extensions of those given in earlier works.

2 Preliminary Results

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [3]) Let h be convex in $\mathbb{U}$ with $\mathrm{Re}(\beta h(z)+\gamma) > 0$. If q is analytic in $\mathbb{U}$ with $q(0) = h(0)$, then

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} \prec h(z) \Longrightarrow q(z) \prec h(z). \tag{2.1}$$

Lemma 2.2 (see [5]) If $f \in S^*(\alpha)$, $0 \le \alpha < 1$ and $|z| = r < 1$, then

$$\frac{r}{(1+r)^{2(1-\alpha)}} \le |f(z)| \le \frac{r}{(1-r)^{2(1-\alpha)}}. \tag{2.2}$$

Lemma 2.3 (see [7]) Let $p \in P_m(\gamma)$ and $|z| = r < 1$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2\, d\theta \le \frac{1+[m^2(1-\gamma)^2-1]r^2}{1-r^2}. \tag{2.3}$$

Lemma 2.4 (see [4]) Let q be univalent in $\mathbb{U}$. Then there exists a point ξ with $|\xi| = r$ such that for all z, $|z| = r$,

$$|z-\xi|\,|q(z)| \le \frac{2r^2}{1-r^2}. \tag{2.4}$$

3 Some Properties of the Classes $R_m^s(b,k,\lambda)$ and $K_m^s(\alpha,b,k,\lambda,\delta)$

We begin by stating the following result which involved the connections between $R_m^s(b,k,\lambda)$ and $R_m(1-b)$.

Theorem 3.1 Let $f \in R_m^s(b,k,\lambda)$. Then

$$(1-\lambda)f_k(z) + \lambda z f_k'(z) \in R_m(1-b) \qquad (z \in \mathbb{U}). \tag{3.1}$$

Proof Let

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z) \tag{3.2}$$

and

$$F_k(z) = (1-\lambda)f_k(z) + \lambda z f_k'(z). \tag{3.3}$$

Then condition (1.2) can be written as

$$1+\frac{1}{b}\left(\frac{zF'(z)}{F_k(z)}-1\right)=p(z) \tag{3.4}$$

for some $p\in P_m$. Substituting z by $\varepsilon^\mu z\,(\mu=0,1,2,\cdots,k-1)$ in (3.4) gives

$$1+\frac{1}{b}\left(\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)+\lambda(\varepsilon^\mu z)^2f''(\varepsilon^\mu z)}{(1-\lambda)f_k(\varepsilon^\mu z)+\lambda\varepsilon^\mu zf_k'(\varepsilon^\mu z)}-1\right)=p(\varepsilon^\mu z). \tag{3.5}$$

We note that $f_k(\varepsilon^\nu z)=\varepsilon^\nu f_k(z)$, $f_k'(\varepsilon^\nu z)=f_k'(z)$ and $\varepsilon^\mu f_k''(\varepsilon^\nu z)=f_k''(z)$. Thus taking $\mu=0,1,2,\cdots,k-1$ in (3.5), respectively, and summing the resulting equations, we get

$$1+\frac{1}{b}\left(\frac{zF_k'(z)}{F_k(z)}-1\right)=\frac{1}{k}\sum_{\mu=0}^{k-1}p(\varepsilon^\mu z). \tag{3.6}$$

Since P_m is a convex set, it is clear that

$$1+\frac{1}{b}\left(\frac{zF_k'(z)}{F_k(z)}-1\right)\in P_m, \tag{3.7}$$

which implies that

$$\frac{zF_k'(z)}{F_k(z)}\in P_m(1-b) \tag{3.8}$$

and hence $F_k(z)\in R_m(1-b)$.

Next, we give the integral representations of functions belonging to the class $R_m^s(b,k,\lambda)$.

Theorem 3.2 Let $f\in R_m^s(b,k,\lambda)$ with $0<\lambda\leq 1$. Then

$$f_k(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z\exp\left(\frac{b}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu u}\frac{p(\xi)-1}{\xi}d\xi\right)u^{\frac{1}{\lambda}-1}du \tag{3.9}$$

for some $p\in P_m$.

Proof Suppose that $f\in R_m^s(b,k,\lambda)$. From (1.2), we get

$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f_k(z)+\lambda zf_k'(z)}=b\left(p(z)-1\right)+1 \tag{3.10}$$

for some $p\in P_m$. Substituting z by $\varepsilon^\mu z$ $(\mu=0,1,2,\cdots,k-1)$ in (3.10), we have

$$\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)+\lambda(\varepsilon^\mu z)^2f''(\varepsilon^\mu z)}{(1-\lambda)f_k(\varepsilon^\mu z)+\lambda\varepsilon^\mu zf_k'(\varepsilon^\mu z)}=b\left(p(\varepsilon^\mu z)-1\right)+1. \tag{3.11}$$

By observing that $f_k(\varepsilon^\mu z)=\varepsilon^\mu f_k(z)$ and $f_k'(\varepsilon^\mu z)=f_k'(z)$, we know that (3.11) can be written as

$$\frac{zf'(\varepsilon^\mu z)+\lambda\varepsilon^\mu z^2f''(\varepsilon^\mu z)}{(1-\lambda)f_k(z)+\lambda zf_k'(z)}=b\left(p(\varepsilon^\mu z)-1\right)+1. \tag{3.12}$$

Taking $\mu = 0, 1, 2, \cdots, k-1$ in (3.12), respectively, and summing the resulting equations, we obtain

$$\frac{zf_k'(z) + \lambda z^2 f_k''(z)}{(1-\lambda)f_k(z) + \lambda z f_k'(z)} = \frac{1}{k}\sum_{\mu=0}^{k-1}\left(b\left(p(\varepsilon^\mu z) - 1\right) + 1\right), \tag{3.13}$$

which follows that

$$\frac{(1-\lambda)f_k'(z) + \lambda(zf_k'(z))'}{(1-\lambda)f_k(z) + \lambda z f_k'(z)} - \frac{1}{z} = \frac{b}{k}\sum_{\mu=0}^{k-1}\frac{p(\varepsilon^\mu z) - 1}{z}. \tag{3.14}$$

Integrating (3.14), we get

$$\log\left(\frac{(1-\lambda)f_k(z) + \lambda z f_k'(z)}{z}\right) = \frac{b}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{p(\xi) - 1}{\xi}d\xi \tag{3.15}$$

or equivalently

$$(1-\lambda)f_k(z) + \lambda z f_k'(z) = z \cdot \exp\left(\frac{b}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{p(\xi) - 1}{\xi}d\xi\right). \tag{3.16}$$

The assertion of Theorem 3.2 can now be derived from (3.16).

Theorem 3.3 Let $f \in R_m^s(b, k, \lambda)$ with $0 < \lambda \leq 1$. Then

$$f(z) = \frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z\int_0^u \exp\left(\frac{b}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \xi}\frac{p(t) - 1}{t}dt\right)\cdot\left(b\left(p(\xi) - 1\right) + 1\right)d\xi u^{\frac{1}{\lambda}-2}du \tag{3.17}$$

for some $p \in P_m$.

Proof Suppose that $f \in R_m^s(b, k, \lambda)$. From (1.2) and (3.16), we have

$$\begin{aligned}(1-\lambda)f'(z) + \lambda(zf'(z))' &= \frac{(1-\lambda)f_k(z) + \lambda z f_k'(z)}{z}\cdot\left(b\left(p(z) - 1\right) + 1\right)\\ &= \exp\left(\frac{b}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{p(t) - 1}{t}dt\right)\cdot\left(b\left(p(z) - 1\right) + 1\right).\end{aligned} \tag{3.18}$$

Integrating (3.18) yields

$$(1-\lambda)f(z) + \lambda z f'(z) = \int_0^z \exp\left(\frac{b}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \xi}\frac{p(t) - 1}{t}dt\right)\cdot\left(b\left(p(\xi) - 1\right) + 1\right)d\xi. \tag{3.19}$$

From (3.19), we can get (3.17) easily.

In what follows, we provide some coefficient inequalities and covering theorems for functions in the class $R_m^s(b, k, \lambda)$.

Theorem 3.4 Let $f \in R_m^s(b, k, \lambda)$ with $k \geq 2$. Then

$$|a_2| \leq \frac{m\,|b|}{2(1+\lambda)}. \tag{3.20}$$

Proof Suppose that $f \in R_m^s(b,k,\lambda)$. In view of Theorem 3.1, there exists a function $\phi \in R_m(1-b)$, $\phi(z) = (1-\lambda)f_k(z) + \lambda z f_k'(z)$ such that

$$zf'(z) + \lambda z^2 f''(z) = \phi(z)p(z) \tag{3.21}$$

for some $p \in P_m(1-b)$. Using the fact that

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu}f(\varepsilon^\nu z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\left(\varepsilon^\nu z + \sum_{n=2}^{\infty}a_n(\varepsilon^\nu z)^n\right) = z + \sum_{l=2}^{\infty}a_{(l-1)k+1}z^{(l-1)k+1}, \tag{3.22}$$

we have

$$\phi(z) = z + \sum_{l=2}^{\infty}[1+\lambda(l-1)k]a_{(l-1)k+1}z^{(l-1)k+1}. \tag{3.23}$$

Let

$$p(z) = 1 + \sum_{n=1}^{\infty}c_n z^n. \tag{3.24}$$

Then we find from (3.21) that

$$z + \sum_{n=2}^{\infty}n[1+\lambda(n-1)]a_n z^n = \left(z + \sum_{l=2}^{\infty}[1+\lambda(l-1)k]a_{(l-1)k+1}z^{(l-1)k+1}\right)\left(1+\sum_{n=1}^{\infty}c_n z^n\right). \tag{3.25}$$

Comparing the coefficients z^2 in both sides of (3.25), we get $2(1+\lambda)a_2 = c_1$, which follows that

$$|a_2| \le \frac{|c_1|}{2(1+\lambda)}. \tag{3.26}$$

Since $|c_1| \le m\,|b|$ for $p \in P_m(1-b)$, we get the desired assertion of Theorem 3.4.

Theorem 3.5 Let $f \in R_m^s(b,k,\lambda)$ with $k \ge 2$. Then the unit disk $\mathbb{U}$ is mapped by every univalent function f onto a domain that contains the disk $|\omega| < r_1$, where

$$r_1 = \frac{2(1+\lambda)}{4(1+\lambda)+m\,|b|}. \tag{3.27}$$

Proof Suppose that $f \in R_m^s(b,k,\lambda)$. Also, let ω_0 be any complex number such that $f(z) \ne \omega_0$ for $z \in \mathbb{U}$, then $\omega_0 \ne 0$ and

$$\frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right)z^2 + \cdots \tag{3.28}$$

for every univalent function f. This leads to

$$\left|a_2 + \frac{1}{\omega_0}\right| \le 2 \tag{3.29}$$

and hence

$$|\omega_0| \ge \frac{1}{|a_2|+2}. \tag{3.30}$$

Using (3.30) and Theorem 3.4, we obtain the required result.

Let $L_r f(z)$ denote the length of the image of the circle $|z| = r$ under $f(z)$. we finally show some basic properties of functions in the class $K_m^s(\alpha, b, k, \lambda, \delta)$ including arc-length and coefficient problems.

Theorem 3.6 Suppose that $f \in K_m^s(\alpha, b, k, \lambda, \delta)$ with $0 < \lambda \leq 1$ and $0 < b \leq 1$. Then

$$L_r f(z) \leq \begin{cases} C(\alpha, b, \delta, m) M(r)^{1-\alpha} \left(\frac{1}{1-r}\right)^{\frac{4\alpha b+1}{2}} & (0 < \alpha \leq 1), \\ C(\alpha, b, \delta, m) N(r)^{1-\alpha} \left(\frac{1}{1-r}\right)^{\frac{4\alpha b+1}{2}} & (\alpha > 1), \end{cases} \tag{3.31}$$

where $N(r) = \min\limits_{|z|=r} |f(z)|$, $M(r) = \max\limits_{|z|=r} |f(z)|$, and $C(\alpha, b, \delta, m)$ is a constant which is determined by the parameters α, b, δ and m.

Proof Suppose that $f \in K_m^s(\alpha, b, k, \lambda, \delta)$. From definition (1.4), we know that

$$\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{g_k(z)}\right)^{\alpha} = p(z) \tag{3.32}$$

for some $p \in P_m(\delta)$. It follows that

$$zf'(z) = (f(z))^{1-\alpha} (g_k(z))^{\alpha} p(z). \tag{3.33}$$

For $0 < \alpha \leq 1$, we find from (3.33) that

$$L_r f(z) = \int_0^{2\pi} |zf'(z)| \, d\theta \leq \int_0^{2\pi} |f(z)|^{1-\alpha} |g_k(z)|^{\alpha} |p(z)| \, d\theta \leq M^{1-\alpha}(r) \int_0^{2\pi} |g_k(z)|^{\alpha} |p(z)| \, d\theta, \tag{3.34}$$

where $M(r) = \max\limits_{|z|=r} |f(z)|$. Since $g \in R_2^s(b, k, \lambda)$, from Theorem 3.1, we have

$$(1-\lambda) g_k(z) + \lambda z g_k'(z) = G_k(z) \in R_2(1-b) \equiv S^*(1-b). \tag{3.35}$$

Let $q(z) = \frac{zg_k'(z)}{g_k(z)}$. It follows from (3.35) that

$$\frac{G_k(z)}{g_k(z)} = 1 - \lambda + \lambda q(z). \tag{3.36}$$

Differentiate both sides of (3.36) logarithmically, we obtain

$$q(z) + \frac{zq'(z)}{q(z) + \frac{1-\lambda}{\lambda}} = \frac{zG_k'(z)}{G_k(z)} \prec \frac{1 + (2b-1)z}{1-z}. \tag{3.37}$$

By noting that

$$\mathrm{Re}\left[\frac{1 + (2b-1)z}{1-z} + \frac{1-\lambda}{\lambda}\right] > 0 \qquad (0 < b \leq 1;\ 0 < \lambda \leq 1),$$

an application of Lemma 2.1 to (3.37) yields

$$q(z) \prec \frac{1 + (2b-1)z}{1-z}, \tag{3.38}$$

which implies that $g_k(z) \in S^*(1-b)$. By Lemma 2.2, we have

$$\frac{r}{(1+r)^{2b}} \le |g_k(z)| \le \frac{r}{(1-r)^{2b}}. \tag{3.39}$$

Using (3.39) and Lemma 2.3, we deduce from (3.34) that

$$\begin{aligned} L_r f(z) &\le M(r)^{1-\alpha}\frac{r^\alpha}{(1-r)^{2\alpha b}}\int_0^{2\pi}|p(z)|\,d\theta \\ &\le 2\pi M(r)^{1-\alpha}\frac{r^\alpha}{(1-r)^{2\alpha b}}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2\,d\theta\right)^{\frac{1}{2}} \\ &\le 2\pi M(r)^{1-\alpha}\frac{r^\alpha}{(1-r)^{2\alpha b}}\left(\frac{1+[m^2(1-\delta)^2-1]r^2}{1-r^2}\right)^{\frac{1}{2}} \\ &= C(\alpha,b,\delta,m)M(r)^{1-\alpha}\left(\frac{1}{1-r}\right)^{\frac{4\alpha b+1}{2}}. \end{aligned} \tag{3.40}$$

Similarly, for $\alpha > 1$, we have

$$L_r f(z) \leqq C(\alpha,b,\delta,m)N(r)^{1-\alpha}\left(\frac{1}{1-r}\right)^{\frac{4\alpha b+1}{2}}.$$

Theorem 3.7 Let $f \in K_m^s(\alpha,b,k,\lambda,\delta)$ with $0<\lambda\le 1$ and $0<b\le 1$. Then

$$|a_n| \le \begin{cases} C_1(\alpha,b,\delta,m)M(n)^{1-\alpha}n^{\frac{4\alpha b-1}{2}} & (0<\alpha\le 1), \\ C_1(\alpha,b,\delta,m)N(n)^{1-\alpha}n^{\frac{4\alpha b-1}{2}} & (\alpha>1). \end{cases} \tag{3.41}$$

Proof Suppose that $f \in K_m^s(\alpha,b,k,\lambda,\delta)$. For $n \ge 1$ and $z = re^{i\theta}$, Cauchy's Theorem gives that

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)e^{-in\theta}d\theta. \tag{3.42}$$

Using Theorem 3.6 for $0<\alpha\le 1$, we get

$$n\,|a_n| \le \frac{1}{2\pi r^n}C(\alpha,b,\delta,m)M(r)^{1-\alpha}\left(\frac{1}{1-r}\right)^{\frac{4\alpha b+1}{2}}. \tag{3.43}$$

Taking $r = 1-\frac{1}{n}$ in (3.43), we obtain

$$|a_n| \le C_1(\alpha,b,\delta,m)M(n)^{1-\alpha}n^{\frac{4\alpha b-1}{2}}. \tag{3.44}$$

Using the similar techniques, we can prove the corresponding result for $\alpha > 1$.

Theorem 3.8 Let $f \in K_m^s(\alpha,b,k,\lambda,\delta)$ with $0<\lambda\le 1$ and $0<b\le 1$. Then

$$||a_{n+1}|-|a_n|| \le \begin{cases} C_2(\alpha,b,\delta,m)M(r)^{1-\alpha}\left(\frac{1}{1-r}\right)^{\frac{1}{2}} & (0<\alpha\le 1), \\ C_2(\alpha,b,\delta,m)N(r)^{1-\alpha}\left(\frac{1}{1-r}\right)^{\frac{4(\alpha-1)b+1}{2}} & (\alpha>1). \end{cases} \tag{3.45}$$

Proof It is known that for $\xi \in \mathbb{U}, z = re^{i\theta}$ and $n \geq 1$, one has

$$|(n+1)\xi a_{n+1} - na_n| \leq \int_0^{2\pi} |z-\xi|\, |zf'(z)|\, d\theta. \tag{3.46}$$

Since $f \in K_m^s(\alpha, b, k, \lambda, \delta)$, we obtain

$$zf'(z) = (f(z))^{1-\alpha} (g_k(z))^{\alpha} p(z) \tag{3.47}$$

and

$$\frac{r}{(1+r)^{2b}} \leq |g_k(z)| \leq \frac{r}{(1-r)^{2b}}. \tag{3.48}$$

For $0 < \alpha \leq 1$, combining (3.47), (3.48) and (3.46), we get

$$|(n+1)\xi a_{n+1} - na_n| \leq M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \int_0^{2\pi} |z-\xi|\, |g_k(z)|\, |p(z)|\, d\theta. \tag{3.49}$$

By Lemmas 2.3 and 2.4, we deduce that

$$\begin{aligned}
|(n+1)\xi a_{n+1} - na_n| &\leq M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \frac{2r^2}{1-r^2} \int_0^{2\pi} |p(z)|\, d\theta \\
&\leq 2\pi M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \frac{2r^2}{1-r^2} \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2\, d\theta \right)^{\frac{1}{2}} \\
&\leq 2\pi M(r)^{1-\alpha} \frac{r^{\alpha-1}}{(1+r)^{2(\alpha-1)b}} \frac{2r^2}{1-r^2} \left(\frac{1 + [m^2(1-\delta)^2 - 1]r^2}{1-r^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

Putting $|\xi| = r = \frac{n}{n+1}$, it follows that

$$||a_{n+1}| - |a_n|| \leq C_2(\alpha, b, \delta, m) M(n)^{1-\alpha} n^{\frac{1}{2}}.$$

Similarly, we can get the required result for $\alpha > 1$.

References

[1] Arif M, Noor K I, Khan R. On subclass of analytic functions with respect to symmetrical points[J]. Abstr. Appl. Anal., 2012, Article ID 790689, doi:10.1155/2012/790689.

[2] Das R N, Singh P. On a subclass of schlicht mapping[J]. Indian J. Pure Appl. Math., 1977, 8: 864–872.

[3] Eenigenburg P J, Miller S S, Mocanu P T, Reade O M. Second order differential inequalities in the complex plane[J]. J. Math. Anal. Appl., 1978, 65: 289–305.

[4] G. M. Golusin. On distortion theorems and coefficients of univalent functions[J]. Matematicheskii Sbornik, 1946, 19:183–202.

[5] I. Graham and G. Kohr. Geometric fuction theory in one and higher dimensions[M]. New Nork: Marcel Dekker, 2003.

[6] Huang Y Y, Liu M S. Properties of certain subclasses of multivalent analytic functions involving the Dziok-Srivastava operator[J]. Appl. Math. Comput., 2008, 204: 137–149.

[7] Noor K I. On subclasses of close-to-convex functions of higher order[J]. Intern. J. Math. Math. Sci., 1992, 15: 279–290.

[8] Noor K I. On some subclasses of m-fold symmetric analytic functions[J]. Comput. Math. Appl., 2010, 60: 14–22.

[9] Noor K I, Malikz B, Saima Mustafax. A survey on functions of bounded boundary and bounded radius rotation[J]. Appl. Math. E-Notes, 2012, 12: 136–152.

[10] Noor K I, Mustafa S. Some classes of analytic functions related with functions of bounded radius rotation with respect to symmetrical points[J]. J. Math. Inequ., 2009, 3: 267–276.

[11] Padmanabhan K S, Parvatham R. Properties of a class of functions with bounded boundary rotation[J]. Ann. Polonici Math., 1975, 31: 311–323.

[12] Pinchuk B. Functions of bounded boundary rotation[J]. Israel J. Math., 1971, 10: 7–16.

[13] Sakaguchi K. On a certain univalent mapping[J]. Journal of the Math. Soc. Japan, 1959, 11: 72–75.

[14] Wang Z G, Gao C Y, Yuan S M. On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points[J]. J. Math. Anal. Appl., 2006, 322: 97–106.

[15] Wang Z G, Jiang Y P, Srivastava H M. Some subclasses of multivalent analytic functions involving Dziok-Srivastava operator[J]. Integ. Trans. Special Funct., 2008, 19: 129–146.

[16] Yuan S M, Liu Z M. Some properties of α-convex and α-quasiconvex functions with respect to n-symmetric points[J]. Appl. Math. Comput., 2007, 188: 1142–1150.

[17] Yuan S M, Liu Z M. Some properties of two subclasses of k-fold symmetric functions associated with Srivastava-Attiya operator[J]. Appl. Math. Comput., 2011, 218: 1136–1141.

与k折对称点有关的解析函数族的一些新子族

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摘 要: 本文引入了两类与k折对称点有关的解析函数族的新子族. 利用从属理论, 得到了这些函数族的积分表示、系数不等式、覆盖定理、弧长估计等结果. 所得结果推广了一些相关文献的结论.

关键词: 解析函数; k折对称点; 从属; 弧长

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