ON COMPLETE SHRINKING RICCI-HARMONIC SOLITONS

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Abstract: In this paper, we study the geometry of shrinking Ricci-harmonic solitons. By utilizing the method of Manola, Gabriele and Carlo [4] under the Ricci soliton, we prove the result that every compact shrinking Ricci-harmonic soliton is a gradient one, which extends the result in the case of Ricci soliton. Moreover, by utilizing the method of Zhang [14], we prove a more precise volume growth estimate than that of at most Euclidean growth for the complete non-compact gradient shrinking Ricci-harmonic soliton, which extends the result of Zhang [14] in the case of Ricci soliton.

Keywords: shrinking Ricci-harmonic soliton; gradient; volume growth

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1 Introduction

Let \((M^n, g)\) be a complete smooth Riemannian manifold, the metric \(g\) is called a Ricci-harmonic soliton if there exists a vector field \(X\) and a constant \(\lambda\), such that

\[
\begin{align*}
R_{ij} - \alpha \nabla_i \phi \nabla_j \phi + \frac{1}{2} L_X g &= \lambda g_{ij}, \\
\tau g \phi &= \nabla X \phi,
\end{align*}
\]

(1.1)

where \(\phi : (M^n, g) \to (N^m, h)\) is a map between the Riemannian manifolds \((M^n, g)\) and \((N^m, h)\), \(Rc\) is the Ricci curvature of \((M, g)\), \(\tau g \phi = \text{trace} \nabla d\phi\), and \(\alpha\) is a nonnegative constant.

We call the Ricci-harmonic soliton (1.1) a shrinking, steady, expanding Ricci-harmonic soliton if \(\lambda > 0\), \(\lambda = 0\), or \(\lambda < 0\). If \(X\) is a gradient of some function \(f\), then \(L_X g = \nabla^2 f\), we call the Ricci-harmonic soliton a gradient Ricci-harmonic soliton with potential function \(f\).

Similar to the Ricci soliton, the Ricci-harmonic soliton is a self-similar solution to the Ricci-harmonic flow,

\[
\begin{align*}
\frac{\partial}{\partial t} g &= -2 Rc + 2 \alpha(t) \nabla \phi \otimes \nabla \phi, \\
\frac{\partial}{\partial t} \phi &= \tau g \phi,
\end{align*}
\]

(1.2)

Cao and Zhou [2] proved that a complete noncompact gradient shrinking Ricci soliton has at most Euclidean volume growth by estimating the bounds for the potential function. Zhang [14] proved a more precise estimate, \( \text{Vol}(B(x_0, r)) \leq C(r + 1)^{-2\delta} \), when the scalar curvature is bounded below by a positive constant \( \delta \). Yang and Shen [12] found that the complete noncompact gradient shrinking Ricci-harmonic soliton still has at most Euclidean volume growth by proving \( R - \alpha |\nabla \phi|^2 \) is nonnegative and estimating the bounds of the potential function \( f \).

In Section 2, we prove the result that every compact shrinking Ricci-harmonic soliton is a gradient one. The method is inspired by Manola, Gabriele and Carlo’s work [4] and different from that in [8]. In Section 3, we extend Zhang’s work [14] to the case of complete noncompact Ricci-harmonic soliton.

Our main theorems in this paper are below.

**Theorem 1.1** Every compact shrinking Ricci-harmonic soliton is a gradient one.

**Theorem 1.2** Let \((M^n, g)\) be the complete noncompact shrinking gradient Ricci-harmonic soliton structure (3.1), if there exists a nonnegative constant \( \delta \) such that \( R - \alpha |\nabla \phi|^2 \geq \delta \), then there is a constant \( C < +\infty \) depending only on \( g \) and \( x_0 \) such that

\[
\text{Vol}(B_{x_0}(r)) \leq C(r + 1)^{-2\delta}
\]

for all \( r > r_0 \), where \( B_{x_0}(r) \) is a geodesic ball with radius \( r \) and \( r_0 \) is a positive constant.

**Remark 1.1** The condition \( R - \alpha |\nabla \phi|^2 \geq \delta \) added in Theorem 1.3 is reasonable for Yang and Shen [12] proved \( R - \alpha |\nabla \phi|^2 \geq 0 \).

2 The Compact Case

**Lemma 2.1** (Log Sobolev inequality, see [3]) Let \((M^n, g)\) be a compact Riemannian manifold. For any \( a > 0 \), there exists a constant \( C(a, g) \) such that if \( \varphi > 0 \) satisfies

\[
\int_M \varphi^2 dVol = 1,
\]

then

\[
\int_M \varphi^2 \log \varphi dVol \leq a \int_M |\nabla \varphi|^2 dVol + C(a, g).
\]

**Lemma 2.2** Let \((M^n, g)\) be a compact Riemannian manifold, \( F : M \to \mathbb{R} \) be a smooth function and \( \lambda \) be a positive constant, then there exists a smooth function \( f : M \to \mathbb{R} \) satisfies the equation

\[
F + 2\Delta f - |\nabla f|^2 + 2\lambda f = \text{Const}.
\]
**Proof** Define a functional $W$

$$W(g, f) = \int_M (F + 2\Delta f - |\nabla f|^2 + 2\lambda f)e^{-f}dVol$$

and

$$\mu(g) = \inf \{ W(g, f) : f \in C^\infty(M) \text{ with } \int_M e^{-f}dVol = 1 \}.$$  

Let $\omega = e^{-\frac{f}{2}}$, we have

$$\int_M \omega^2dVol = 1$$

and

$$W(g, f) = \int_M (F + |\nabla f|^2 + 2\lambda f)e^{-f}dVol$$

$$= \int_M [(F - 4\lambda \log \omega)\omega^2 + 4|\nabla \omega|^2]dVol := H(g, \omega).$$

Since $F$ is bounded below on $M$ and from Log Sobolev inequality (Lemma 2.1), there exist a constant $C < +\infty$ such that

$$\int_M \omega^2\log \omega dVol \leq \frac{1}{\lambda} \int_M |\nabla \omega|^2 + C.$$  

Then the positive minimizer $\omega_1$ realizing $\mu(g)$ is the lowest positive eigenvalue of the nonlinear operator

$$\Theta(\omega) := -4\Delta \omega + (F - 4\lambda \log \omega)\omega = \mu(g)\omega.$$  

Choose $\omega$ such that $H(g, \omega) \leq C_1$, then $\int_M \omega^2dVol = 1$, and there exists a positive constant $C_2$ with

$$C_1 \geq H(g, \omega) \geq 2 \int_M |\nabla \omega|^2dVol - C_2.$$  

Hence any minimizing sequence for $H(g, \cdot)$ is bounded in $W_2^1(M)$. We get a minimizer $\omega_1 \in W_2^1(M)$ and $\omega_1$ is a weak solution to

$$-4\Delta \omega + (F - 4\lambda \log \omega)\omega = \mu(g)\omega.$$  

By elliptic regularity theory (see Gilbarg and Trudinger [5]), we have $\omega_1 \in C^\infty$. It’s easy to verify that $\omega_1 > 0$. Then there exists a smooth function $f_1 = -2\log \omega_1$ realizing $\mu(g)$, i.e.,

$$F + 2\Delta f_1 - |\nabla f_1|^2 + 2\lambda f_1 = \mu(g)$$

for $\lambda > 0$.

**Proof of Theorem 1.1** Considering the compact shrinking Ricci-harmonic soliton

$$\begin{cases}
R_{ij} - \alpha \nabla_i \phi \nabla_j \phi + \frac{1}{2} L_X g = \lambda g_{ij}, \\
\tau g \phi = \nabla X \phi.
\end{cases}$$

(2.3)
From Lemma 2.2, there exists a smooth function $f : M \to \mathbb{R}$ satisfying
\[ R - \alpha|\nabla \phi|^2 + 2\Delta f - |\nabla f|^2 + 2\lambda f = \text{Const.}, \]
we have
\[
\nabla_j[2(R_{ij} - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f - \lambda g_{ij})e^{-f}]
\]
\[=\nabla_j(2\nabla_j f - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f - \lambda g_{ij})e^{-f} + 2\alpha|\nabla \phi|^2 e^{-f} + 2\alpha|\nabla f - X\phi|^2 e^{-f}
\]
\[=\nabla_i(\nabla_j f - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f - \lambda g_{ij})e^{-f} + 2\alpha|\nabla \phi|^2 e^{-f} + 2\alpha|\nabla f - X\phi|^2 e^{-f}
\]
\[=\nabla_i(\nabla_j f - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f - \lambda g_{ij})e^{-f} + 2\alpha|\nabla \phi|^2 e^{-f} + 2\alpha|\nabla f - X\phi|^2 e^{-f}
\]
\[\geq 0.\]

Denote $|R_{ij} - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f - \lambda g_{ij}|^2 e^{-f} + 2\alpha|\nabla f - X\phi|^2 e^{-f}$ by $Q$, we conclude that
\[0 \leq Q = \text{div}[(\nabla_j f - X\phi)(R_{ij} - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f - \lambda g_{ij})e^{-f}]. \tag{2.4}\]

Integrating $Q$, we have $Q \equiv 0$ by Stokes’s theorem and the compactness of $M^n$. This implies compact shrinking Ricci-harmonic soliton $(2.3)$ is a gradient Ricci-harmonic soliton with $X = \nabla f$.

Similar to the proof of Theorem 1.1, we have the direct corollary.

**Corollary 2.1** Every compact steady Ricci-harmonic soliton is a gradient one.

**Proposition 2.1** For the compact shrinking Ricci-harmonic soliton $(2.3)$, the potential function $f$ equals a Hodge-de Rham potential up to a constant.

**Proof** By the Hodge-de Rham decomposition theorem, there exists a divergence-free vector field $Y$ and a function $b$ on $M^n$, such that
\[ X = Y + \nabla b, \tag{2.5} \]
we deduce $\text{div}X = \Delta b$. By Theorem 1.1, we can find a potential function $f$ to $(M, g, X)$ satisfying $X = \nabla f$, then $\text{div}X = \Delta f$.

We conclude that $f = b + \text{Const.}$ for $\Delta(f - b) = 0$ and $M$ is compact.
Remark 2.1 Proposition 2.1 provides another way to find the potential function \( f \) to the generic compact shrinking Ricci-harmonic soliton structure \((M^n, g, X)\). Normalizing \( f \), we can replace \( f \) by the Hodge-de Rham potential to vector field \( X \).

3 The Complete Noncompact Case

In this section, we consider the complete noncompact gradient shrinking Ricci-harmonic soliton

\[
\begin{cases}
R_{ij} - \alpha \nabla_i \phi \nabla_j \phi + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}, \\
\tau g \phi = \langle \nabla \phi, \nabla f \rangle.
\end{cases}
\tag{3.1}
\]

Lemma 3.1 Let \((M^n, g)\) be the complete noncompact gradient shrinking Ricci-harmonic soliton structure (3.1), we have the following four equalities

\[
\begin{align*}
R - \alpha |\nabla \phi|^2 + \triangle f &= \frac{n}{2}, \\
R - \alpha |\nabla \phi|^2 + |\nabla f|^2 - f &= 0.
\end{align*}
\tag{3.2}
\tag{3.3}
\]

Proof Taking trace of the first equation of (3.1), (3.2) is obtained.

Taking covariant derivatives and using the commutation formula for the covariant derivatives, we have

\[
\nabla_i R_{jk} - \nabla_j R_{ik} - \alpha (\nabla_j \phi \nabla_i \phi - \nabla_i \phi \nabla_j \phi) + R_{ijkl} \nabla_l f = 0.
\]

Taking the trace on \( j \) and \( k \), we have

\[
\nabla_i R - \nabla_j R_{ij} - \alpha \nabla_j \phi \nabla_i \phi + \alpha \nabla_i \phi \tau g \phi - R_{ij} \nabla_i f = 0.
\]

Using the fact that

\[
\nabla_i R - 2 \nabla_j R_{ij} = 0,
\]

\[
\nabla_i \nabla_j \phi \nabla_j \phi = \frac{1}{2} \nabla_i (\nabla_j \phi \nabla_j \phi) = \frac{1}{2} \nabla_i |\nabla \phi|^2
\]

and

\[
\alpha \nabla_i \phi \tau g \phi - R_{ij} \nabla_i f = \nabla_i \nabla_j f \nabla_j f - \frac{1}{2} g_{ij} \nabla_j f = \frac{1}{2} \nabla_i (|\nabla f|^2 - f),
\]

we obtain \( \frac{1}{2} \nabla (R - \alpha |\nabla \phi|^2 + |\nabla f|^2 - f) = 0 \), hence \( R - \alpha |\nabla \phi|^2 + |\nabla f|^2 - f = \text{Const} \). Normalizing \( f \) by adding a constant, (3.3) follows.

Lemma 3.2 Let \((M^n, g)\) be the complete noncompact shrinking Ricci-harmonic soliton structure (3.1) and \( \mathbf{V}(r) := \int_{\{f<r\}} dV \) and \( \mathbf{V}_R(r) := \int_{\{f<r\}} \frac{R - \alpha |\nabla \phi|^2}{2} dV \), we have

\[
n \mathbf{V}(r) - 2r \mathbf{V}'(r) = 2 \mathbf{V}_R(r) - 2 \mathbf{V}'_R(r).
\tag{3.4}
\]

Proof Integrating by parts and using eq. (3.2),

\[
\frac{n}{2} \mathbf{V}(r) - \mathbf{V}_R(r) = \int_{\{f<r\}} \Delta f dV = \int_{\{f=r\}} \nabla f \cdot \frac{\nabla f}{|\nabla f|} dA = \int_{\{f=r\}} |\nabla f| dA,
\tag{3.5}
\]
which implies
\[
\frac{n}{2} V(r) \geq V_R(r) . \tag{3.6}
\]
By co-Area formula (see [11]), we have
\[
V'(r) = \int_{\{f=r\}} \frac{1}{|\nabla f|} dA \tag{3.7}
\]
and
\[
V'_R(r) = \int_{\{f=r\}} \frac{R - a|\nabla f|^2}{|\nabla f|} dA . \tag{3.8}
\]
Using (3.3) and combining (3.5), (3.7) and (3.8), we obtain
\[
\frac{n}{2} V(r) - V_R(r) = r V'(r) - V'_R(r) .
\]

**Proof of Theorem 1.2** Calculating directly,
\[
\frac{d}{dr} \left( \log \left( r^{\frac{n-2\delta}{2}} V(r) \right) \right) = \frac{r V'(r) - \frac{n-2\delta}{2} V(r)}{r V(r)} \leq \frac{V'_R(r)}{r V(r)} , \tag{3.9}
\]
where the last inequality comes from (3.4) and $V_R(r) \geq \delta V(r)$.

Fixed $r_0 > 0$, for any $r_1 > r_0$, integrating (3.9) by parts on $[r_0, r_1]$ yields
\[
\log \frac{r_1}{r_0} \frac{r^{-\frac{n-2\delta}{2}} V(r_1)}{r^{-\frac{n-2\delta}{2}} V(r_0)} \leq \int_{r_0}^{r_1} \frac{1}{r V(r)} dV_R(r) \leq \frac{V_R(r_1)}{r V(r_1)} \left| r_0 \right| + \int_{r_0}^{r_1} \frac{V_R(r)}{r^2 V(r)} dr + \int_{r_0}^{r_1} \frac{V_R(r) V'(r)}{r V^2(r)} dr \leq \frac{n}{2} + \frac{n}{2} \int_{r_0}^{r_1} \frac{\log V(r_1)}{r_1} - \frac{\log V(r_0)}{r_0} \left( \frac{1}{r_1} - \frac{1}{r_0} \right) + \frac{n}{2} \int_{r_0}^{r_1} \frac{\log V(r)}{r^2} dr . \tag{3.10}
\]

When $r_0$ is large enough, by the proof of Theorem 1.1 in Yang and Shen [11], we have
\[
V(r) \leq C_1 r^n \tag{3.11}
\]
for some positive constant $C_1$ and for $r \geq r_0$. Plugging inequality (3.11) into (3.10), we have
\[
\log \frac{r_1}{r_0} \frac{r^{-\frac{n-2\delta}{2}} V(r_1)}{r^{-\frac{n-2\delta}{2}} V(r_0)} \leq +\infty \tag{3.12}
\]
for any $r_1 > r_0$. Moreover, there is a positive constant $C_2$ such that
\[
V(r_1) \leq C_2 r_1^{\frac{n-2\delta}{\delta}} . \tag{3.13}
\]

For $f(x) \leq \frac{1}{4} (d(x_0, x) + 2 \sqrt{f(x_0)^2})^2$ (see Proposition 4.1 in [12]), $B_{x_0}(r) \subset \{ f \leq \frac{1}{4} (r + C_2)^2 \}$. We have
\[
Vol(B_{x_0}(r)) \leq V \left( \frac{1}{4} (d(x_0, x) + 2 \sqrt{f(x_0)^2}) \leq C(r + 1)^{n-2\delta} \tag{3.14}
\]
for some positive constant $C$ depends only on $g$ and $x_0$. 

References


关于完备收缩的Ricci-harmonic孤子的研究

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关键词: 收缩的Ricci-harmonic孤子; 梯度; 体积增长