SOME REMARKS ON GEOMETRIC INEQUALITIES FOR SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD OF QUASI-CONSTANT CURVATURE

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Abstract: In this paper, we study Chen’s inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature. By using algebraic techniques, we establish Chen’s general inequalities, Chen-Ricci inequalities and inequalities between the warping function and the squared mean curvature, which generalize several results of Özgür and Chen’s.

Keywords: Chen’s inequalities; Chen-Ricci inequalities; warped product; quasi-constant curvature

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1 Introduction

According to Chen [1], one of the most important problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Related with famous Nash embedding theorem [2], Chen introduced a new type of Riemannian invariants, known as $\delta$-invariants [3, 4, 5]. The author’s original motivation was to provide answers to a question raised by Chern concerning the existence of minimal isometric immersions into Euclidean space [6]. Therefore, Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established inequalities for submanifolds in real space forms in terms of the sectional curvature, the scalar curvature and the squared mean curvature [7]. Later, he established general inequalities relating $\delta(n_1, \cdots, n_k)$ and the squared mean curvature for submanifolds in real space forms [8]. Similar inequalities also hold for Lagrangian submanifolds of complex space forms. In [9], Chen proved that, for any $\delta(n_1, \cdots, n_k)$, the equality case holds if and only if the Lagrangian submanifold is minimal. This interesting phenomenon inspired people to look for a more sharp inequality. In 2007, Oprea improved the inequality on $\delta(2)$ for Lagrangian submanifolds in complex space forms[10]. Recently, Chen and Dillen established general inequalities for Lagrangian

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submanifolds in complex space forms and provided some examples showing these new improved inequalities are best possible [11]. Such invariants and inequalities have many nice applications to several areas in mathematics [12].

Afterwards, many papers studied similar problems for different submanifolds in various ambient spaces, like complex space forms [13], Sasakian space forms [14], \((\kappa, \mu)\)-contact space forms [15], Lorentzian manifold [16], Euclidean space [17] and locally conformal almost cosymplectic manifolds [18].

This paper is organized as follows. In Section 2, the basic elements of the theory of \(\delta\)-invariants are briefly presented. In Section 3, we establish general inequalities of \(\delta\)-invariants for submanifolds of a Riemannian manifold of quasi-constant curvature [19], which generalize a result of paper [20]. In Section 4, we obtain an inequality between the Ricci curvature and the squared mean curvature for submanifolds of the ambient space by using an algebraic lemma. Finally, in Section 5, we establish inequalities between the warping function \(f\) (intrinsic structure) and the squared mean curvature (extrinsic structure) for warped product submanifolds \(M_1 \times_f M_2\) in a Riemannian manifold of quasi-constant curvature, as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

2 Preliminaries

In [19], Chen and Yano introduced the notion of a Riemannian manifold \((N, g)\) of quasi-constant curvature as a Riemannian manifold with the curvature tensor satisfying the condition

\[
\overline{R}(X, Y, Z, W) = a[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] + b[g(X, Z)T(Y)T(W) - g(X, W)T(Y)T(Z) + g(Y, W)T(X)T(Z) - g(Y, Z)T(X)T(W)],
\]

(2.1)

where \(a, b\) are scalar functions and \(T\) is a 1-form defined by

\[
g(X, P) = T(X)
\]

(2.2)

and \(P\) is a unit vector field. If \(b = 0\), it can be easily seen that the manifold reduces to a space of constant curvature.

Decomposing the vector field \(P\) on \(M\) uniquely into its tangent and normal components \(P^T\) and \(P^\perp\), respectively, we have

\[
P = P^T + P^\perp.
\]

(2.3)

Let \(M\) be an \(n\)-dimensional submanifold of an \((n+p)\)-dimensional Riemannian manifold of quasi-constant curvature \(N^{n+p}\). The Gauss equation is given by

\[
R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z))
\]

(2.4)

for all \(X, Y, Z, W \in TM\), where \(R\) and \(\overline{R}\) are the curvature tensors of \(M\) and \(N^{n+p}\), respectively, and \(h\) is the second fundamental form.
In $N^{n+p}$ we choose a local orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ such that, restricting to $M^n$, $e_1, \cdots, e_n$ are tangent to $M^n$. We write $h_{ij} = g(h(e_i, e_j), e_r)$. The mean curvature vector $\zeta$ is given by $\zeta = \frac{1}{n} \sum_{r=1}^{n} h_{i}^{r} e_{r}$, then the mean curvature $H$ is given by $H = \| \zeta \|$. 

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by $e_i$ and $e_j$. Then the scalar curvature of $M^n$ is given by

$$
\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j). 
$$

(2.5)

Let $L$ be an $l$-dimensional subspace of $T_x M$, $x \in M$, $l \geq 2$ and $\{e_1, \cdots, e_l\}$ an orthonormal basis of $L$. We define the scalar curvature $\tau(L)$ of the $l$-plane $L$ by

$$
\tau(L) = \sum_{1 \leq a < b \leq l} K(e_a \wedge e_b). 
$$

(2.6)

For simplicity we put

$$
\Psi(L) = \sum_{1 \leq i < j \leq l} [g(P^T, e_i)^2 + g(P^T, e_j)^2]. 
$$

(2.7)

For an integer $k \geq 0$ we denote by $S(n, k)$ the set of $k$-tuples $(n_1, \cdots, n_k)$ of integers $\geq 2$ satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. We denote by $S(n)$ the set of unordered $k$-tuples with $k \geq 0$ for a fixed $n$. For each $k$-tuples $(n_1, \cdots, n_k) \in S(n)$, Chen defined a Riemannian invariant $\delta(n_1, \cdots, n_k)$ as follows $[8]

$$
\delta(n_1, \cdots, n_k)(x) = \tau(x) - S(n_1, \cdots, n_k)(x), 
$$

(2.8)

where $S(n_1, \cdots, n_k)(x) = \inf \{\tau(L_1) + \cdots + \tau(L_k)\}$, and $L_1, \cdots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_x M$ such that $\dim L_j = n_j$, $j \in \{1, \cdots, k\}$. For each $(n_1, \cdots, n_k) \in S(n)$, we put

$$
c(n_1, \cdots, n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)},

\quad d(n_1, \cdots, n_k) = \frac{1}{2}[n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1)].
$$

For a differentiable function $f$ on $M$, the Laplacian $\triangle f$ of $f$ is defined by

$$
\triangle f = \sum_{i=1}^{n} [\nabla_{e_i} e_i] f - e_i e_i f. 
$$

We shall use the following lemmas.

**Lemma 2.1** $[7]$ Let $a_1, a_2, \cdots, a_n, b$ be $(n + 1)(n \geq 2)$ real numbers such that

$$
(\sum_{i=1}^{n} a_i)^2 = (n - 1)(\sum_{i=1}^{n} a_i^2 + b),
$$

We have

$$
\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i^2 + b} = \frac{1}{n - 1}.
$$

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$$
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$$

We have

$$
\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i^2 + b} = \frac{1}{n - 1}.
$$
then \(2a_1a_2 \geq b\), with the equality holding if and only if \(a_1 + a_2 = a_3 = \cdots = a_n\).

**Lemma 2.2** Let \(f(x_1, x_2, \cdots, x_n)\) be a function in \(\mathbb{R}^n\) defined by

\[
f(x_1, x_2, \cdots, x_n) = x_1 \sum_{i=2}^{n} x_i.
\]

If \(x_1 + x_2 + \cdots + x_n = 2\lambda\), then we have \(f(x_1, x_2, \cdots, x_n) \leq \lambda^2\), with the equality holding if and only if \(x_1 = x_2 + x_3 + \cdots + x_n = \lambda\).

**Proof** From \(x_1 + x_2 + \cdots + x_n = 2\lambda\), we have \(\sum_{i=2}^{n} x_i = 2\lambda - x_1\). It follows that

\[
f(x_1, x_2, \cdots, x_n) = x_1(2\lambda - x_1) = -(x_1 - \lambda)^2 + \lambda^2,
\]

which represents Lemma 2.2 to prove.

### 3 Chen’s General Inequalities

**Theorem 3.1** If \(M^n (n \geq 3)\) is a submanifold of a Riemannian manifold of quasi-constant curvature \(N^{n+p}\), then we have

\[
\delta(n_1, \cdots, n_k) \leq c(n_1, \cdots, n_k)H^2 + d(n_1, \cdots, n_k)a + b((n - 1) \| P^T \|^2 - \sum_{j=1}^{k} \Psi(L_j)) \quad (3.1)
\]

for any \(k\)-tuples \((n_1, \cdots, n_k) \in S(n)\). The equality case of (3.1) holds at \(x \in M^n\) if and only if there exist an orthonormal basis \(\{e_1, \cdots, e_n\}\) of \(T_xM\) and an orthonormal basis \(\{e_{n+1}, \cdots, e_{n+p}\}\) of \(T^\perp_xM\) such that the shape operators of \(M^n\) in \(N^{n+p}\) at \(x\) have the following forms

\[
A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad A_{e_r} = \begin{pmatrix} A^r_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A^r_k & 0 \\ 0 & 0 & \cdots & \mu_r I \end{pmatrix}, \quad r = n + 2, \cdots, n + p,
\]

where \(a_1, \cdots, a_n\) satisfy

\[
a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n
\]

and each \(A^r_j\) is a symmetric \(n_j \times n_j\) submatrix satisfying \(\text{trace}(A^r_1) = \cdots = \text{trace}(A^r_k) = \mu_r\), \(I\) is an identity matrix.

**Remark 3.2** For \(\delta(2)\), inequality (3.1) is due to Cihan Özgür [20, Theorem 3.1].

**Proof** Let \(x \in M^n\) and \(\{e_1, e_2, \cdots, e_n\}\) and \(\{e_{n+1}, e_{n+2}, \cdots, e_{n+p}\}\) be orthonormal basis of \(T_xM^n\) and \(T^\perp_xM^n\), respectively, such that the mean curvature vector \(\zeta\) is in the
From (3.4) we deduce

\[ a_i = h_{ii}^{n+1}, \quad i = 1, 2, \ldots, n, \]

\[ b_1 = a_1, \quad b_2 = a_2 + \cdots + a_{n_1}, \quad b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \ldots, \]

\[ b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k}, \]

\[ b_{k+2} = a_{n_1+\cdots+n_k+1}, \ldots, b_{γ+1} = a_n, \]

\[ Δ_1 = \{1, \ldots, n_1\}, \ldots, \]

\[ Δ_k = \{(n_1 + \cdots + n_{k-1}) + 1, \ldots, n_1 + \cdots + n_k\}, \]

\[ Δ_{k+1} = (Δ_1 \times Δ_1) \cup \cdots \cup (Δ_k \times Δ_k). \]

Let \( L_1, \ldots, L_k \) be mutually orthogonal subspaces of \( T_x M \) with \( \dim L_j = n_j \), defined by

\[ L_j = \text{Span}\{e_{n_1+\cdots+n_{j-1}+1}, \ldots, e_{n_1+\cdots+n_j}\}, \quad j = 1, \ldots, k. \]

From (2.4), (2.6) and (2.7) we have

\[ τ(L_j) = \frac{n_j(n_j-1)}{2} a + b\Psi(L_j) + \sum_{r=n+2}^{n+p} \sum_{1<j<j'} [h_{α_jα_j}^r h_{β_jβ_j}^r - (h_{α_jβ_j}^r)^2], \tag{3.2} \]

\[ 2τ = n(n-1)a + 2b(n-1) \| P^T \|^2 + n^2 H^2 - \| h \|^2. \tag{3.3} \]

We can rewrite (3.3) as \( n^2 H^2 = (\| h \|^2 + η)γ \), or equivalently,

\[ (\sum_{i=1}^{n} h_{ii}^{n+1})^2 = γ[\sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + η], \tag{3.4} \]

where

\[ η = 2τ - 2c(n_1, \ldots, n_k)H^2 - n(n-1)a - 2(n-1)b \| P^T \|^2, \tag{3.5} \]

\[ γ = n + k - \sum_{j=1}^{k} n_j. \]

From (3.4) we deduce

\[ (\sum_{i=1}^{γ+1} b_i)^2 = γ[η + \sum_{i=1}^{γ+1} b_i^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - 2 \sum_{α_1<β_1} a_{α_1} a_{β_1} - \cdots - 2 \sum_{α_k<β_k} a_{α_k} a_{β_k}], \]

where \( α_j, β_j ∈ Δ_j \), for all \( j = 1, \ldots, k \). Applying Lemma 2.1, we derive

\[ \sum_{j=1}^{k} \sum_{α_j<β_j} a_{α_j} a_{β_j} ≥ \frac{1}{2γ}[η + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^r)^2], \]
it follows that
\[ \sum_{j=1}^{k} \sum_{\alpha_j \beta_j} \left[ h_{\alpha_j \beta_j} - (h_{\alpha_j \beta_j})^2 \right] \geq \frac{n}{2} + \frac{1}{2} \sum_{r=n+1}^{n+p} \sum_{(\alpha, \beta) \notin \Delta_{k+1}} (h_{\alpha \beta})^2 + \sum_{r=n+2}^{n+p} \sum_{\alpha_j \beta_j \in \Delta_j} (h_{\alpha_j \beta_j})^2 \geq \frac{n}{2}. \]

From (3.2) and (3.6) we have
\[ \sum_{j=1}^{k} \tau(L_j) \geq \sum_{j=1}^{k} \left( \frac{n_j(n_j - 1)}{2} a + b \Psi(L_j) \right) + \frac{1}{2} \eta. \]  

Using (2.8), (3.5) and (3.7), we derive the desired inequality.

The equality case of (3.1) at a point \( x \in M \) holds if and only if we have equality in all the previous inequalities and also in Lemma 2.1, thus, the shape operators take the desired forms.

4 Chen-Ricci Inequalities

In [21], Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for any \( n \)-dimensional Riemannian submanifold of a real space form \( R^m(c) \) of constant sectional curvature \( c \) as follows

**Theorem 4.1** (see [21, Theorem 4]) Let \( M \) be an \( n \)-dimensional submanifold of a real space form \( R^m(c) \). Then the following statements are true.

1. For each unit vector \( X \in T_p M \), we have
   \[ \| \zeta \|^2 \geq \frac{4}{n^2} [\mathrm{Ric}(X) - (n - 1)c]. \]  

2. If \( \zeta(p) = 0 \), then a unit vector \( X \in T_p M \) satisfies the equality case of (4.1) if and only if \( X \) belongs to the relative null space \( N(p) \) given by
   \[ N(p) = \{ X \in T_p M \mid h(X, Y) = 0, \forall Y \in T_p M \}. \]

3. The equality case of (4.1) holds for all unit vectors \( X \in T_p M \) if and only if either \( p \) is a geodesic point or \( n = 2 \) and \( p \) is an umbilical point.

Afterwards, many papers studied similar problems for different submanifolds in various ambient manifolds [22–24]. Thus, after putting an extra condition on the ambient manifold, like semi-symmetric metric connections in the case of real space forms [25] and curvature-like tensors in the case of a Riemannian manifold [26], one proves the results similar to that of Theorem 4.1.

In [20], Özgür obtained several Chen’s inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature. However, he didn’t establish an inequality between the classical Ricci-curvature and the squared mean curvature. Under these circumstances it
becomes necessary to give a theorem, which could present an inequality between the Ricci-curvature and the squared mean curvature for submanifolds in the ambient manifold.

**Theorem 4.2** Let \( M^n \) be an \( n \)-dimensional submanifold of an \((n + p)\)-dimensional Riemannian manifold of quasi-constant curvature \( \mathcal{N}^{n+p} \). For each unit vector \( X \) in \( T_x M \) we have

\[
\text{Ric}(X) \leq (n - 1)a + (n - 2)b g(P^T, X)^2 + b \parallel P^T \parallel^2 + \frac{n^2}{4}H^2. \tag{4.2}
\]

The equality sign holds for any tangent vector \( X \) in \( T_x M \) if and only if either \( x \) is a totally geodesic point or \( n = 2 \) and \( x \) is an umbilical point.

**Remark 4.3** For \( b = 0 \), inequality (4.2) is due to (4.1).

**Remark 4.4** We should point out that our approach is different from Chen’s.

**Proof** Let \( x \in M^n \) and \( \{e_1, e_2, \ldots, e_n\} \) and \( \{e_{n+1}, e_{n+2}, \ldots, e_{n+p}\} \) be orthonormal bases of \( T_x M^n \) and \( T_x^\perp \mathcal{N}^n \), respectively, such that \( X = e_1 \). From equations (2.1), (2.2), (2.3) and (2.4) it follows that

\[
R_{ijij} = a + b[g(P^T, e_i)^2 + g(P^T, e_j)^2] + \sum_{r=n+1}^{n+p} [h^r_{ij} h^r_{ij} - (h^r_{ij})^2]. \tag{4.3}
\]

Using (4.3) one derives

\[
\text{Ric}(X) = \sum_{i=2}^{n} R_{1i1i} = (n - 1)a + (n - 1)b g(P^T, e_1)^2 \\
+ b \sum_{i=2}^{n} g(P^T, e_i)^2 + \sum_{r=n+1}^{n+p} \sum_{i=2}^{n} [h^r_{ii} h^r_{ii} - (h^r_{ii})^2] \\
\leq (n - 1)a + (n - 2)b g(P^T, X)^2 + b \parallel P^T \parallel^2 + \sum_{r=n+1}^{n+p} \sum_{i=2}^{n} h^r_{ii} h^r_{ii}. \tag{4.4}
\]

Let us consider the quadratic forms \( f_r : R^n \rightarrow R \), defined by

\[
f_r(h^r_{11}, h^r_{22}, \ldots, h^r_{nn}) = \sum_{i=2}^{n} h^r_{ii} h^r_{ii}.
\]

We consider the problem \( \max f_r \), subject to \( \Gamma : h^r_{11} + h^r_{22} + \cdots + h^r_{nn} = k^r \), where \( k^r \) is a real constant.

From Lemma 2.2, we see that the solution \( (h^r_{11}, h^r_{22}, \ldots, h^r_{nn}) \) of the problem in question must satisfy

\[
h^r_{11} = \sum_{j=2}^{n} h^r_{jj} = \frac{k^r}{2}, \tag{4.5}
\]

which implies

\[
f_r \leq \frac{(k^r)^2}{4}. \tag{4.6}
\]
From (4.4) and (4.6) we have

\[
\text{Ric}(X) \leq (n-1)a + (n-2)bg(P^T, X)^2 + b \| P^T \|^2 + \sum_{r=n+1}^{n+p} \frac{(k_r)^2}{4}.
\]

\[
= (n-1)a + (n-2)bg(P^T, X)^2 + b \| P^T \|^2 + \frac{n^2}{4} H^2.
\]

Next, we shall study the equality case.

For each unit vector \( X \) at \( x \), if the equality case of inequality (4.2) holds, from (4.4), (4.5) and (4.6) we have

\[ h_{r1} = 0, \quad i \neq 1, \quad \forall \ r, \quad (4.7) \]

\[ h_{r1} + h_{r2} + \cdots + h_{rn} - 2h_{11} = 0, \quad \forall r. \quad (4.8) \]

For any unit vector \( X \) at \( x \), if the equality case of inequality (4.2) holds, noting that \( X \) is arbitrary, by computing \( \text{Ric}(e_j), j = 2, 3, \cdots, n \) and combining (4.7) and (4.8) we have

\[ h_{ij} = 0, \quad i \neq j, \quad \forall r; \quad h_{11} + h_{22} + \cdots + h_{nn} - 2h_{11} = 0, \quad \forall i, r. \]

We can distinguish two cases:

(1) \( n \neq 2 \), \( h_{ij} = 0, \quad i, j = 1, 2, \cdots, n, \quad r = n + 1, \cdots, n + p \) or

(2) \( n = 2 \), \( h_{11} = h_{22}, \quad h_{12} = 0, \quad r = 3, \cdots, 2 + p. \)

The converse is trivial.

We immediately have the following

**Corollary 4.5** Let \( M^n \) be an \( n \)-dimensional submanifold of an \( (n + p) \)-dimensional Riemannian manifold of quasi-constant curvature \( N^{n+p} \). The equality case of inequality (4.2) holds for any tangent vector \( X \) of \( M^n \) if and only if either \( M^n \) is a totally geodesic submanifold in \( N^{n+p} \) or \( n = 2 \) and \( M^n \) is a totally umbilical submanifold.

**Corollary 4.6** If \( \zeta(x) = 0 \), then a unit vector \( X \in T_x M \) satisfies the equality case of (4.2) if and only if \( X \) belongs to the relative null space \( N(x) \) given by

\[ N(x) = \{ X \in T_x M \mid h(X, Y) = 0, \quad \forall Y \in T_x M \}. \]

**Proof** Assume \( \zeta(x) = 0 \). For each unit vector \( X \in T_x M \), equality holds in (4.2) if and only if (4.5) and (4.7) hold. Then \( h_{1i} = 0, \quad \forall i, r \), i.e., \( X \in N(x) \).

## 5 Warped Product Submanifolds

Related with famous Nash embedding theorem[2], Chen established a general sharp inequality for warped products in real space form [27]. Later, he studied warped products in complex hyperbolic spaces [28] and complex projective spaces [29], respectively. Afterwards, many papers studied similar problems for different submanifolds in various ambient spaces [30–32]. In the present paper, we establish an inequality for warped product submanifolds of a Riemannian manifold of quasi-constant curvature.
The study of warped product manifolds was initiated by Bishop and O’Neill [33]. Following [33], we have

Let \((M_1, g_1)\) and \((M_2, g_2)\) be two Riemannian manifolds and \(f\) a positive differentiable function on \(M_1\), where \(\dim M_1 = n_1\) \((i = 1, 2)\), \(n_1 + n_2 = n\). The warped product of \(M_1\) and \(M_2\) is the Riemannian manifold \(M_1 \times_f M_2 = (M_1 \times M_2, g)\), where \(g = g_1 + f^2g_2\). More explicitly, if vector fields \(X\) and \(Y\) tangent to \(M_1 \times_f M_2\) at \((x, y)\), then

\[
g(X, Y) = g_1(\pi_1X, \pi_1Y) + f^2(x)g_2(\pi_2X, \pi_2Y),
\]

where \(\pi_i(i = 1, 2)\) are the canonical projections of \(M_1 \times_f M_2\) onto \(M_1\) and \(M_2\), respectively, and * stands for derivative map.

For a warped product \(M_1 \times_f M_2\), we denote by \(D_1\) and \(D_2\) the distributions given by the vectors tangent to leaves and fibres, respectively, where \(D_1\) is obtained from the tangent vectors of \(M_1\) via the horizontal lift and \(D_2\) by tangent vectors of \(M_2\) via the vertical lift.

Let \(\phi : M^n = M_1 \times_f M_2 \rightarrow N^{n+p}\) be an isometric immersion of a warped product \(M_1 \times_f M_2\) into a Riemannian manifold of quasi-constant curvature. Denote by \(h\) the second fundamental form of \(\phi\). Denote by \(\text{tr}h_1\) and \(\text{tr}h_2\) the trace of \(h\) restricted to \(M_1\) and \(M_2\), respectively. The immersion \(\phi\) is called mixed totally geodesic if \(h(X,Z) = 0\) for any \(X\) in \(D_1\) and \(Z\) in \(D_2\).

Since \(M_1 \times_f M_2\) is a warped product, we have \(\nabla_XZ = \nabla_ZX = \frac{1}{f}(Xf)Z\) for any unit vector fields \(X, Z\) tangent to \(M_1, M_2\), respectively. It follows that

\[
K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}[(\nabla_X f) X - X^2 f]. \tag{5.1}
\]

We set \(\|P^T\|_{\mathcal{M}_1}^2 = \sum_{j=1}^{n_1} g(P^T, e_j)^2\), \(\|P^T\|_{\mathcal{M}_2}^2 = \sum_{s=n_1+1}^{n} g(P^T, e_s)^2\).

**Theorem 5.1** Let \(\phi : M_1 \times_f M_2 \rightarrow N^{n+p}\) be an isometric immersion of a warped product into a Riemannian manifold of quasi-constant curvature, then we have

\[
\frac{\triangle f}{f} \leq \frac{n^2 H^2}{4n_2} + n_1 a + \frac{b}{n_2} \|P^T\|_{\mathcal{M}_1}^2 + n_1 \|P^T\|_{\mathcal{M}_2}^2, \tag{5.2}
\]

where \(H^2\) is the squared mean curvature of \(\phi\), and \(\triangle\) is the Laplacian operator of \(M_1\).

The equality case of (5.2) holds if and only if \(\phi\) is a mixed totally geodesic immersion with \(\text{tr}h_1 = \text{tr}h_2\).

**Proof** In \(N^{n+p}\) we choose a local orthonormal frame \(\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}\), such that \(e_1, \cdots, e_{n_1}\) are tangent to \(M_1\), \(e_{n+1}, \cdots, e_n\) are tangent to \(M_2\), \(e_{n+1}\) is parallel to the mean curvature vector \(\zeta\).

Using (5.1) and the definition of \(\triangle f\), we get

\[
\frac{\triangle f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s) \tag{5.3}
\]

for each \(s \in \{n_1 + 1, \cdots, n\}\).
Using (2.1), (2.3) and (2.4) we have
\[ 2\tau + \| h \|^2 - n^2H^2 = 2b(n-1) \| P^T \|^2 + (n^2 - n)a. \] (5.4)

We set
\[ \delta = 2\tau - (n^2 - n)a - 2b(n-1) \| P^T \|^2 - \frac{n^2}{2}H^2. \] (5.5)

Then (5.4) can be written as
\[ n^2H^2 = 2(\delta + \| h \|^2). \] (5.6)

If we put \( a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}, a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}, \) from (5.6) we have
\[
\left( \sum_{i=1}^{3} a_i \right)^2 = 2\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2
+ \sum_{r=n+2}^{n+p} \sum_{i=1}^{n} (h_{rj}^{n+1})^2 - \sum_{2 \leq j < k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq i < j \leq n} h_{ii}^{n+1} h_{tt}^{n+1}].
\]

From Lemma 2.1 we get
\[
\sum_{1 \leq j < k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \geq \frac{\delta}{2} + \sum_{1 \leq i < j \leq n} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{n+p} \sum_{i=1}^{n} (h_{rj}^{n+1})^2
\] (5.7)

with the equality holding if and only if
\[
\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}.
\] (5.8)

From (5.3) we have
\[
\frac{n_2 \Delta f}{f} = \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \leq s < t \leq n} K(e_s \wedge e_t)
= \tau - \frac{n_1(n_1-1)a}{2} - (n_1-1)b \sum_{j=1}^{n_1} g(P^T, e_j)^2 - \sum_{r=n+1}^{n+p} \sum_{1 \leq j < k \leq n_1} [h_{jj}^r h_{kk}^r - (h_{jk}^r)^2]
- \frac{n_2(n_2-1)a}{2} - (n_2-1)b \sum_{s=n_1+1}^{n} g(P^T, e_s)^2 - \sum_{r=n+1}^{n+p} \sum_{n_1+1 \leq s < t \leq n} [h_{ss}^r h_{tt}^r - (h_{st}^r)^2]
= \tau - \frac{n(n-1)a}{2} + n_1 n_2 a - b[(n_1-1) \sum_{j=1}^{n_1} g(P^T, e_j)^2 + (n_2-1) \sum_{s=n_1+1}^{n} g(P^T, e_s)^2]
- \sum_{r=n+1}^{n+p} \sum_{1 \leq j < k \leq n_1} [h_{jj}^r h_{kk}^r - (h_{jk}^r)^2] - \sum_{r=n+1}^{n+p} \sum_{n_1+1 \leq s < t \leq n} [h_{ss}^r h_{tt}^r - (h_{st}^r)^2].
\] (5.9)
Combining (5.7) and (5.9) we have

\[
\frac{n_2 \Delta f}{f} \leq \tau - \frac{n(n - 1)a}{2} + n_1 n_2 a - b[(n_1 - 1) \sum_{j=1}^{n_1} g(P^T, e_j)^2 + (n_2 - 1) \sum_{s=n_1+1}^{n} g(P^T, e_s)^2] - \frac{\delta}{2} - \frac{1}{n_2} \sum_{1 \leq j \leq n_1 \leq t \leq n} (h_{jt}^{n+1})^2 - \frac{1}{2} \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^r)^2
\]

\[
- \sum_{r=n+2}^{n+p} \sum_{1 \leq k < t \leq n_1} [h_{jt}^r h_{kk}^r - (h_{jk}^r)^2] - \sum_{r=n+2}^{n+p} \sum_{1 \leq s \leq n_1 \leq t \leq n} [h_{ss}^r h_{tt}^r - (h_{st}^r)^2].
\]

\[
= \tau - \frac{n(n - 1)a}{2} + n_1 n_2 a - b[(n_1 - 1) \sum_{j=1}^{n_1} g(P^T, e_j)^2 + (n_2 - 1) \sum_{s=n_1+1}^{n} g(P^T, e_s)^2] - \frac{\delta}{2}
\]

\[
\leq \frac{n^2 H^2}{4} + n_1 n_2 a - b[(n_1 - 1) \sum_{j=1}^{n_1} g(P^T, e_j)^2]
\]

\[
+ (n_2 - 1) \sum_{s=n_1+1}^{n} g(P^T, e_s)^2] + b(n-1) \| P^T \|^2
\]

\[
= \frac{n^2 H^2}{4} + n_1 n_2 a + b \| P^T \|^2 - n_1 \sum_{j=1}^{n_1} g(P^T, e_j)^2 - n_2 \sum_{s=n_1+1}^{n} g(P^T, e_s)^2],
\]

which proves inequality.

Next, we shall study the equality case.

From (5.7) and (5.10) we know that the equality case of (5.2) holds if and only if

\[
h_{jt}^r = 0, \quad 1 \leq j \leq n_1, \quad n_1 + 1 \leq t \leq n, \quad n + 1 \leq r \leq n + p,
\]

(5.11)

\[
\sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^{n} h_{tt}^r = 0, \quad n + 2 \leq r \leq n + p.
\]

(5.12)

Obviously (5.11) is equivalent to \( h(D_1, D_2) = 0 \), thus, the immersion \( \phi \) is mixed totally geodesic. Further on, from (5.8) and (5.12), we have

\[
\sum_{i=1}^{n_1} h_{ii}^r = \sum_{s=n_1+1}^{n} h_{ss}^r, \quad \forall r,
\]

it follows that \( \text{tr} h_1 = \text{tr} h_2 \).

**Remark 5.2** If \( b = 0 \), inequality (5.2) is due to Chen [28, Theorem 1.4].

As applications of Theorem 5.1, we have
Corollary 5.3 Under the same assumption as in Theorem 5.1, if \( f \) is a harmonic function, there are no isometric minimal immersion of \( M_1 \times_f M_2 \) into \( N^{n+p} \) with \( a < 0, b \leq 0 \).

Corollary 5.4 Under the same assumption as in Theorem 5.1, if \( f \) is an eigenfunction of the Laplacian on \( M_1 \) with eigenvalue \( \lambda > 0 \), there are no isometric minimal immersion of \( M_1 \times_f M_2 \) into \( N^{n+p} \) with \( a < 0, b \leq 0 \).

Remark 5.5 In [34, Theorem 4.1], Ganchev and Mihova proved that a Riemannian manifold of quasi-constant curvature \( N^{n+p}(n+p \geq 4) \) with \( a < 0, b \neq 0 \), can be locally \( \xi \)-isometric to a canal space-like hypersurface in the Minkowski space \( \mathbb{R}^{n+p+1}_1 \), \( \xi \) is a unit vector field on \( N^{n+p} \).

References


拟常曲率黎曼流形中子流形的几何不等式的一些注记

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摘要: 本文研究了拟常曲率黎曼流形中子流形的Chen不等式。利用代数技巧，建立了Chen广义不等式。Chen-Ricci不等式和关于卷积函数和平均曲率平方的不等式，推广了Özgür和Chen的一些结果。

关键词: Chen不等式; Chen-Ricci不等式; 卷积; 拟常曲率