ISOPERIMETRIC ESTIMATE OF THE FIRST EIGENVALUES FOR THE WEIGHTED $p$-LAPLACIAN ON MANIFOLDS

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Abstract: In this paper, we estimate the lower bounds of the first eigenvalues for the weighted $p$-Laplacian on manifolds. By using the coarea formula, the Cavalieri principle and the Federer-Fleming theorem, we obtain the estimation of the lower bounds for the first eigenvalues by the Cheeger constant or the isoperimetric constant.

Keywords: weighted $p$-Laplacian; weighted manifold; isoperimetric constant; first eigenvalue; lower bound

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1 Introduction

Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in an $n$-dimensional Riemannian manifold $(M, g)$. The $p$-Laplacian is defined by

$$\Delta_p : W_0^{1,p}(\Omega) \mapsto W^{-1,q}(\Omega),$$

$$u \mapsto \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u),$$

where $W_0^{1,p}(\Omega)$ is the Sobolev space given by the closure of $C_0^{\infty}(\Omega)$ with norm

$$\|u\|_{1,p} = \int_\Omega |u|^p dV + \int_\Omega |\nabla u|^p dV,$$

and $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ and $1 < p, q < \infty, \frac{1}{p} + \frac{\frac{1}{q}}{n} = 1$. As a generalization of the usual Laplacian, the $p$-Laplacian is widely used in many subjects, especially $\Delta_p$ models the non-Newtonian fluids in physics. It describes dilatant fluids when $p > 2$ and pseudoplastics when $p < 2$, whereas $p = 2$ corresponds to Newtonian fluids. The operator

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\( \Delta_p \) with \( p \neq 2 \) also appears in many other applications, such as reaction-diffusion problems, flow through porous media, nonlinear elasticity, etc., see [14] for more details.

Let \((M, g, d\mu)\) be a weighted manifold, that is, a Riemannian manifold \((M, g)\) endowed with a weighted volume form \(d\mu = e^{-\varphi} dV\), where \(\varphi \in C^\infty(M)\) and \(dV\) is the volume element induced by the metric \(g\). With respect to the weighted measure, the weighted \(p\)-Laplacian is defined as follows

\[
\Delta_{p,\varphi} u = \text{div}(\|\nabla u\|^{p-2}\nabla u) - \|\nabla u\|^{(p-2)} \langle \nabla u, \nabla \varphi \rangle = \Delta_p u - \|\nabla u\|^{(p-2)} \langle \nabla u, \nabla \varphi \rangle.
\]

We are interested in the following nonlinear eigenvalue problem (the Dirichlet eigenvalue problem)

\[
\begin{cases}
\Delta_{p,\varphi} u + \lambda |u|^{p-2} u = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases}
\tag{1.1}
\]

We recall that the first eigenvalue for the weighted \(p\)-Laplacian has the following variational characterisation

\[
\lambda_{p,\varphi}(\Omega) = \min_{\substack{u \in W^{1,p}_0(\Omega) \\
u \equiv 0}} \frac{\int_{\Omega} \|\nabla u\|^p e^{-\varphi} dV}{\int_{\Omega} |u|^p e^{-\varphi} dV}.
\tag{1.2}
\]

The problems of partial differential equations involving weighted \(p\)-Laplacian have been studied by many mathematicians, see [1, 18], etc.. For more researches on eigenvalue problems, we refer the readers to [6, 15, 16] etc.

For the following nonlinear eigenvalue problem

\[
\begin{cases}
\Delta_p f + \lambda |f|^{p-2} f = 0 \text{ in } \Omega, \\
f|_{\partial \Omega} = 0.
\end{cases}
\]

The first eigenvalue associated with a Riemannian metric \(g\) on a manifold \(M\) has been extensively studied in recent mathematical literature, such as [7–11], etc.. In [8] and [9], Kawohl-Fridman and Lefton-Wei used the coarea formula and the Cavalieri principle to estimate the lower bound of the first eigenvalue about this problem by the Cheeger constant

\[
\lambda_{1,p}(\Omega) \geq \left( \frac{h(\Omega)}{p} \right)^p,
\]

where \(h(\Omega) = \mathcal{H}_\infty(\Omega)\) is the Cheeger constant of domain \(\Omega\). This wonderful result inspires us to estimate the lower bounds of the first eigenvalues for the weighted \(p\)-Laplacian operator eigenvalue problems by the Cheeger constant.

In this paper, we use the coarea formula, the Cavalieri principle and the Federer-Fleming Theorem to investigate the first eigenvalues of problem (1.1). We obtain the lower bounds estimations of the first eigenvalues for the weighted \(p\)-Laplace operator eigenvalue problems by the Cheeger constant and isoperimetric constant.
2 Main Results

In this section, our main goal is to estimate the lower bounds of the first eigenvalues for the weighted $p$-Laplacian eigenvalue problems on weighted manifolds. First, we recall some preliminary knowledge of the isoperimetric constant, Cavalieri's Principle and the coarea formula for later use.

Definition 2.1 Let $M$ be an $n$-dimensional Riemannian manifold with $n \geq 2$. For each $\nu > 1$, the $\nu$-isoperimetric constant of $M$, $J_\nu(M)$, is defined to be the infimum

$$J_\nu(M) = \inf_{\Omega} \frac{A(\partial \Omega)}{V(\Omega)^{1-\frac{1}{\nu}}}$$

where $\Omega$ varies over open submanifolds of $M$ possessing compact closure and $C^\infty$ boundary. If $\nu = \infty$, $J_\infty(M)$ is called the Cheeger constant, that is

$$J_\infty(M) = \inf_{\Omega} \frac{A(\partial \Omega)}{V(\Omega)}$$

Remark 2.2 As stated in [3], the fact that $J_\nu(M) > 0$ is only possible for $n \leq \nu \leq \infty$. Indeed, let $\nu < n$, and consider a small geodesic ball $B(x; \epsilon)$, with center $x \in M$ and radius $\epsilon > 0$, for the isoperimetric quotient of $B(x; \epsilon)$,

$$\lim_{\epsilon \to 0} \frac{A(\partial \Omega)}{V(\Omega)^{1-\frac{1}{\nu}}} \sim \lim_{\epsilon \to 0} \text{const.} \epsilon^{\frac{n}{\nu} - 1} = 0.$$ 

So it seems at first glance that one only has a discussion of isoperimetric constants for $\nu \geq n$.dimM.

Definition 2.3 Let $M$ be an $n$-dimensional Riemannian manifold, $n \geq 2$. For each $\nu > 1$, the Sobolev constant of $M$, $S_\nu(M)$, is defined to be the infimum

$$S_\nu(M) = \inf_{f} \frac{\|\nabla f\|_{1}}{\|f\|_{\nu}}$$

where $f \in C^\infty_0(M)$.

The isoperimetric constant and the Sobolev constant have the following famous relationship:

Lemma 2.4 (The Federer-Fleming Theorem) The isoperimetric and Sobolev constants are equal, that is,

$$J_\nu(M) = S_\nu(M).$$  \hspace{1cm} (2.1)

The detailed proof of the Federer-Fleming theorem can be found in [3, 4] and [12]. This elegant result was first proven in [4] by Federer and Fleming, and in [12] independently by Maz'ya in 1960.

Lemma 2.5 (see [3] The coarea Formula) Let $M$ be a $C^n$ Riemannian manifold, and let $\Phi : M \to \mathbb{R}$ be a $C^n$ function. Then for any measurable function $u : M \to \mathbb{R}$ that is everywhere nonnegative or is in $L^1(M)$, one has

$$\int_M u|\nabla \Phi|dV = \int_R dV_1(y) \int_{\Phi^{-1}(y)} (u|_{\Phi^{-1}(y)})dA.$$
Lemma 2.6 (see [3] Cavalieri’s Principle) Let $\nu$ be a measure on Borel sets in $[0, \infty]$, $\phi$ its indefinite integral, given by

$$\phi(t) = \nu([0,t)) < +\infty, \forall t > 0,$$

$(\Omega, \Sigma, \mu)$ a measure space, and $u$ a nonnegative $\Sigma$-measurable function on $\Omega$. Then

$$\int_\Omega \phi(u(x))d\mu(x) = \int_0^\infty \mu(u > t)d\nu(t)$$

or equivalently

$$\int_\Omega d\mu(x) \int_0^{u(x)} d\nu(t) = \int_0^\infty d\nu(t) \int_\Omega I_{\{u(t)\}}d\mu.$$

Using the coarea formula and the Cavalieri principle, we can get the following lower bound estimation of the first eigenvalue for the weighted $p$-Laplacian on weighted Riemannian manifold by the Cheeger constant.

Theorem 2.7 Let $\Omega$ be a connected domain with smooth boundary $\partial \Omega$ in an $n$-dimensional weighted Riemannian manifold $(M, g, d\mu)$. Assume $\lambda_{p, \varphi}(\Omega)$ is the first eigenvalue of problem (1.1) for $\varphi \in C^\infty(\Omega)$. Then

$$\lambda_{p, \varphi}^\frac{1}{p}(\Omega) \geq \frac{1}{p} \left( h(\Omega) - C_\varphi \right), \quad (2.2)$$

where $C_\varphi = \max_{x \in \Omega} |\nabla \varphi|$ and $h(\Omega) = \mathcal{H}_n(\Omega)$ are the the Cheeger constant of domain $\Omega$.

Proof For any $u \in C^\infty_0(\Omega)$, set

$$\Omega(t) = \{ x \in \Omega : |u|^p e^{-\varphi} > t \}$$

and

$$V(t) = V(\Omega(t)), \quad A(t) = A(\partial \Omega(t)).$$

It follows from the Hölder inequality that

$$\int_\Omega |\nabla (u^p e^{-\varphi})|dV \leq p \int_\Omega |u|^{p-1} |\nabla u| e^{-\varphi} dV + \int_\Omega |u|^p |\nabla \varphi| e^{-\varphi} dV$$

$$\leq p \left\{ \int_\Omega |u|^p e^{-\varphi} dV \right\}^{\frac{p-1}{2}} \left\{ \int_\Omega |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{2}} + \int_\Omega |u|^p |\nabla \varphi| e^{-\varphi} dV. \quad (2.3)$$

From the coarea formula, the Cavalieri principle and the definition of Cheeger constant, we
can get
\[
\int_{\Omega} |\nabla (u^p e^{-\varphi})| dV = \int_{0}^{\infty} A(t) dt
\]
\[
= \int_{0}^{\infty} \frac{A(t)}{V(t)} V(t) dt \geq h(\Omega) \int_{0}^{\infty} V(t) dt
\]
\[
= h(\Omega) \int_{0}^{\infty} dt \int_{\Omega} I_{\{|u|^p e^{-\varphi} > t\}} dV
\]
\[
= h(\Omega) \int_{\Omega} dV \int_{0}^{\infty} \frac{|u|^p e^{-\varphi}}{dt}
\]
\[
= h(\Omega) \int_{\Omega} |u|^p e^{-\varphi} dV,
\]
since \( C_{\infty}^c(\Omega) \) is dense in \( W^{1,p}_{0}(\Omega) \), the above relation holds also for any \( u \in W^{1,p}_{0}(\Omega) \), which together with (2.3) implies
\[
\frac{\left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}}}{\left\{ \int_{\Omega} |u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}}} \geq \frac{1}{p} (h(\Omega) - C_{\varphi}),
\]
this inequality and (1.2) imply
\[
\lambda_{\rho, \varphi}^{\frac{1}{p}}(\Omega) = \min_{\substack{u \in W^{1,p}_{0}(\Omega) \\
u \geq 0}} \frac{\left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}}}{\left\{ \int_{\Omega} |u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}}} \geq \frac{1}{p} (h(\Omega) - C_{\varphi}),
\]
which completes the proof.

Using the Federer-Fleming theorem, we can get the following lower bound estimation of the first eigenvalue by the isoperimetric constant.

**Theorem 2.8** Assume that \( \Omega \) satisfies the conditions of Theorem 2.7, and the isoperimetric constant \( \mathcal{J}_\nu(\Omega) \) is positive for some \( \nu > 1 \). Then
\[
\lambda_{\rho, \varphi}^{\frac{1}{p}}(\Omega) \geq \frac{1}{p} \left( \frac{\mathcal{J}_\nu(\Omega)}{V(\Omega)^{\frac{\nu}{p}}} - C_{\varphi} \right), \tag{2.4}
\]

**Proof** For any \( u \in W^{1,p}_{0}(\Omega) \), let \( f(u) = |u|^{p-1} u e^{-\varphi} \), then, we first have by the Hölder inequality that
\[
\int_{\Omega} |f| dV \leq \left\{ \int_{\Omega} |f|^\frac{\nu}{\nu-1} dV \right\}^{\frac{\nu-1}{\nu}} \left\{ \int_{\Omega} 1 dV \right\} \frac{1}{\nu} = \left\{ \int_{\Omega} |f|^\frac{\nu}{\nu-1} dV \right\}^{\frac{\nu-1}{\nu}} V(\Omega)^{\frac{\nu}{\nu-1}}. \tag{2.5}
\]
According to the Federer-Fleming theorem (2.1) and the definition of the sobolev constant, we deduce
\[
\frac{\mathcal{J}_\nu(\Omega)}{V(\Omega)^{\frac{\nu}{p}}} \left\{ \int_{\Omega} |f|^\frac{\nu}{\nu-1} dV \right\}^{\frac{\nu-1}{\nu}} \leq \int_{\Omega} |\nabla f| dV,
\]
which together with (2.5) gives us
\[ \int_{\Omega} |f| dV \leq \frac{V(\Omega)^{\frac{1}{p}}}{J_{\nu}(\Omega)} \int_{\Omega} |\nabla f| dV. \] (2.6)

Again, by the Hölder inequality, we have
\[ \int_{\Omega} |\nabla f| dV = p \int_{\Omega} |u|^{p-1}|\nabla u| e^{-\varphi} dV + \int_{\Omega} |u|^p |\nabla \varphi| e^{-\varphi} dV \]
\[ \leq p \left\{ \int_{\Omega} |u|^p e^{-\varphi} dV \right\}^{\frac{p-1}{p}} \left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}} + \int_{\Omega} |u|^p |\nabla \varphi| e^{-\varphi} dV. \] (2.7)

The combination of (2.6) and (2.7) can yield
\[ \int_{\Omega} |u|^p e^{-\varphi} dV \]
\[ \leq V(\Omega)^{\frac{1}{p}} \left\{ p \left\{ \int_{\Omega} |u|^p e^{-\varphi} dV \right\}^{\frac{p-1}{p}} \left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}} + \int_{\Omega} |u|^p |\nabla \varphi| e^{-\varphi} dV \right\}. \]

this inequality implies
\[ \frac{\left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}}}{\left\{ \int_{\Omega} |u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}}} \geq \frac{1}{p} \left( \frac{J_{\nu}(\Omega)}{V(\Omega)^{\frac{1}{p}}} - C_\varphi \right), \]

from this inequality and (1.2), it is obvious that
\[ \lambda_{p,\varphi}^{\frac{1}{p}}(\Omega) = \min_{u \in W_{0}^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}} \leq \frac{1}{p} \left( \frac{J_{\nu}(\Omega)}{V(\Omega)^{\frac{1}{p}}} - C_\varphi \right), \]

which completes the proof.

**Remark 2.9** It is obvious that, if we take \( \nu = \infty \), then from (2.4) we have
\[ \lambda_{p,\varphi}^{\frac{1}{p}}(\Omega) = \min_{u \in W_{0}^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla u|^p e^{-\varphi} dV \right\}^{\frac{1}{p}} \geq \lim_{\nu \to \infty} \frac{1}{p} \left( \frac{J_{\nu}(\Omega)}{V(\Omega)^{\frac{1}{p}}} - C_\varphi \right) = \frac{1}{p} \left( h(\Omega) - C_\varphi \right). \]

**Corollary 2.10** Let \( \Omega \) be a connected domain with smooth boundary \( \partial \Omega \) in the Euclidean space \( \mathbb{R}^n \). Then
\[ \lambda_{p,\varphi}^{\frac{1}{p}}(\Omega) \geq \frac{n}{p} \left( \frac{\omega_n}{V(\Omega)} \right)^{\frac{1}{n}} - nC_\varphi, \] (2.8)
where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \).

**Proof** It is well known that

\[
J_n(\Omega) = n\omega_n^\frac{1}{n}
\]

for any domain \( \Omega \subseteq \mathbb{R}^n \), where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). From this fact and (2.4), we can get

\[
\lambda_{p,\varphi}^{\frac{1}{p}}(\Omega) \geq \frac{n}{p} \left( \left( \frac{\omega_n}{V(\Omega)} \right)^{\frac{1}{n}} - nC_\varphi \right),
\]

which completes the proof.

**Example 1** If \( \Omega = B_n(R) \) is a ball in \( \mathbb{R}^n \) with radius \( R \), then the volume of \( \Omega \) is \( V(\Omega) = \omega_n R^n \), and we can get

\[
\lambda_{p,\varphi}^{\frac{1}{p}}(\Omega) \geq \frac{n}{p} \left( \frac{1}{R} - nC_\varphi \right)
\]

directly by (2.8). Since any ball is trivial Cheeger set (see [2]), by simply calculation, we can obtain

\[
h(\Omega) = \frac{A(\partial \Omega)}{V(\Omega)} = \frac{n}{R}
\]

from inequality (2.2), thus, we can get the same inequality as above.

**Example 2** Let \( S^n \) be a unit sphere with sectional curvature 1, and \( \Omega \subseteq S^n \) (small enough) be a relatively compact domain with smooth boundary \( \partial \Omega \). Then the Ricci curvature of \( S^n \) is \( n - 1 \). From [17, Theorem 1.4], we know that for any connected domain \( \Omega \subset S^n \), \( n = 2, 3, 4, 5 \),

\[
\frac{A(\partial \Omega)}{V(\Omega)^{1-\frac{2}{n}}} \geq n\omega_n^\frac{2}{n} \left( 1 - \tau V(\Omega)^{\frac{2}{n}} \right)^\frac{1}{n},
\]

where \( \tau = \frac{n(n-1)}{2(n+2)\omega_n^2} \). According to Definition 2.1, we derive

\[
J_n(\Omega) \geq n\omega_n^\frac{1}{n} \left( 1 - \tau V(\Omega)^{\frac{2}{n}} \right)^\frac{1}{n}.
\]

Then from (2.4), we have

\[
\lambda_{p,\varphi}^{\frac{1}{p}}(\Omega) \geq \frac{1}{p} \left( \frac{n\omega_n^\frac{2}{n} \left( 1 - \tau V(\Omega)^{\frac{2}{n}} \right)^\frac{1}{n}}{V(\Omega)^\frac{2}{n}} - C_\varphi \right).
\]

**References**


加权流形上加权$p$-Laplace特征值问题的第一特征值下界估计

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摘要: 本文研究了加权流形上加权$p$-Laplacian特征值问题的第一特征值下界估计问题. 利用余面积公式, Cavalieri原理以及Federer-Fleming定理, 获得了由Cheeger常数或等周常数确定的第一特征值的下界估计.

关键词: 加权$p$-Laplacian; 加权流形; 等周常数; 第一特征值; 下界