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INTEGRAL REPRESENTATIONS FOR BI-REGULAR FUNCTIONS AND HARMONIC FUNCTIONS OVER PLANE IN CLIFFORD ANALYSIS

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Abstract: In this article, we study the integral representations over hyper-complex plane Π for bi-regular functions and harmonic functions with values in a Clifford algebra. By constructing the kernel functions, we give the integral representation formulas over hyper-complex plane Π for bi-regular functions and harmonic functions with values in a Clifford algebra. These results are extensions of integral representations over hyper-complex plane Π for regular functions.

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1 Introduction

Integral representation formulas are very powerful tools for solving boundary value problems in Clifford analysis. In [1–15, 18–28] etc., a great deal of work about integral representation formulas and boundary value problems in Clifford analysis was well presented. In [16–17], classical theories of boundary value problems and singular integral equations were systematically built.

However, most of the work about integral representation formulas was built over bounded domains. Naturally, developing integral representation formulas over unbounded domains is important and interesting, it will serve to study the Riemann-Hilbert boundary value problems for $k-$regular functions over unbounded domains in Clifford analysis. Similar to Cauchy type integrals over the real axis in classical complex analysis, Cauchy type integrals over the plane in Clifford analysis framework are also valuable. In [8], Cauchy transform and Hilbert transform over $\mathcal{R}^m$ were introduced; In [12–13] etc., by constructing the new Cauchy kernel function, some integral representation formulas over unbounded domains and its applications were shown. In [27], Cauchy type integral and singular integral over hyper-complex plane Π in the hyper-complex space RQ_3 were studied by using a special Möbius transform, integral representation formulas over hyper-complex plane Π for regular functions were built.

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In this paper, combining the idea in [9] with the technique in [12–13], we construct the kernel functions, and then give the integral representations over hyper-complex plane Π for bi-regular functions and harmonic functions with values in a Clifford algebra.

Let $V_{n,0}$ be an n-dimensional $(n \geq 1)$ real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, $C(V_{n,0})$ be the 2^n-dimensional real linear space with basis

$$\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\},$$

where N stands for the set $\{1, \cdots, n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. We denote $e_\emptyset$ as e_0 and e_A as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. The product on $C(V_{n,0})$ is defined by

$$\begin{cases} e_A e_B = (-1)^{\#(A \cap B)}(-1)^{P(A,B)} e_{A \triangle B}, & \text{if } A, B \in \mathcal{P}N, \\ \lambda\mu = \sum\limits_{A \in \mathcal{P}N} \sum\limits_{B \in \mathcal{P}N} \lambda_A \mu_B e_A e_B, & \text{if } \lambda = \sum\limits_{A \in \mathcal{P}N} \lambda_A e_A,\ \mu = \sum\limits_{B \in \mathcal{P}N} \mu_B e_B, \end{cases} \tag{1.1}$$

where $\#(A)$ is the cardinal number of the set A, the number $P(A,B) = \sum\limits_{j \in B} P(A,j)$, $P(A,j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A \triangle B$ is also order-preserving in the above way, and $\lambda_A \in \mathcal{R}$ is the coefficient of the e_A-component of the Clifford number λ. We also denote λ_0 as $\mathrm{Re}(\lambda)$. Thus $C(V_{n,0})$ is called the Clifford algebra over $V_{n,0}$.

An involution is defined by

$$\begin{cases} \overline{e_A} = (-1)^{\sigma(A)} e_A, & \text{if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A \in \mathcal{P}N} \lambda_A \overline{e_A}, & \text{if } \lambda = \sum\limits_{A \in \mathcal{P}N} \lambda_A e_A, \end{cases} \tag{1.2}$$

where $\sigma(A) = \#(A)(\#(A)+1)/2$. The $C(V_{n.0})$-valued n-differential form

$$\mathrm{d}\sigma = \sum_{k=0}^{n} (-1)^k e_k \mathrm{d}\widehat{x}_k^{N+1}, \ \ \overline{\mathrm{d}\sigma} = \sum_{k=0}^{n} (-1)^k \overline{e_k} \mathrm{d}\widehat{x}_k^{N+1}$$

are exact, where

$$\mathrm{d}\widehat{x}_k^{N+1} = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_{k-1} \wedge \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_n.$$

In this paper, we confine $n = 2$. The real linear space with basis $\{e_0, e_1, e_2\}$ is a subspace of $C(V_{2,0})$, which is called the reduced quaternions and denoted by RQ_3. The operator D which is written as

$$D = \sum_{k=0}^{2} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{2,0})) \to C^{(r-1)}(\Omega, C(V_{2,0})).$$

2 Some Definitions and Lemmas

Let $RQ_3 = \{\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 : x_0, x_1, x_2 \in \mathcal{R}\}$, then RQ_3 is identical with the usual Euclidean space $\mathcal{R}^3$. Denote $\Pi = \{\mathbf{x} \in RQ_3 | x_0 = 0\}$, $RQ_3^+ = \{\mathbf{x} \in RQ_3 | \mathrm{Re}(\mathbf{x}) > 0\}$,

$RQ_3^- = \{\mathbf{x} \in RQ_3 | \mathrm{Re}(\mathbf{x}) < 0\}$, $\partial B(\mathbf{x}, r) = \{\mathbf{y} \in RQ_3 || \mathbf{y} - \mathbf{x}| = r\}$, then Π and $\partial B(0,1)$ are the plane and unit sphere in hyper-complex space RQ_3, respectively. Denote $D(0,R) = \{\mathbf{x} \in \Pi || \mathbf{x}| < R\}$.

Definition 2.1 Denote $f \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$ if f is $\widehat{H}$ in Π. f is called $\widehat{H}$ in Π if f satisfies the following conditions: (i) $|f(\mathbf{x}) - f(\mathbf{x}^*)| \le M_1 |\mathbf{x} - \mathbf{x}^*|^{\mu}$ $\forall \mathbf{x}, \mathbf{x}^* \in \overline{D(0, R_1)}$, where R_1 is any given sufficiently great constant, M_1 is independent of $\mathbf{x}, \mathbf{x}^*$, M_1 depends on R_1, $0 < \mu \le 1$. (ii) $|f(\mathbf{x}) - f(\mathbf{x}^*)| \le M_2 \left| \dfrac{1}{\mathbf{x}} - \dfrac{1}{\mathbf{x}^*} \right|^{\mu}$ $\forall \mathbf{x}, \mathbf{x}^* \in \Pi \setminus D(0, R_2)$, where R_2 is any given sufficiently great constant, M_2 is independent of $\mathbf{x}, \mathbf{x}^*$, M_2 depends on R_2, $0 < \mu \le 1$.

Remark 2.1 $\forall \mathbf{x}, \mathbf{x}^* \in \overline{D(0, R_2)} \setminus D(0, R_1)$, $|f(\mathbf{x}) - f(\mathbf{x}^*)| \le M_1 |\mathbf{x} - \mathbf{x}^*|^{\mu}$ is equivalent to $|f(\mathbf{x}) - f(\mathbf{x}^*)| \le M_2 \left| \dfrac{1}{\mathbf{x}} - \dfrac{1}{\mathbf{x}^*} \right|^{\mu}$, where M_1 and M_2 are given constants.

Remark 2.2 By Definition 2.1, if $f \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$, then $\lim\limits_{|\mathbf{x}| \to \infty} f(\mathbf{x})$ exists, denote $\lim\limits_{|\mathbf{x}| \to \infty} f(\mathbf{x}) = f(\infty)$ and

$$|f(\mathbf{x}) - f(\infty)| \le \frac{M_2}{|\mathbf{x}|^{\mu}}, \ \forall \mathbf{x} \in \Pi \setminus D(0, R_2). \tag{2.1}$$

Definition 2.2 Denote $f \in \widehat{H}_0^{\mu}(\Pi, C(V_{2,0}))$ if f is $\widehat{H}_0$ in Π. f is called $\widehat{H}_0$ in Π if f satisfies the following conditions: (i) $f \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$; (ii) $f(\infty) = 0$.

Definition 2.3 A function $f \in C^{(r)}(\Omega, C(V_{2,0}))(r \ge 2)$ is called bi-regular in Ω if $D^2[f] = 0$ in Ω, which is also called 2-regular in Ω; A function $f \in C^{(r)}(\Omega, C(V_{2,0}))(r \ge 2)$ is called harmonic in Ω if $\triangle[f] = 0$ in Ω, where $\triangle$ is the Laplace operator.

Denote $H_1(\mathbf{y} - \mathbf{x}) = \dfrac{1}{4\pi} \dfrac{\overline{\mathbf{y}} - \overline{\mathbf{x}}}{|\mathbf{y} - \mathbf{x}|^3}$, $H_1^*(\mathbf{y} - \mathbf{x}) = \dfrac{1}{4\pi} \dfrac{\overline{\mathbf{y}} - \overline{S_{\Pi}(\mathbf{x})}}{|\mathbf{y} - S_{\Pi}(\mathbf{x})|^3} = \dfrac{1}{4\pi} \dfrac{\overline{\mathbf{y}} + \mathbf{x}}{|\mathbf{y} + \overline{\mathbf{x}}|^3}$, where $S_{\Pi}(\mathbf{x}) = -\overline{\mathbf{x}}$, $S_{\Pi}(\mathbf{x})$ is just the symmetric point of $\mathbf{x}$ with respect to Π, $\mathbf{y} \ne \mathbf{x}$. Denote $E_1(\mathbf{y} - \mathbf{x}) = H_1(\mathbf{y} - \mathbf{x}) - H_1^*(\mathbf{y} - \mathbf{x})$, $H_2(\mathbf{y} - \mathbf{x}) = H_1(\mathbf{y} - \mathbf{x}) \cdot (y_0 - x_0)$, $H_2^*(\mathbf{y} - \mathbf{x}) = H_1^*(\mathbf{y} - \mathbf{x}) \cdot (y_0 - x_0)$, $E_2(\mathbf{y} - \mathbf{x}) = E_1(\mathbf{y} - \mathbf{x}) \cdot (y_0 - x_0)$.

Lemma 2.1 Let $H_1(\mathbf{y} - \mathbf{x})$, $H_2(\mathbf{y} - \mathbf{x})$, $H_1^*(\mathbf{y} - \mathbf{x})$ and $H_2^*(\mathbf{y} - \mathbf{x})$ be as above, then

$$\begin{cases} D^2[H_2(\mathbf{y} - \mathbf{x})] & = D[H_1(\mathbf{y} - \mathbf{x})] = 0, \\ D^2[H_2^*(\mathbf{y} - \mathbf{x})] & = D[H_1^*(\mathbf{y} - \mathbf{x})] = 0, \\ [H_2(\mathbf{y} - \mathbf{x})]D^2 & = [H_1(\mathbf{y} - \mathbf{x})]D = 0, \\ [H_2^*(\mathbf{y} - \mathbf{x})]D^2 & = [H_1^*(\mathbf{y} - \mathbf{x})]D = 0, \end{cases}$$

where $D = \sum\limits_{k=0}^{2} e_k \dfrac{\partial}{\partial y_k}$.

Lemma 2.2 Let $E_1(\mathbf{y} - \mathbf{x})$ and $E_2(\mathbf{y} - \mathbf{x})$ be as above, then

$$\begin{cases} D^2[E_2(\mathbf{y} - \mathbf{x})] & = D[E_1(\mathbf{y} - \mathbf{x})] = 0, \\ [E_2(\mathbf{y} - \mathbf{x})]D^2 & = [E_1(\mathbf{y} - \mathbf{x})]D = 0, \end{cases}$$

where $D = \sum\limits_{k=0}^{2} e_k \dfrac{\partial}{\partial y_k}$.

Denote $K(\mathbf{y}-\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{\rho(\mathbf{y}-\mathbf{x})}$, $G(\mathbf{y}-\mathbf{x}) = -\frac{1}{4\pi}\left(\frac{1}{\rho(\mathbf{y}-\mathbf{x})} - \frac{1}{\rho(\mathbf{y}+\overline{\mathbf{x}})}\right)$, where $\rho(\mathbf{y}-\mathbf{x}) = \left(\sum\limits_{k=0}^{2}(y_k-x_k)^2\right)^{\frac{1}{2}}$.

Lemma 2.3 Let $K(\mathbf{y}-\mathbf{x})$ be as above, then

$$\begin{cases} \overline{D}[K(\mathbf{y}-\mathbf{x})] & = [K(\mathbf{y}-\mathbf{x})]\overline{D} = H_1(\mathbf{y}-\mathbf{x}), \\ D[K(\mathbf{y}-\mathbf{x})] & = [K(\mathbf{y}-\mathbf{x})]D = \overline{H_1}(\mathbf{y}-\mathbf{x}), \end{cases}$$

where $D = \sum\limits_{k=0}^{2} e_k \frac{\partial}{\partial y_k}$.

Lemma 2.4 Let $G(\mathbf{y}-\mathbf{x})$ be as above, then

$$\begin{cases} \overline{D}[G(\mathbf{y}-\mathbf{x})] & = [G(\mathbf{y}-\mathbf{x})]\overline{D} = E_1(\mathbf{y}-\mathbf{x}), \\ D[G(\mathbf{y}-\mathbf{x})] & = [G(\mathbf{y}-\mathbf{x})]D = \overline{E_1}(\mathbf{y}-\mathbf{x}), \end{cases}$$

where $D = \sum\limits_{k=0}^{2} e_k \frac{\partial}{\partial y_k}$.

Lemma 2.5 Let $E_1(\mathbf{y}-\mathbf{x})$ be as above, then

$$|E_1(\mathbf{y}-\mathbf{x})| \leq \frac{|x_0|}{2\pi}\left(\frac{2}{|\mathbf{y}-\mathbf{x}|^3} + \frac{1}{|\mathbf{y}-\mathbf{x}|^2|\mathbf{y}+\overline{\mathbf{x}}|} + \frac{1}{|\mathbf{y}-\mathbf{x}||\mathbf{y}+\overline{\mathbf{x}}|^2}\right). \tag{2.2}$$

Lemma 2.6 Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $D^2[f] = 0$ in Ω, where Ω is a bounded domain with smooth boundary in RQ_3, then for any $\mathbf{x} \notin \overline{\Omega}$,

$$\int\limits_{\partial\Omega} H_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} H_2(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y D[f](\mathbf{y}) = 0. \tag{2.3}$$

Proof By Lemma 2.1 and Stokes' formula (see [5]), the result follows.

Lemma 2.7 (see [9]) Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $D^2[f] = 0$ in Ω, where Ω is a bounded domain with smooth boundary in RQ_3, then for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = \int\limits_{\partial\Omega} H_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} H_2(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y D[f](\mathbf{y}). \tag{2.4}$$

Lemma 2.8 Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $\triangle[f] = 0$ in Ω , where Ω is a bounded domain with smooth boundary in RQ_3, then for any $\mathbf{x} \notin \overline{\Omega}$,

$$\int\limits_{\partial\Omega} H_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} K(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y}) = 0. \tag{2.5}$$

Proof By Lemma 2.1, Lemma 2.3 and Stokes' formula, the result follows.

Lemma 2.9 Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $\triangle[f] = 0$ in Ω , where Ω is a bounded domain with smooth boundary in RQ_3, then for any $\mathbf{x} \notin \overline{\Omega}$,

$$\int\limits_{\partial\Omega} \overline{H_1}(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} f(\mathbf{y}) - \int\limits_{\partial\Omega} K(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y \overline{D}[f](\mathbf{y}) = 0. \tag{2.6}$$

Proof By Lemma 2.1, Lemma 2.3 and Stokes' formula, the result follows.

Lemma 2.10 Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $\triangle[f] = 0$ in Ω, where Ω is a bounded domain with smooth boundary in RQ_3, then for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = \int\limits_{\partial\Omega} H_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} K(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y}). \tag{2.7}$$

Proof By Lemma 2.8, it can be similarly proved as in Lemma 2.7.

Lemma 2.11 Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $\triangle[f] = 0$ in Ω , where Ω is a bounded domain with smooth boundary in RQ_3, then for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = \int\limits_{\partial\Omega} \overline{H_1}(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} f(\mathbf{y}) - \int\limits_{\partial\Omega} K(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y \overline{D}[f](\mathbf{y}). \tag{2.8}$$

Proof By Lemma 2.9, it can be similarly proved as in Lemma 2.7.

Lemma 2.12 (see [27]) Let $f \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$, then for all $\mathbf{y}_*, \mathbf{y}_{**} \in \Pi$,

$$\lim_{R\to+\infty}\left(\frac{1}{4\pi}\iint\limits_{D(\mathbf{y}_*,R)}\frac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}f(\mathbf{y})\mathrm{d}S - \frac{1}{4\pi}\iint\limits_{D(\mathbf{y}_{**},R)}\frac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}f(\mathbf{y})\mathrm{d}S\right) = 0, \tag{2.9}$$

where $\mathbf{x} \in RQ_3$.

3 Integral Representations over Π for Bi-Regular Functions

In this section, we shall give the integral representations over Π for bi-regular functions. For $f(\mathbf{x}) \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$, the Cauchy type integral Cf over Π is defined by

$$Cf(\mathbf{x}) = -\frac{1}{4\pi}\iint\limits_{\Pi}\frac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}f(\mathbf{y})\mathrm{d}S,\ \mathbf{x} \notin \Pi, \tag{3.1}$$

where

$$-\frac{1}{4\pi}\iint\limits_{\Pi}\frac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}f(\mathbf{y})\mathrm{d}S = \lim_{R\to+\infty} -\frac{1}{4\pi}\iint\limits_{D(0,R)}\frac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}f(\mathbf{y})\mathrm{d}S. \tag{3.2}$$

Lemma 3.13 (see [27]) Let $f(\mathbf{y}) \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$, $Cf(\mathbf{x})$ be defined as in (3.1), then $Cf(\mathbf{x})$ exists and

$$Cf(\mathbf{x}) = \begin{cases} \dfrac{f(\infty)}{2} - \dfrac{1}{4\pi}\iint\limits_{\Pi}\dfrac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}(f(\mathbf{y})-f(\infty))\mathrm{d}S,\ \mathbf{x} \in RQ_3^+, \\ -\dfrac{f(\infty)}{2} - \dfrac{1}{4\pi}\iint\limits_{\Pi}\dfrac{\overline{\mathbf{y}}-\overline{\mathbf{x}}}{|\mathbf{y}-\mathbf{x}|^3}(f(\mathbf{y})-f(\infty))\mathrm{d}S,\ \mathbf{x} \in RQ_3^-. \end{cases} \tag{3.3}$$

Theorem 3.1 Let $f \in C^{(2)}(\Omega, C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega}, C(V_{2,0}))$ and $D^2[f] = 0$ in Ω, where Ω is a bounded domain with smooth boundary in RQ_3^+, then for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = \int\limits_{\partial\Omega} E_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} E_2(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y D[f](\mathbf{y}). \tag{3.4}$$

Proof By Lemma 2.2, Lemma 2.7 and Stokes' formula, the result follows.

Denote $\partial^+ B(\mathbf{x}, R) = \{\mathbf{y} | \mathbf{y} \in \partial B(\mathbf{x}, R),\ \mathrm{Re}(\mathbf{y}) > 0\}$.

Lemma 3.14 For any $\mathbf{x} \in RQ_3^+$,

$$\lim_{R\to+\infty} \int\limits_{\partial^+ B(\mathbf{x},R)} H_1(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y = \lim_{R\to+\infty} \int\limits_{\partial^+ B(\mathbf{x},R)} H_1^*(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y = \frac{1}{2}. \tag{3.5}$$

Proof It can be proved by Lemma 2.5.

Lemma 3.15 Let $f \in \widehat{H}^\mu(\overline{RQ_3^+}, C(V_{2,0}))$, then for any $\mathbf{x} \in RQ_3^+$,

$$\lim_{R\to+\infty} \int\limits_{\partial^+ B(\mathbf{x},R)} H_1(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y f(\mathbf{y}) = \lim_{R\to+\infty} \int\limits_{\partial^+ B(\mathbf{x},R)} H_1^*(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y f(\mathbf{y}) = \frac{1}{2} f(\infty). \tag{3.6}$$

Proof It can be proved by Lemma 3.14.

Lemma 3.16 Let $D[f] \in \widehat{H}_0^\mu(\overline{RQ_3^+}, C(V_{2,0}))$, then for any $\mathbf{x} \in RQ_3^+$,

$$\lim_{R\to+\infty} \int\limits_{\partial^+ B(\mathbf{x},R)} E_2(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y D[f](\mathbf{y}) = 0. \tag{3.7}$$

Proof It can be proved by Lemma 2.5.

Lemma 3.17 Let $f \in \widehat{H}^\mu(\Pi, C(V_{2,0}))$, then for all $\mathbf{y}_*, \mathbf{y}_{**} \in \Pi$,

$$\lim_{R\to+\infty} \left(\iint\limits_{D(\mathbf{y}_*,R)} E_2(\mathbf{y}-\mathbf{x}) f(\mathbf{y}) \mathrm{d}S - \iint\limits_{D(\mathbf{y}_{**},R)} E_2(\mathbf{y}-\mathbf{x}) f(\mathbf{y}) \mathrm{d}S \right) = 0, \tag{3.8}$$

where $\mathbf{x} \in RQ_3$.

Proof By Lemma 2.5, it can be similarly proved as in Lemma 2.12.

Theorem 3.2 Let $f \in C^{(1)}(\overline{RQ_3^+}, C(V_{2,0})) \bigcap C^{(2)}(RQ_3^+, C(V_{2,0}))$, $f \in \widehat{H}^\mu(\overline{RQ_3^+}, C(V_{2,0}))$, $D[f] \in \widehat{H}_0^\mu(\overline{RQ_3^+}, C(V_{2,0}))$ and $D^2[f] = 0$ in RQ_3^+, then for any $\mathbf{x} \in RQ_3^+$,

$$f(\mathbf{x}) = -\iint\limits_{\Pi} E_1(\mathbf{y}-\mathbf{x}) f(\mathbf{y}) \mathrm{d}S + \iint\limits_{\Pi} E_2(\mathbf{y}-\mathbf{x}) D[f](\mathbf{y}) \mathrm{d}S. \tag{3.9}$$

Proof For any $\mathbf{x} \in RQ_3^+$, denote $\Omega = B(\mathbf{x}, R) \bigcap RQ_3^+$, by Theorem 3.1, we have

$$f(\mathbf{x}) = \int\limits_{\partial\Omega} E_1(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} E_2(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y D[f](\mathbf{y}). \tag{3.10}$$

By Lemma 2.12 and Lemma 3.15, it can be proved that

$$\lim_{R\to+\infty} \int\limits_{\partial\Omega} E_1(\mathbf{y}-\mathbf{x}) \mathrm{d}\sigma_y f(\mathbf{y}) = -\iint\limits_{\Pi} E_1(\mathbf{y}-\mathbf{x}) f(\mathbf{y}) \mathrm{d}S. \tag{3.11}$$

In view of

$$\begin{aligned}&\int\limits_{\partial\Omega} E_2(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y D[f](\mathbf{y})\\ =&\int\limits_{\partial^+ B(\mathbf{x},R)} E_2(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y D[f](\mathbf{y})-\iint\limits_{D(\mathrm{Im}\mathbf{x},\sqrt{R^2-x_0^2})} E_2(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S,\end{aligned}\tag{3.12}$$

where $\mathrm{Im}\mathbf{x}=x_1e_1+x_2e_2$. By Lemma 3.17, in view of $D[f]\in\widehat{H}_0^{\mu}(\overline{RQ_3^+},C(V_{2,0}))$, we have

$$\lim_{R\to+\infty}\iint\limits_{D(\mathrm{Im}\mathbf{x},\sqrt{R^2-x_0^2})} E_2(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S=\iint\limits_{\Pi} E_2(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S.\tag{3.13}$$

By Lemma 3.16, Combining (3.12) with (3.13), we have

$$\lim_{R\to+\infty}\int\limits_{\partial\Omega} E_2(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y D[f](\mathbf{y})=-\iint\limits_{\Pi} E_2(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S.\tag{3.14}$$

Combining (3.10), (3.11) with (3.14), taking $R\to+\infty$ in (3.10), the result follows.

4 Integral Representations over Π for Harmonic Functions

In this section, we shall give the integral representations over Π for harmonic functions. Denote $K^*(\mathbf{y}-\mathbf{x})=-\dfrac{1}{4\pi}\dfrac{1}{\rho(\mathbf{y}-S_\Pi(\mathbf{x}))}=-\dfrac{1}{4\pi}\dfrac{1}{\rho(\mathbf{y}+\overline{\mathbf{x}})}$.

Theorem 4.3 Let $f\in C^{(2)}(\Omega,C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega},C(V_{2,0}))$ and $\triangle[f]=0$ in Ω, where Ω is a bounded domain with smooth boundary in RQ_3^+, then for any $\mathbf{x}\in\Omega$,

$$f(\mathbf{x})=\int\limits_{\partial\Omega} E_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y})-\int\limits_{\partial\Omega} G(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y}D[f](\mathbf{y}).\tag{4.1}$$

Proof By Stokes' formula and Lemma 2.3, for any $\mathbf{x}\in\Omega$, we have

$$\int\limits_{\partial\Omega} H_1^*(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y})-\int\limits_{\partial\Omega} K^*(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y}D[f](\mathbf{y})=0.\tag{4.2}$$

Combining Lemma 2.10 with (4.2), the result follows.

Theorem 4.4 Let $f\in C^{(2)}(\Omega,C(V_{2,0}))\bigcap C^{(1)}(\overline{\Omega},C(V_{2,0}))$ and $\triangle[f]=0$ in Ω, where Ω is a bounded domain with smooth boundary in RQ_3^+, then for any $\mathbf{x}\in\Omega$,

$$f(\mathbf{x})=\int\limits_{\partial\Omega} \overline{E_1}(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y}f(\mathbf{y})-\int\limits_{\partial\Omega} G(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y\overline{D}[f](\mathbf{y}).\tag{4.3}$$

Proof By Stokes' formula and Lemma 2.3, for any $\mathbf{x}\in\Omega$, we have

$$\int\limits_{\partial\Omega} \overline{H_1^*}(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y}f(\mathbf{y})-\int\limits_{\partial\Omega} K^*(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y\overline{D}[f](\mathbf{y})=0.\tag{4.4}$$

Combining Lemma 2.11 with (4.4), the result follows.

Lemma 4.18 Let $f \in \widehat{H}^{\mu}(\Pi, C(V_{2,0}))$, then for all $\mathbf{y}_*, \mathbf{y}_{**} \in \Pi$,

$$\lim_{R\to+\infty}\left(\iint\limits_{D(\mathbf{y}_*,R)} G(\mathbf{y}-\mathbf{x})f(\mathbf{y})\mathrm{d}S - \iint\limits_{D(\mathbf{y}_{**},R)} G(\mathbf{y}-\mathbf{x})f(\mathbf{y})\mathrm{d}S\right) = 0, \tag{4.5}$$

where $\mathbf{x} \in RQ_3$.

Proof It can be similarly proved as in Lemma 2.12.

Theorem 4.5 Let $f \in C^{(1)}(\overline{RQ_3^+}, C(V_{2,0})) \bigcap C^{(2)}(RQ_3^+, C(V_{2,0}))$, $f \in \widehat{H}^{\mu}(\overline{RQ_3^+}, C(V_{2,0}))$, $D[f] \in \widehat{H}_0^{\mu}(\overline{RQ_3^+}, C(V_{2,0}))$ and $\triangle[f] = 0$ in RQ_3^+, then for any $\mathbf{x} \in RQ_3^+$,

$$f(\mathbf{x}) = -\iint\limits_{\Pi} E_1(\mathbf{y}-\mathbf{x})f(\mathbf{y})\mathrm{d}S + \iint\limits_{\Pi} G(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S. \tag{4.6}$$

Proof For any $\mathbf{x} \in RQ_3^+$, denote $\Omega = B(\mathbf{x}, R) \bigcap RQ_3^+$, by Theorem 4.3, we have

$$f(\mathbf{x}) = \int\limits_{\partial\Omega} E_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) - \int\limits_{\partial\Omega} G(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y}). \tag{4.7}$$

By Lemma 2.12 and Lemma 3.15, it can be proved that

$$\lim_{R\to+\infty}\int\limits_{\partial\Omega} E_1(\mathbf{y}-\mathbf{x})\mathrm{d}\sigma_y f(\mathbf{y}) = -\iint\limits_{\Pi} E_1(\mathbf{y}-\mathbf{x})f(\mathbf{y})\mathrm{d}S. \tag{4.8}$$

In view of

$$\begin{aligned}&\int\limits_{\partial\Omega} G(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y})\\ =&\int\limits_{\partial^+ B(\mathbf{x},R)} G(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y}) - \iint\limits_{D(\mathrm{Im}\mathbf{x},\sqrt{R^2-x_0^2})} G(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S,\end{aligned} \tag{4.9}$$

where $\mathrm{Im}\mathbf{x} = x_1e_1 + x_2e_2$. In view of $D[f] \in \widehat{H}_0^{\mu}(\overline{RQ_3^+}, C(V_{2,0}))$, it can be proved that

$$\lim_{R\to+\infty}\int\limits_{\partial^+ B(\mathbf{x},R)} G(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y}) = 0. \tag{4.10}$$

By Lemma 4.18, in view of $D[f] \in \widehat{H}_0^{\mu}(\overline{RQ_3^+}, C(V_{2,0}))$, we have

$$\lim_{R\to+\infty}\iint\limits_{D(\mathrm{Im}\mathbf{x},\sqrt{R^2-x_0^2})} G(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S = \iint\limits_{\Pi} G(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S. \tag{4.11}$$

Combining (4.9), (4.10) with (4.11), we have

$$\lim_{R\to+\infty}\int\limits_{\partial\Omega} G(\mathbf{y}-\mathbf{x})\overline{\mathrm{d}\sigma_y} D[f](\mathbf{y}) = -\iint\limits_{\Pi} G(\mathbf{y}-\mathbf{x})D[f](\mathbf{y})\mathrm{d}S. \tag{4.12}$$

Combining (4.7), (4.8) with (4.12), taking $R \to +\infty$ in (4.7), the result follows.

Corollary 4.1 Let $f \in C^{(1)}(\overline{RQ_3^+}, C(V_{2,0})) \bigcap C^{(2)}(RQ_3^+, C(V_{2,0}))$, $f \in \widehat{H}^\mu(\overline{RQ_3^+}, C(V_{2,0}))$ and $D[f] = 0$ in RQ_3^+, then for any $\mathbf{x} \in RQ_3^+$,

$$f(\mathbf{x}) = -\iint\limits_{\Pi} E_1(\mathbf{y} - \mathbf{x}) f(\mathbf{y}) \mathrm{d}S. \tag{4.13}$$

Theorem 4.6 Let $f \in C^{(1)}(\overline{RQ_3^+}, C(V_{2,0})) \bigcap C^{(2)}(RQ_3^+, C(V_{2,0}))$, $f \in \widehat{H}^\mu(\overline{RQ_3^+}, C(V_{2,0}))$, $\overline{D}[f] \in \widehat{H}_0^\mu(\overline{RQ_3^+}, C(V_{2,0}))$ and $\triangle[f] = 0$ in RQ_3^+, then for any $\mathbf{x} \in RQ_3^+$,

$$f(\mathbf{x}) = -\iint\limits_{\Pi} \overline{E_1}(\mathbf{y} - \mathbf{x}) f(\mathbf{y}) \mathrm{d}S + \iint\limits_{\Pi} G(\mathbf{y} - \mathbf{x}) \overline{D}[f](\mathbf{y}) \mathrm{d}S. \tag{4.14}$$

Proof By Theorem 4.4, it can be similarly proved as in Theorem 4.5.

Corollary 4.2 Let $f \in C^{(1)}(\overline{RQ_3^+}, C(V_{2,0})) \bigcap C^{(2)}(RQ_3^+, C(V_{2,0}))$, $f \in \widehat{H}^\mu(\overline{RQ_3^+}, C(V_{2,0}))$ and $\overline{D}[f] = 0$ in RQ_3^+, then for any $\mathbf{x} \in RQ_3^+$,

$$f(\mathbf{x}) = -\iint\limits_{\Pi} \overline{E_1}(\mathbf{y} - \mathbf{x}) f(\mathbf{y}) \mathrm{d}S. \tag{4.15}$$

References

[1] Begehr H. Iterations of Pompeiu operators[J]. Mem. Diff. Eq. Math. Phys., 1997, 12(1): 3–21.

[2] Begehr H. Iterated integral operators in Clifford analysis[J]. J. Anal. Appl., 1999, 18(2): 361–377.

[3] Begehr H. Representation formulas in Clifford analysis[A]. Acoustics, mechanics and the related topics of mathematical analysis[C]. Singapore: World Sci. Publ., 2002: 8–13.

[4] Begehr H, Zhang Zhongxiang, Du Jinyuan. On Cauchy-Pompeiu formula for functions with values in a universal Clifford algebra[J]. Acta Math. Sci., 2003, 23B(1): 95–103.

[5] Brack F, Delanghe R, Sommen F. Clifford Analysis[M]. Research Notes Math. 76, London: Pitman Books Ltd, 1982.

[6] Delanghe R. On regular analytic functions with values in a Clifford algebra[J]. Math. Ann., 1970, 185(1): 91–111.

[7] Delanghe R. On the singularities of functions with values in a Clifford algebra[J]. Math. Ann., 1972, 196(2): 293–319.

[8] Delanghe R. Clifford analysis: History and perspective[J]. Comput. Meth. Funct. The., 2001, 1: 107–153.

[9] Delanghe R, Brackx F. Hypercomplex function theory and Hilbert modules with reproducing kernel[J]. Proc. London Math. Soc., 1978, 37: 545–576.

[10] Delanghe R, Sommen F, Soucek V. Clifford algebra and spinor-valued functions[M]. Dordrecht: Kluwer Press, 1992.

[11] Eriksson S. L, Leutwiler H. Hypermonogenic functions and Möbius transformations[J]. Adv. Appl. Clifford Alg., 2001, 11(s2): 67–76.

[12] Franks E, Ryan J. Bounded monogenic functions on unbounded domains[J]. Contemporary Math., 1998, 212(1): 71–79.

[13] Gürlebeck K, Kähler U, Ryan J, Sprössig W. Clifford analysis over unbounded Domains[J]. Adv. Appl. Math., 1997, 19(2): 216–239.

[14] Gürlebeck K, Sprössig W. Quaternionic analysis and elliptic boundary value problems[M]. Berlin: Akademie-Verlag, 1989.

[15] Iftimie V. Functions hypercomplex[J]. Bull. Math. Soc. Sci. Math. R. S. Romania, 1965, 57(9): 279–332.

[16] Lu Jianke. Boundary value problems of analytic functions[M]. Singapor: World Sci. Publ., 1993.

[17] Muskhelishvilli N. I. Singular integral equations[M]. Moscow: NauKa Press, 1968.

[18] Obolashvili E. Higher order partial differential equations in Clifford analysis[M]. Berlin: Birkhauser, Boston, Basel, 2002.

[19] Olea E. M. Morera type problems in Clifford analysis[J]. Rev. Mat. Iberoam, 2001, 17(4): 559–585.

[20] Peetre J, Qian T. Möbius covariance of iterated Dirac operators[J]. J. Austral. Math. Soc. (Ser. A), 1994, 56(3): 403–414.

[21] Qian T, Ryan J. Conformal transformations and Hardy spaces arising in Clifford analysis[J]. J. Oper. The., 1996, 35(3): 349–372.

[22] Xu Zhenyuan, Zhou Chiping. On boundary value problems of Riemann-Hilbert type for monogenic functions in a half space of $\mathcal{R}^m$ $(m \geq 2)$[J]. Complex Variables, 1993, 22(2): 181–193.

[23] Zhang Zhongxiang. On k-regular functions with values in a universal Clifford algebra[J]. J. Math. Anal. Appl., 2006, 315(2): 491–505.

[24] Zhang Zhongxiang. Some properties of operators in Clifford analysis[J]. Complex Var., Elliptic Eq., 2007, 52(6): 455–473.

[25] Zhang Zhongxiang. Möbius transform and Poisson integral representation for monogenic functions[J]. Acta Math. Sinica A, 2013, 56(4): 487–504.

[26] Zhang Zhongxiang. The Schwarz lemma in Clifford analysis[J]. Proc. American Math. Soc., 2014, 142(4): 1237–1248.

[27] Zhang Zhongxiang. Some integral representations and singular integral over plane in Clifford analysis[J]. Adv. Appl. Clifford Algebras, 2014, 24: 1145–1157.

[28] Zhang Zhongxiang. On singular integral equations with translations[J]. J. Math., 2001, 12(2): 161–167.

Clifford分析中双正则函数及调和函数在平面上的积分表示

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摘要: 本文研究了取值在Clifford代数上双正则函数及调和函数在超复平面上积分表示的问题. 利用构造核函数的方法, 获得了双正则函数及调和函数在超复平面上的积分表示公式, 这些结果推广了Clifford分析中正则函数在超复平面上的积分表示公式.

关键词: Clifford代数; 积分表示; 双正则函数; 调和函数

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