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ASYMPTOPIC SOLUTION FOR SINGULARLY PERTURBED FRACTIONAL ORDER DIFFERENTIAL EQUATION

FENG Yi-hu[1], MO Jia-qi[2]

(*1.Department of Electronics and Information Engineering, Bozhou Teachers College, Bozhou 236800, China*)

(*2.Department of Mathematic, Anhui Normal University, Wuhu 241003, China*)

Abstract: In this paper, a class of initial value problem for the singularly perturbed fractional order nonlinear differential equation is considered. Using the stretched variable method, a formal solution and its asymptotic expansion are constructed. And the uniformly valid asymptotic expansion of solution is proved by using the theory of differential inequalities. From obtained result, we know that this approximate solution possesses good accuracy.

Keywords: fractional order differential equation; singular perturbation; asymptotic solution

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1 Introduction

In the natural, many physical problems can be solved using the fractional order derivative. For example, in many complicated seepage flow, heat conduction phenomena and so on, they can be solved using the idea of fractional order derivative [1]. Fractional order derivative possesses its broad practice sense. However, solving the fractional order differential equation is very difficult. In this paper, we obtain asymptotic solution for the fractional order differential equation using the singularly perturbed theory, and get its valid estimation using the theory of differential inequalities.

The nonlinear singularly perturbed problem was a very attractive object in the academic circles [2]. During the past decade, many asymptotic methods were developed, including the boundary layer method, the methods of matched asymptotic expansion, the method of averaging and multiple scales. Recently, many scholars such as Hovhannisyan and Vulanovic [3], Abid, Jieli and Trabelsi [4], Graef and Kong [5], Guarguaglini and Natalini [6] and Barbu

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and Cosma [7] did a great deal of work. Using the method of singular perturbation and others Mo et al. studied also a class of nonlinear boundary value problems for the reaction diffusion equations, a class of activator inhibitor system, the shock wave, the soliton, the laser pulse and the problems of atmospheric physics and so on [8–19]. In this paper, we constructed asymptotic solution for the fractional order differential equation, and proved it's uniformly valid.

The α-th order fractional order derivative $D_x^\alpha u$ of $u(x)$ is defined by

$$D_x^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} u(t) dt,$$

where Γ is the Gamma function, α is a positive fraction less than 1.

Consider the following singularly perturbed initial value problem,

$$\varepsilon D_x^\alpha D_x^\alpha u + a(x) \frac{du}{dx} = f(x, u, \varepsilon), \quad x \geq 0, \tag{1}$$

$$u(0, \varepsilon) = A(\varepsilon), \tag{2}$$

$$\varepsilon \frac{du}{dx}(0, \varepsilon) = B(\varepsilon), \tag{3}$$

where ε is a positive small parameters.

We need the following hypotheses:

[H_1] $\alpha > 0$, $f(x, u, \varepsilon), A(\varepsilon)$ and $B(\varepsilon)$ are sufficiently smooth with respect to their arguments in corresponding domains;

[H_2] $f_\varepsilon > 0, f_u \leq -c < 0$, where c is a constant.

2 Outer Solution

From the hypotheses, there is a solution $U_0(x)$ of the reduced equation for original problem (1)–(3):

$$a(x) \frac{dU_0}{dx} = f(x, U_0, 0), \quad x \geq 0, \tag{4}$$

$$U_0(0) = A(0). \tag{5}$$

We construct the outer solution U of the original problem (1)–(3). Set

$$U(x, \varepsilon) \sim \sum_{i=0}^{\infty} U_i(x) \varepsilon^i. \tag{6}$$

Substituting (6) into eq. (1), developing the nonlinear term $f(x, U, \varepsilon)$ in ε, and equating coefficients of the same powers of ε both sides, from the solution $U_0(x)$ of the reduced equations (4), (5), we obtain that

$$a(x) \frac{dU_i}{dx} = F_i - D_x^\alpha D_x^\alpha U_{i-1}, \quad x \geq 0, \quad i = 1, 2, \cdots, \tag{7}$$

$$U_i(0) = A_i, \quad i = 1, 2, \cdots, \tag{8}$$

where $A_i = \frac{1}{i!}[\frac{\partial^i A}{\varepsilon^i}]_{\varepsilon=0}$ and F_i $(i = 1, 2, \cdots)$ are determined functions, their constructions are omitted.

From the defined of the α-th order fractional order derivative $D_x^\alpha u$, it is easy to see that eq. (7) can translated by the following Volterra integral equation:

$$\int_0^x (x-t)^{-\alpha} U_i(t)dt = \int_0^x \Gamma(1-\alpha)\frac{F_i(t) - D_t^\alpha D_t^\alpha U_{i-1}(t)}{a(t)}dt + D_0, \tag{9}$$

where D_0 is an arbitrary constant, which is decided by condition (8). From Volterra integral equation (9) and condition (8), we can obtain the solution $U_i(x)$ $(i = 1, 2, \cdots)$, successively. Substituting $U_i(x)$ into eq. (6), then we can obtain the outer solution $U(x, \varepsilon)$ for the original problem (1)–(3). But it may not satisfy initial condition (3), so that we need to construct the initial layer corrective term V near $x = 0$.

3 Initial Layer Correction

Let the solution of the initial value problem (1)–(3) is the form

$$u \sim U(x, \varepsilon) + V(\tau, \varepsilon) \tag{10}$$

with

$$V(\tau, \varepsilon) \sim \sum_{i=0}^{\infty} V_\tau \varepsilon^i, \tag{11}$$

where $\tau = x/\varepsilon$ is a stretched variable [2].

Substituting (10) and (11) into eqs. (1)–(3), developing the nonlinear term $f, A(\varepsilon)$ and $B(\varepsilon)$ in ε and equating coefficients of the same powers of ε in both sides of the equations, we obtain

$$D_\tau^\alpha D_\tau^\alpha V_i + a(x)\frac{dV_i}{d\tau} = G_i, \quad \tau \geq 0, \ i = 0, 1, 2, \cdots, \tag{12}$$

$$V_i|_{\tau=0} = 0, \quad i = 0, 1, 2, \cdots, \tag{13}$$

$$V_i'|_{\tau=0} = B_i - U_i'|_{\tau=0} = 0, \quad i = 0, 1, 2, \cdots, \tag{14}$$

where $B_i = \frac{1}{i!}[\frac{\partial^i B}{\varepsilon^i}]_{\varepsilon=0}$, and G_i are determined functions which constructions are omitted too.

From the fractional order differential equation (12), we can obtain the following Volterra integral systems for (V_i, Z_i) $(i = 0, 1, 2, \cdots)$:

$$\int_0^\tau [(\tau - t)^{-\alpha} - \Gamma(1-\alpha)a(t)]Z_i(t)dt = \int_0^\tau \Gamma(1-\alpha)G_i(t)dt + D_1, \tag{15}$$

$$\int_0^\tau \Gamma(\tau - t)V_i(t)dt \int_0^\tau \Gamma(1-\alpha)Z_i(t)dt + D_2, \tag{16}$$

where D_i $(i = 1, 2)$ are arbitrary constants, which are decided by conditions (13) and (14). From the linear Volterra integral system (15), (16) and conditions (13), (14), we can obtain the solutions (V_i, Z_i) $(i = 0, 1, 2, \cdots)$, successively.

Substituting V_i $(i = 0, 1, 2, \cdots)$, into eq. (11), we can obtain the initial layer corrective term $V(\tau, \varepsilon)$ of the solution $u(x, \varepsilon)$ for the original problem (1)–(3).

From eqs. (6) and (11) and the above obtained U_i and V_i $(i = 0, 1, 2, \cdots)$, we obtain the asymptotic expansion of solution for the initial value problem of 2α-th order fractional order differential equations (1)–(3):

$$u \sim \sum_{i=0}^{\infty} [U_i + V_i]\varepsilon^i, \quad x \geq 0, \ 0 < \varepsilon \ll 1. \tag{17}$$

And we can prove inductively that $V_i(\tau)$ $(i = 0, 1, 2, \cdots)$ possess initial layer behavior near $x = 0$.

$$V_i(\tau) = O(\exp(-k_i\tau)) = O(\exp(-k_i x/\varepsilon)), \quad 0 < \varepsilon \ll 1, \ i = 0, 1, 2, \cdots, \tag{18}$$

where k_i $(i = 0, 1, 2, \cdots)$ are positive constants.

4 Uniform Validity of the Asymptotic Solution

Now we prove that the asymptotic solution obtained above is a uniformly valid asymptotic expansion in ε.

Definition There are two smooth functions $\overline{u}$ and $\underline{u}$, if $\overline{u} \geq \underline{u}$ and they satisfy inequalities, respectively:

$$\varepsilon D_x^{\alpha} D_x^{\alpha} \overline{u} + a(x)\frac{d\overline{u}}{dx} - f(x, \overline{u}, \varepsilon) \leq 0, \ \overline{u}(0, \varepsilon) \geq A(\varepsilon), \ \varepsilon \overline{u}'(0, \varepsilon) \geq B(\varepsilon), \quad 0 \leq x \leq X_0,$$

$$\varepsilon D_x^{\alpha} D_x^{\alpha} \underline{u} + a(x)\frac{d\underline{u}}{dx} - f(x, \underline{u}, \varepsilon) \geq 0, \ \underline{u}(0, \varepsilon) \leq A(\varepsilon), \ \varepsilon \underline{u}'(0, \varepsilon) \leq B(\varepsilon), \quad 0 \leq x \leq X_0,$$

where X_0 is a constant large enough, then we say that $\overline{u}$ and $\underline{u}$ are upper and lower solutions of the problem (1)–(3), respectively.

Theorem 1 Assume that $[\mathrm{H}_1], [\mathrm{H}_2]$ hold. If $\overline{u}(x, \varepsilon)$ and $\underline{u}(x, \varepsilon)$ are upper and lower solutions of the initial value problem (1)–(3) for the singularly perturbed fractional order differential equation, respectively, then there is a solution $u(x, \varepsilon)$ of the initial value problem (1)–(3) such that

$$\underline{u}(x, \varepsilon) \leq u(x, \varepsilon) \leq \overline{u}(x, \varepsilon).$$

Proof We construct two function sequences which are decided by the following recurrence relations:

$$\varepsilon D_x^{\alpha} D_x^{\alpha} u_{n+1} + a(x)\frac{du_{n+1}}{dx} = f(x, u_n, \varepsilon), \quad n = 0, 1, 2, \cdots, \tag{19}$$

$$u_{n+1}(0, \varepsilon) = A(\varepsilon) \ \ \varepsilon \frac{du_{n+1}}{dx}(0, \varepsilon) = B(\varepsilon), \quad n = 0, 1, 2, \cdots. \tag{20}$$

Let $\overline{u}_0 = \overline{u}$ and $\underline{u}_0 = \underline{u}$ are initial iteration functions of eqs. (19) and (20), respectively, and we have $\overline{u}_n$ and $\underline{u}_n$, successively. Thus we obtain two function sequences $\{\overline{u}_n\}$ and $\{\underline{u}_n\}$. Now we consider their convergence.

Let $\overline{y}_0 = \overline{u}_0 - \overline{u}_1$, from hypothesis $[H_2]$, we have

$$\begin{aligned} & \varepsilon D_x^\alpha D_x^\alpha \overline{y}_0 + a(x)\overline{y}_0' = \varepsilon D_x^\alpha D_x^\alpha \overline{u}_0 + a(x)\overline{u}_0' - \varepsilon D_x^\alpha D_x^\alpha \overline{u}_1 - a(x)\overline{u}_1' \\ \leq \quad & f(x, \overline{u}_0, 0) - f(x, \overline{u}_0, 0) = 0, \\ & \overline{y}_0|_{x=0} = \overline{u}_0|_{x=0} - \overline{u}_1|_{x=0} = 0, \quad \overline{y}_0'|_{x=0} = \overline{u}_0'|_{x=0} - \overline{u}_1'|_{x=0} = 0. \end{aligned}$$

Thus from the extremum principle [2], we have $\overline{y}_0 \geq 0$. That is $\overline{u}_0 \geq \overline{u}_1$.

If $\overline{u}_n \geq \overline{u}_{n+1}$, set $\overline{y}_0 = \overline{u}_n - \overline{u}_{n+1}$, then we have

$$\begin{aligned} & \varepsilon D_x^\alpha D_x^\alpha \overline{y}_n + a(x)\overline{y}_n' = \varepsilon D_x^\alpha D_x^\alpha \overline{u}_n + a(x)\overline{u}_n' - \varepsilon D_x^\alpha D_x^\alpha \overline{u}_{n+1} - a(x)\overline{u}_{n+1}' \\ \leq \quad & f(x, \overline{u}_{(n-1)}, \varepsilon) - f(x, \overline{u}_n, \varepsilon) \leq 0, \\ & \overline{y}_n|_{x=0} = \overline{u}_n|_{x=0} - \overline{u}_{n-1}|_{x=0}, \quad \overline{y}_n'|_{x=0} = \overline{u}_n'|_{x=0} - \overline{u}_{n-1}'|_{x=0}). \end{aligned}$$

Thus $\overline{y}_n \geq 0$, that is $\overline{u}_n \geq \overline{u}_{n+1}$ $(n \geq 1)$.

From inductive method, we know $\overline{u} = \overline{u}_0 \geq \overline{u}_1 \geq \cdots \geq \overline{u}_n \geq \overline{u}_{n+1} \geq \cdots$.

Analogously, we have $\underline{u} = \underline{u}_0 \leq \underline{u}_1 \leq \cdots \leq \underline{u}_n \leq \underline{u}_{n+1} \leq \cdots$ and $\overline{u}_n \geq \underline{u}_n$, $n = 0, 1, 2, \cdots$.

From the above, and the Arzela-Ascoli theorem, there is a solution $u(x, \varepsilon)$ of the initial value problem (1)–(3) such that $\underline{u}(x, \varepsilon) \leq u(x, \varepsilon) \leq \overline{u}(x, \varepsilon)$. The proof of Theorem 1 is completed.

Theorem 2 Under hypotheses $[H_1], [H_2]$, there is a solution $u(x.\varepsilon)$ of the initial value problem (1)–(3) for the singularly perturbed fractional order differential equation, which possesses the following uniformly valid asymptotic expansion in ε on $x \in [0, X_0]$.

$$u = \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + O(\varepsilon^{m+1}), \quad x \in [0, X_0], \ 0 < \varepsilon \ll 1. \tag{21}$$

Proof First, we construct the auxiliary functions $\alpha(x, \varepsilon), \beta(x, \varepsilon)$:

$$\alpha(x, \varepsilon) = \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)]\varepsilon^i - r\varepsilon^{m+1} \quad x \in [0, X_0], \tag{22}$$

$$\beta(x, \varepsilon) = \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + r\varepsilon^{m+1} \quad x \in [0, X_0], \tag{23}$$

where r is a positive constant large enough to be chosen below, m is an arbitrary positive integer.

Obviously, we have

$$\alpha(x, \varepsilon) \leq \beta(x, \varepsilon), \tag{24}$$

and for ε small enough, there is a positive constant δ_1 such that

$$\begin{aligned} \beta(0, \varepsilon) \quad &= \quad \sum_{i=0}^{m} [U_i(0) + V_i(0)]\varepsilon^i + r\varepsilon^{m+1}, \\ &\geq \quad A(\varepsilon) - \delta_1\varepsilon^{m+1} + r\varepsilon^{m+1} = A(\varepsilon) + (r - \delta_1)\varepsilon^{m+1}. \end{aligned}$$

Thus selecting $r \geq \delta_1$, we have

$$\beta(0,\varepsilon) \geq A(\varepsilon). \tag{25}$$

Analogously, we have

$$\alpha(0,\varepsilon) \leq A(\varepsilon), \quad \varepsilon\alpha'(0,\varepsilon) \leq B(\varepsilon) \leq \varepsilon\beta'(0,\varepsilon). \tag{26}$$

Now we prove that

$$\varepsilon D_x^\alpha D_x^\alpha \alpha + a(x)\frac{d\alpha}{dx} - f(x,\alpha,\varepsilon) \geq 0, \quad 0 < x \leq X_0, \tag{27}$$

$$\varepsilon D_x^\alpha D_x^\alpha \beta + a(x)\frac{d\beta}{dx} - f(x,\beta,\varepsilon) \leq 0, \quad 0 < x \leq X_0. \tag{28}$$

In fact, from the hypotheses and eq. (18), for ε small enough, there is a positive constant δ_2, such that

$$\begin{aligned}
& \varepsilon D_x^\alpha D_x^\alpha \beta + a(x)\frac{d\beta}{dx} - f(x,\beta,\varepsilon) \\
= \;& \varepsilon D_x^\alpha D_x^\alpha \sum_{i=0}^{m}[U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + a(x)\sum_{i=0}^{m}[U_i'(x) + V_i'(x/\varepsilon)]\varepsilon^i \\
& - f(x, \sum_{i=0}^{m}[U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + r\varepsilon^{m+1}, \varepsilon) \\
\leq \;& -f(x,U_0,0) - \sum_{i=0}^{m}[f_u(x,U_0)U_i + F_i - D_x^\alpha D_x^\alpha U_{i-1} - a(x)U_i']\varepsilon^i \\
& + \sum_{i=0}^{m}[D_\tau^\alpha D_\tau^\alpha V_i + a(x)V_i' - G_i]\varepsilon^i + [f(x, \sum_{i=0}^{m}[U_i(x) + V_i(x/\varepsilon)]\varepsilon^i, \varepsilon) \\
& - f(x, \sum_{i=0}^{m}[U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + r\varepsilon^{m+1}, \varepsilon)] + \delta_2\varepsilon^{m+1} \leq (-cr + \delta_2)\varepsilon^{m+1}.
\end{aligned}$$

Selecting $r \geq \delta_2/c$, then we have eq. (28). Analogously, we can prove eq. (27). From eqs. (24)–(27), α and β are upper and lower solutions respectively. From Theorem 1, there is a solution $u(x,\varepsilon)$ of the initial value problem (1)–(3) such that $\alpha(x,\varepsilon) \leq u(x,\varepsilon) \leq \beta(x,\varepsilon)$. And from eqs. (22) and (23), we have relation (21). The proof of Theorem 2 is completed.

References

[1] Delbosco D. Existence and uniqueness for nonlinear fractional differential equation[J]. J. Math. Anal. Appl., 1996, 204(2): 609–625.

[2] de Jager E M, Jiang Furu. The theory of singular perturbation[M]. Amsterdam: North-Holland Publishing Co., 1996.

[3] Hovhannisyan G, Vulanovic R. Stability inequalities for one-dimensional singular perturbation problems[J]. Nonl. Stud., 2008, 15(4): 297–322.

[4] Abid I, Jleli M. Trabelsi N. Weak solutions of quasilinear biharmonic problems with positive, increasing and convex nonlinearities[J]. Anal. Appl. Singap., 2008, 6(3): 213–227.

[5] Graef J R, Kong L. Solutions of second order multi-point boundary value problems[J]. Math. Proc. Camb. Philos. Soc., 2008, 145(2): 489–510.

[6] Guarguaglini F R; Natalini R. Fast reaction limit and large time behavior of solutions to a nonlinear model of sulphation phenomena[J]. Commun. Par. Differ. Equ., 2007, 32(2): 163–189.

[7] Barbu L, Cosma,E. Elliptic regularizations for the nonlinear heat equation[J]. J. Math. Anal. Appl., 2009, 351(2), 392–399.

[8] Mo Jiaqi. A class of singulaely perturbed differential-difference reaction diffusion equations[J]. Adv. Math., 2009, 38(2): 227–230.

[9] Mo Jiaqi, Lin Wantao. Asymptotic solution of activator inhibitor systems for nonlinear reaction diffusion equations[J]. J. Sys. Sci. Com., 2008, 20(1): 119–128.

[10] Mo Jiaqi. Approximate solution of homotopic mapping to solitary for generalized nonlinear KdV system[J] Chin. Phys. Lett., 2009, 26(1): 010204.

[11] Mo Jiaqi. A variational iteration solving method for a class of generalized Boussinesq equations[J]. Chin Phys. Lett., 2009, 26(6): 060202.

[12] Mo Jiaqi. Homotopic mapping solving method for gain fluency of laser pulse amplifier[J]. Sci. China, Ser. G, 2009, 39(5): 568–661.

[13] Mo Jiaqi, Lin Wantao. Asymptotic solution for a class of sea-air oscillator model for El-Nino-southern oscillation[J]. Chin. Phys., 2008, 17(2): 370–372.

[14] Mo Jiaqi. A class of homotopic solving method for ENSO mpdel[J]. Acta Math. Sci., 2009, 29(1): 101–109.

[15] Mo Jiaqi, Chen Huaijun. The corner layer solution of Robin problem for semilinear equation[J]. Math, Appl., 2012, 25(1): 1–4.

[16] Mo Jiaq, Lin Yihua, Lin Wantao, Chen Lihua. Perturbed solving method for interdecadal sea-air oscillator model[J]. Chin. Geo. Sci., 2012, 22(1): 1, 42–47.

[17] Liu Shude. Spike layer solutions of some quadratic singular perturbation problems with high-order turning points[J]. Math. Appl., 2014, 27(1): 50–51

[18] Chen Lihua, Yao Jingsun, Wen Zhaohui, Mo Jiaqi. Shock wave solution for the nonlinear reaction diffusion problems[J]. J. Math., 2013, 33(3): 383–387.

[19] 莫嘉琪, 姚静荪. 一类双参数奇摄动非线性反应扩散方程[J]. 数学杂志, 2011, 31(1): 133–137.

奇摄动分数阶微分方程的渐近解

冯依虎[1], 莫嘉琪[2]

(1. 亳州师范高等专科学校电子与信息工程系, 安徽 亳州 236800)

(2. 安徽师范大学数学系, 安徽 芜湖 241003)

摘要: 本文研究了一类奇摄动非线性分数阶微分方程初值问题. 利用伸长变量构造出解的形式展开式, 并利用微分不等式理论, 证明了解的一致有效的渐近式. 所得的结果具有较好精度的近似解.

关键词: 分数阶微分方程; 奇摄动; 渐近解

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