ASYMPTOPIC SOLUTION FOR SINGULARLY PERTURBED FRACTIONAL ORDER DIFFERENTIAL EQUATION

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Abstract: In this paper, a class of initial value problem for the singularly perturbed fractional order nonlinear differential equation is considered. Using the stretched variable method, a formal solution and its asymptotic expansion are constructed. And the uniformly valid asymptotic expansion of solution is proved by using the theory of differential inequalities. From obtained result, we know that this approximate solution possesses good accuracy.

Keywords: fractional order differential equation; singular perturbation; asymptotic solution

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1 Introduction

In the natural, many physical problems can be solved using the fractional order derivative. For example, in many complicated seepage flow, heat conduction phenomena and so on, they can be solved using the idea of fractional order derivative [1]. Fractional order derivative possesses its broad practice sense. However, solving the fractional order differential equation is very difficult. In this paper, we obtain asymptotic solution for the fractional order differential equation using the singularly perturbed theory, and get its valid estimation using the theory of differential inequalities.

The nonlinear singularly perturbed problem was a very attractive object in the academic circles [2]. During the past decade, many asymptotic methods were developed, including the boundary layer method, the methods of matched asymptotic expansion, the method of averaging and multiple scales. Recently, many scholars such as Hovhannisyan and Vulanovic [3], Abid, Jieli and Trabelsi [4], Graef and Kong [5], Guarguaglini and Natalini [6] and Barbu [7]...
and Cosma [7] did a great deal of work. Using the method of singular perturbation and others Mo et al. studied also a class of nonlinear boundary value problems for the reaction diffusion equations, a class of activator inhibitor system, the shock wave, the soliton, the laser pulse and the problems of atmospheric physics and so on [8–19]. In this paper, we constructed asymptotic solution for the fractional order differential equation, and proved it’s uniformly valid.

The $\alpha$-th order fractional order derivative $D^\alpha_x u$ of $u(x)$ is defined by

$$D^\alpha_x u = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} u(t) dt,$$

where $\Gamma$ is the Gamma function, $\alpha$ is a positive fraction less than 1.

Consider the following singularly perturbed initial value problem,

$$\varepsilon D^\alpha_x D^\alpha_x u + a(x) \frac{du}{dx} = f(x, u, \varepsilon), \quad x \geq 0,$$

(1)

$$u(0, \varepsilon) = A(\varepsilon),$$

(2)

$$\varepsilon \frac{du}{dx}(0, \varepsilon) = B(\varepsilon),$$

(3)

where $\varepsilon$ is a positive small parameters.

We need the following hypotheses:

[H1] $\alpha > 0$, $f(x, u, \varepsilon), A(\varepsilon)$ and $B(\varepsilon)$ are sufficiently smooth with respect to their arguments in corresponding domains;

[H2] $f_\varepsilon > 0$, $f_u \leq -c < 0$, where $c$ is a constant.

2 Outer Solution

From the hypotheses, there is a solution $U_0(x)$ of the reduced equation for original problem (1)–(3):

$$a(x) \frac{dU_0}{dx} = f(x, U_0, 0), \quad x \geq 0,$$

(4)

$$U_0(0) = A(0).$$

(5)

We construct the outer solution $U$ of the original problem (1)–(3). Set

$$U(x, \varepsilon) \sim \sum_{i=0}^{\infty} U_i(x) \varepsilon^i.$$  

(6)

Substituting (6) into eq. (1), developing the nonlinear term $f(x, U, \varepsilon)$ in $\varepsilon$, and equating coefficients of the same powers of $\varepsilon$ both sides, from the solution $U_0(x)$ of the reduced equations (4), (5), we obtain that

$$a(x) \frac{dU_i}{dx} = F_i - D^\alpha_x D^\alpha_x U_{i-1}, \quad x \geq 0, \quad i = 1, 2, \ldots,$$

(7)

$$U_i(0) = A_i, \quad i = 1, 2, \ldots,$$

(8)
where \( A_i = \frac{1}{n^i} \frac{\partial^i A}{\partial x^i} \bigg|_{x=0} \) and \( F_i \) \((i = 1, 2, \cdots)\) are determined functions, their constructions are omitted.

From the defined of the \( \alpha \)-th order fractional order derivative \( D^\alpha_x u \), it is easy to see that eq. (7) can be translated by the following Volterra integral equation:

\[
\int_0^x (x-t)^{-\alpha} U_i(t) dt = \int_0^\tau \Gamma(1-\alpha) \frac{F_i(t) - D^\alpha_x D^\alpha_x U_{i-1}(t)}{a(t)} dt + D_0, \tag{9}
\]

where \( D_0 \) is an arbitrary constant, which is decided by condition (8). From Volterra integral equation (9) and condition (8), we can obtain the solution \( U_i(x) \) \((i = 1, 2, \cdots)\), successively. Substituting \( U_i(x) \) into eq. (6), then we can obtain the outer solution \( U(x, \epsilon) \) for the original problem (1)–(3). But it may not satisfy initial condition (3), so that we need to construct the initial layer corrective term \( V \) near \( x = 0 \).

### 3 Initial Layer Correction

Let the solution of the initial value problem (1)–(3) is the form

\[
u \sim U(x, \epsilon) + V(\tau, \epsilon) \tag{10}
\]

with

\[
V(\tau, \epsilon) \sim \sum_{i=0}^\infty V_i \epsilon^i, \tag{11}
\]

where \( \tau = x/\epsilon \) is a stretched variable [2].

Substituting (10) and (11) into eqs. (1)–(3), developing the nonlinear term \( f, A(\epsilon) \) and \( B(\epsilon) \) in \( \epsilon \) and equating coefficients of the same powers of \( \epsilon \) in both sides of the equations, we obtain

\[
D^\alpha_x D^\alpha_x V_i + a(x) \frac{dV_i}{d\tau} = G_i, \quad \tau \geq 0, \quad i = 0, 1, 2, \cdots, \tag{12}
\]

\[
V_i|_{\tau=0} = 0, \quad i = 0, 1, 2, \cdots, \tag{13}
\]

\[
V_i'|_{\tau=0} = B_i - U_i'|_{\tau=0} = 0, \quad i = 0, 1, 2, \cdots, \tag{14}
\]

where \( B_i = \frac{1}{n!} \left[ \frac{\partial^n B}{\partial x^n} \right]_{x=0} \), and \( G_i \) are determined functions which constructions are omitted too.

From the fractional order differential equation (12), we can obtain the following Volterra integral systems for \((V_i, Z_i)\) \((i = 0, 1, 2, \cdots)\):

\[
\int_0^\tau [(\tau-t)^{-\alpha} - \Gamma(1-\alpha)a(t)] Z_i(t) dt = \int_0^\tau \Gamma(1-\alpha) G_i(t) dt + D_1, \tag{15}
\]

\[
\int_0^\tau \Gamma(\tau-t) V_i(t) dt \int_0^\tau \Gamma(1-\alpha) Z_i(t) dt + D_2, \tag{16}
\]

where \( D_i \) \((i = 1, 2)\) are arbitrary constants, which are decided by conditions (13) and (14). From the linear Volterra integral system (15), (16) and conditions (13), (14), we can obtain the solutions \((V_i, Z_i)\) \((i = 0, 1, 2, \cdots)\), successively.
Substituting $V_i (i = 0, 1, 2, \cdots)$, into eq. (11), we can obtain the initial layer corrective term $V(\tau, \varepsilon)$ of the solution $u(x, \varepsilon)$ for the original problem (1)–(3).

From eqs. (6) and (11) and the above obtained $U_i$ and $V_i (i = 0, 1, 2, \cdots)$, we obtain the asymptotic expansion of solution for the initial value problem of $2\alpha$-th order fractional order differential equations (1)–(3):

$$u \sim \sum_{i=0}^{\infty} [U_i + V_i] \varepsilon^i, \quad x \geq 0, \quad 0 < \varepsilon \ll 1.$$  \hspace{1cm} (17)

And we can prove inductively that $V_i(\tau) (i = 0, 1, 2, \cdots)$ possess initial layer behavior near $x = 0$.

$$V_i(\tau) = O(\exp(-k_i\tau)) = O(\exp(-k_i x/\varepsilon)), \quad 0 < \varepsilon \ll 1, \quad i = 0, 1, 2, \cdots,$$  \hspace{1cm} (18)

where $k_i (i = 0, 1, 2, \cdots)$ are positive constants.

4 Uniform Validity of the Asymptotic Solution

Now we prove that the asymptotic solution obtained above is a uniformly valid asymptotic expansion in $\varepsilon$.

**Definition** There are two smooth functions $\underline{u}$ and $\overline{u}$ if $\underline{u} \geq u$ and they satisfy inequalities, respectively:

$$\varepsilon D_x^\alpha D_x^\alpha u + a(x) \frac{du}{dx} - f(x, u, \varepsilon) \leq 0, \quad \underline{u}(0, \varepsilon) \geq A(\varepsilon), \quad \varepsilon \overline{u}'(0, \varepsilon) \geq B(\varepsilon), \quad 0 \leq x \leq X_0,$$

$$\varepsilon D_x^\alpha D_x^\alpha \overline{u} + a(x) \frac{d\overline{u}}{dx} = f(x, \overline{u}, \varepsilon) \geq 0, \quad \overline{u}(0, \varepsilon) \leq A(\varepsilon), \quad \varepsilon \overline{u}'(0, \varepsilon) \leq B(\varepsilon), \quad 0 \leq x \leq X_0,$$

where $X_0$ is a constant large enough, then we say that $\underline{u}$ and $\overline{u}$ are upper and lower solutions of the problem (1)–(3), respectively.

**Theorem 1** Assume that $[H_1], [H_2]$ hold. If $\underline{u}(x, \varepsilon)$ and $\overline{u}(x, \varepsilon)$ are upper and lower solutions of the initial value problem (1)–(3) for the singularly perturbed fractional order differential equation, respectively, then there is a solution $u(x, \varepsilon)$ of the initial value problem (1)–(3) such that

$$\underline{u}(x, \varepsilon) \leq u(x, \varepsilon) \leq \overline{u}(x, \varepsilon).$$

**Proof** We construct two function sequences which are decided by the following recurrence relations:

$$\varepsilon D_x^\alpha D_x^\alpha u_{n+1} + a(x) \frac{du_{n+1}}{dx} = f(x, u_n, \varepsilon), \quad n = 0, 1, 2, \cdots,$$  \hspace{1cm} (19)

$$u_{n+1}(0, \varepsilon) = A(\varepsilon), \quad \varepsilon \frac{du_{n+1}}{dx}(0, \varepsilon) = B(\varepsilon), \quad n = 0, 1, 2, \cdots.$$  \hspace{1cm} (20)

Let $\underline{u}_0 = \underline{u}$ and $\overline{u}_0 = \overline{u}$ are initial iteration functions of eqs. (19) and (20), respectively, and we have $\underline{u}_n$ and $\overline{u}_n$, successively. Thus we obtain two function sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$. Now we consider their convergence.
No. 2 Asymptotic solution for singularly perturbed fractional order differential equation 243

Let \( y_0 = \pi_0 - \pi_1 \), from hypothesis \([H_2]\), we have

\[
\varepsilon D_x^\alpha D_x^\beta y_0 + a(x)y_0' = \varepsilon D_x^\alpha D_x^\beta \pi_0 + a(x)\pi_0' - \varepsilon D_x^\alpha D_x^\beta \pi_1 - a(x)\pi_1',
\]

\[
\leq f(x, \pi_0, 0) - f(x, \pi_0, 0) = 0,
\]

\[
y_0|_{x=0} = \pi_0|_{x=0} - \pi_1|_{x=0} = 0, \quad y_0'|_{x=0} = \pi_0'|_{x=0} - \pi_1'|_{x=0} = 0.
\]

Thus from the extremum principle [2], we have \( y_0 \geq 0 \). That is \( y_0 \geq \pi_1 \).

If \( y_n \geq y_{n+1} \), set \( y_0 = \pi_n - \pi_{n+1} \), then we have

\[
\varepsilon D_x^\alpha D_x^\beta y_n + a(x)y_n' = \varepsilon D_x^\alpha D_x^\beta \pi_n + a(x)\pi_n' - \varepsilon D_x^\alpha D_x^\beta \pi_{n+1} - a(x)\pi_{n+1}'
\]

\[
\leq f(x, \pi_{n-1}, \varepsilon) - f(x, \pi_n, \varepsilon) \leq 0,
\]

\[
y_n|_{x=0} = \pi_n|_{x=0} - \pi_{n-1}|_{x=0}, \quad y_n'|_{x=0} = \pi_n'|_{x=0} - \pi_{n-1}'|_{x=0}.
\]

Thus \( y_n \geq 0 \), that is \( y_n \geq y_{n+1} \) \((n \geq 1)\).

From inductive method, we know \( \pi = \pi_0 \geq \pi_1 \geq \cdots \geq \pi_n \geq \pi_{n+1} \geq \cdots \).

Analogously, we have \( u = u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \) and \( \pi_n \geq u_n \), \( n = 0, 1, 2, \cdots \).

From the above, and the Arzela-Ascoli theorem, there is a solution \( u(x, \varepsilon) \) of the initial value problem (1)–(3) such that \( u(x, \varepsilon) \leq u(x, \varepsilon) \leq \pi(x, \varepsilon) \). The proof of Theorem 1 is completed.

**Theorem 2** Under hypotheses \([H_1], [H_2]\), there is a solution \( u(x, \varepsilon) \) of the initial value problem (1)–(3) for the singularly perturbed fractional order differential equation, which possesses the following uniformly valid asymptotic expansion in \( \varepsilon \) on \( x \in [0, X_0] \).

\[
u = \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + O(\varepsilon^{m+1}), \quad x \in [0, X_0], \quad 0 < \varepsilon \ll 1.
\]

**Proof** First, we construct the auxiliary functions \( \alpha(x, \varepsilon), \beta(x, \varepsilon) \):

\[
\alpha(x, \varepsilon) = \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)]\varepsilon^i - r\varepsilon^{m+1} \quad x \in [0, X_0], \quad (22)
\]

\[
\beta(x, \varepsilon) = \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)]\varepsilon^i + r\varepsilon^{m+1} \quad x \in [0, X_0], \quad (23)
\]

where \( r \) is a positive constant large enough to be chosen below, \( m \) is an arbitrary positive integer.

Obviously, we have \( \alpha(x, \varepsilon) \leq \beta(x, \varepsilon) \), \( (24) \)

and for \( \varepsilon \) small enough, there is a positive constant \( \delta_1 \) such that

\[
\beta(0, \varepsilon) \geq A(\varepsilon) - \delta_1 \varepsilon^{m+1} + r\varepsilon^{m+1} = A(\varepsilon) + (r - \delta_1)\varepsilon^{m+1}.
\]
Thus selecting \( r \geq \delta_1 \), we have
\[
\beta(0, \varepsilon) \geq A(\varepsilon).
\]
(25)

Analogously, we have
\[
\alpha(0, \varepsilon) \leq A(\varepsilon), \quad \varepsilon \alpha'(0, \varepsilon) \leq B(\varepsilon) \leq \varepsilon \beta'(0, \varepsilon).
\]
(26)

Now we prove that
\[
\varepsilon D_x^m \alpha + a(x) \frac{d\alpha}{dx} - f(x, \alpha, \varepsilon) \geq 0, \quad 0 < x \leq X_0,
\]
(27)
\[
\varepsilon D_x^m \beta + a(x) \frac{d\beta}{dx} - f(x, \beta, \varepsilon) \leq 0, \quad 0 < x \leq X_0.
\]
(28)

In fact, from the hypotheses and eq. (18), for \( \varepsilon \) small enough, there is a positive constant \( \delta_2 \), such that
\[
\varepsilon D_x^m \alpha + a(x) \frac{d\alpha}{dx} - f(x, \alpha, \varepsilon) = \varepsilon D_x^m \alpha + a(x) \frac{d\alpha}{dx} - f(x, \alpha, \varepsilon) \leq - f(x, U_0, \varepsilon) - \sum_{i=0}^{m} [f_i(x, U_0) U_i + F_i - D_x^m D_x^m U_{i-1} - a(x) U_i] \varepsilon^i
\]
\[
+ \sum_{i=0}^{m} [D_x^m D_x^m V_i + a(x) V_i' - G_i] \varepsilon^i + [f(x, \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)] \varepsilon^i, \varepsilon) - f(x, \sum_{i=0}^{m} [U_i(x) + V_i(x/\varepsilon)] \varepsilon^i, \varepsilon) + \delta_2 \varepsilon^{m+1} \leq (\varepsilon \alpha'(0, \varepsilon) - \beta'(0, \varepsilon)) \leq \varepsilon \beta'(0, \varepsilon)
\]
Selecting \( r \geq \delta_2/c \), then we have eq. (28). Analogously, we can prove eq. (27). From eqs. (24)–(27), \( \alpha \) and \( \beta \) are upper and lower solutions respectively. From Theorem 1, there is a solution \( u(x, \varepsilon) \) of the initial value problem (1)–(3) such that \( \alpha(x, \varepsilon) \leq u(x, \varepsilon) \leq \beta(x, \varepsilon) \). And from eqs. (22) and (23), we have relation (21). The proof of Theorem 2 is completed.

References


奇摄动分数阶微分方程的渐近解

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摘要: 本文研究了一类奇摄动非线性分数阶微分方程初值问题. 利用长变量构造出解的形式展开式, 并利用微分不等式理论, 证明了解的一致有效的渐近式. 所得的结果具有较好精度的近似解.

关键词: 分数阶微分方程; 奇摄动; 渐近解