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PARAMETER ESTIMATION FOR α-WEIGHTED FRACTIONAL BRIDGE WITH DISCRETE OBSERVATIONS

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Abstract: In this paper, we consider the problem of estimating the unknown parameter $\alpha > 0$ of the weighted fractional bridge $dX_t = -\alpha\frac{X_t}{T-t}dt + dB_t^{a,b}$, $0 \le t < T$, where $B^{a,b}$ is a weighted fractional Brownian motion of parameters $a > -1, |b| < 1, |b| \le a+1$. Assume that the process is observed at discrete time $t_i = i\triangle_n, i = 0, \cdots, n$ and $T_n = n\triangle_n$, we construct a least squares estimator $\widehat{\alpha}_n$ of α and prove that $\widehat{\alpha}_n$ converges to α in probability as $n \to \infty$.

Keywords: weighted fractional bridge; least squares; α-weighted fractional bridge; parameter estimation estimator

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1 Introduction

The long-range dependence property has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing and finance. The best known and most widely used process that exhibits the long-range dependence property is fractional Brownian motion (fBm in short). The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity and it has stationary increments.

Recently, Es-Sebaiy and Nourdin [9] studied the asymptotic properties of a least squares estimator for the parameter α of a fractional bridge defined as

$$X_0 = 0, \ dX_t = -\alpha\frac{X_t}{T-t}dt + dB_t^H, \ 0 \le t < T, \tag{1.1}$$

where B^H is a fBm with Hurst parameter $H > 1/2$, and the process X was observed continuously. Especially, when $H = \frac{1}{2}$, Barczy and Pap [3, 4] studied the various problems related to the α-Wiener bridge.

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In applications usually the process cannot be observed continuously. Only discrete-time observations are available. There exists a rich literature on the parameter estimation problem for diffusion processes driven by fBm based on discrete observations (see, for example, Hu [10], Hu and Song [11], Es-Sebaiy [8] and the reference.)

Motivated by all these results, in this paper, we will consider the α weighted fractional bridge (1.1). Assume that the process X is observed equidistantly in time with the step size $t_i = i\triangle_n, i = 0, \cdots, n$, and $T_n = n\triangle_n$ denotes the length of the 'observation window'. We also assume that $T_n + \triangle_n = T$, and $\triangle_n \to 0$ when $n \to \infty$. Our goal is to study the asymptotic behavior of the least squares estimator (LSE for short) $\widehat{\alpha}_n$ of α based on the sampling data $X_{t_i}, i = 0, \cdots, n$. Our technics used in this work were inspired from Es-Sebaiy [8].

The least squares estimator $\widehat{\alpha}_n$ aims to minimize

$$\alpha \mapsto \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\dot{X}_t + \alpha \frac{X_{t_{i-1}}}{T - t_{i-1}}|^2 dt.$$

This is a quadratic function of α. The minimum is achieved when

$$\widehat{\alpha}_n = -\frac{\sum\limits_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{T - t_{i-1}} \delta^{a,b} X_t}{\triangle_n \sum\limits_{i=1}^{n} \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}. \tag{1.2}$$

By (1.1), we can get the following result

$$\widehat{\alpha}_n - \alpha = -\frac{\sum\limits_{i=1}^{n} M_i}{\triangle_n \sum\limits_{i=1}^{n} \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}, \tag{1.3}$$

where $M_i = \alpha \frac{X_{t_{i-1}}}{T-t_{i-1}} \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{T - t_{i-1}} - \frac{X_s}{T - s} \right) ds + \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{T - t_{i-1}} \delta^{a,b} B_t^{a,b}, i = 1, \cdots, n.$

The paper is organized as follows. In Section 2 some known results that we will use are recalled. The consistency of estimator is proved Section 3.

2 Preliminaries

In this section we introduce some basic facts on the Malliavin calculus for the weighted fractional Brownian motion $B^{a,b}$. Recall that the weighted fractional Brownian motion $B^{a,b}$ with parameters $a > -1, |b| < 1, |b| \leq a + 1$ is a centered and self-similar Gaussian process with long/short-range dependence. It admits the relatively simple covariance function as follows

$$E\left[B_t^{a,b} B_s^{a,b}\right] = R^{a,b}(t, s) := \int_0^{s \wedge t} u^a[(t - u)^b + (s - u)^b] du, \quad s, t \geq 0. \tag{2.1}$$

Clearly, for $a = 0,\ b = 0$, $B^{a,b}$ coincides with the standard Brownian motion B. For $a = 0$, we get

$$E\left[B_t^{a,b}B_s^{a,b}\right] = \frac{1}{b+1}[t^{b+1} + s^{b+1} - |s-t|^{b+1}], \tag{2.2}$$

which for $-1 < b < 1$ corresponds to the covariance of the well-known fBm with Hurst index $\frac{b+1}{2}$ and it admits the explicit significance. Hence, wfBm's are a family of processes which extend fBm's, perhaps it may be useful in some applications. This process $B^{a,b}$ appeared in Bojdecki et al. [5] in a limit of occupation time fluctuations of a system of independent particles moving in $\mathbb{R}^d$ according a symmetric α-stable Lévy process, $0 < \alpha \leq 2$ (see also Bojdecki et al. [6, 7]), and it is neither a semimartingale nor a Markov process unless $a = 0, b = 0$, so many of the powerful techniques from stochastic analysis are not available when dealing with $B^{a,b}$. The wfBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths). However, in comparison with fBm, the wfBm has non-stationary increments and satisfies the following estimates (see Bojdecki et al. [6], Yan et al. [15]):

$$c_{a,b}(t \vee s)^a|t-s|^{b+1} \leq E\left[\left(B_t^{a,b} - B_s^{a,b}\right)^2\right] \leq C_{a,b}(t \vee s)^a|t-s|^{b+1} \tag{2.3}$$

for $s, t \geq 0$, where $C_{a,b}$ and $c_{a,b}$ stand for positive constants and whose value may be different in different appearance. Thus Kolmogorov's continuity criterion implies that wfBm is Hölder continuous of order δ for any $\delta < \frac{1}{2}(1+b)$. The process $B^{a,b}$ is $\frac{a+b+1}{2}$ self-similar, and certainly, the self similar index $\frac{b+1}{2}$ does not coincide with the Hölder index $\frac{1+b}{2}$. However, the Hölder indices of many popular self-similar Gaussian processes coincide with their self similar indices such as fractional Brownian motion, sub-fractional Brownian motion and bi-fractional Brownian motion. That is causing trouble for the research, and it is also our a motivation to study the weighted-fractional Brownian motion. More studies on wfBm could be found in Garzón [13], Shen et al. [14], Yan et al. [15].

As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to the Gaussian process $B^{a,b}$, which will be related to the Malliavin calculus. Some surveys and complete literatures could be found in Alós et al. [1] and Nualart [12]. We recall here the basic definitions and results of this calculus. The crucial ingredient is the canonical Hilbert space $\mathcal{H}$ (is also said to be reproducing kernel Hilbert space) associated to the wfBm which is defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R^{a,b}(t,s)$. The application $\mathcal{E} \ni \varphi \mapsto B(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{a,b}$ and it can be extended to $\mathcal{H}$. The Hilbert space $\mathcal{H}$ can be written as

$$\mathcal{H} = \{\varphi : [0,T] \to \mathbb{R} \ \mid \ \|\varphi\|_{\mathcal{H}} < \infty\},$$

where

$$\|\varphi\|_{\mathcal{H}}^2 := \int_0^T \int_0^T \varphi(t)\varphi(s)\phi(t,s)dtds$$

with $\phi(t,s) = b(t \wedge s)^a (t \vee s - t \wedge s)^{b-1}$. Notice that the elements of the Hilbert space $\mathcal{H}$ may not be functions but distributions of negative order. We can use the subspace $|\mathcal{H}|$ of $\mathcal{H}$ which is defined as the set of measurable function φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := \int_0^T \int_0^T |\varphi(s)||\varphi(r)|\phi(s,r)dsdr < \infty. \tag{2.4}$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\varphi\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$, and

$$L^2([0,T]) \subset L^{\frac{2}{1+a+b}} \subset |\mathcal{H}| \subset \mathcal{H}. \tag{2.5}$$

For $b > 0$ we denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^{a,b}(\varphi_1), \cdots, B^{a,b}(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}, i = 1, 2, \cdots, n$. The Malliavin derivative of a function $F \in \mathcal{S}$ as above is given by

$$D^{a,b}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^{a,b}(\varphi_1), \cdots, B^{a,b}(\varphi_n))\varphi_i.$$

The derivative operator $D^{a,b}$ is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^{a,b}F\|_{\mathcal{H}}^2}.$$

The divergence integral $\delta^{a,b}$ is the adjoint operator of $D^{a,b}$. That is, we say that a random variable u in $L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator $\delta^{a,b}$, denoted by $\mathrm{Dom}(\delta^{a,b})$, if $E\left|\langle D^{a,b}F, u\rangle_{\mathcal{H}}\right| \le c\|F\|_{L^2(\Omega)}$ for every $F \in \mathcal{S}$. In this case $\delta^{a,b}(u)$ is defined by the duality relationship

$$E\left[F\delta^{a,b}(u)\right] = E\langle D^{a,b}F, u\rangle_{\mathcal{H}} \tag{2.6}$$

for any $u \in \mathbb{D}^{1,2}$. We will use the notation $\delta^{a,b}(u) = \int_0^T u_s dB_s^{a,b}$ to express the Skorohod integral of a process u, and the indefinite Skorohod integral is defined as $\int_0^t u_s dB_s^{a,b} = \delta^{a,b}(u\mathbb{1}_{[0,t]})$.

If $u \in D^{1,2}(|\mathcal{H}|)$, $u \in \mathrm{Dom}\delta$, then we have (see Nualart [12])

$$E|\delta(u)|^2 \le C_{a,b}\left(\|Eu\|_{|\mathcal{H}|}^2 + E(\|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2)\right),$$

where the constant $C_{a,b}$ depends only on a, b. As a consequence, applying (2.5) we have

$$E|\delta(u)|^2 \le C_{a,b}\left(\|Eu\|_{L^{\frac{2}{1+a+b}}([0,T])}^2 + E(\|Du\|_{L^{\frac{2}{1+a+b}}([0,T]^2)}^2)\right). \tag{2.7}$$

For every $n \geq 1$, let $\mathcal{H}_n$ be the n-th Wiener chaos of $B^{a,b}$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B^{a,b}(h)), h \in \mathcal{H}, ||h||_{\mathcal{H}} = 1\}$, where H_n is the n-th Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n!H_n(B^{a,b}(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\|.\|_{\mathcal{H}^{\odot n}} = \frac{1}{\sqrt{n!}}\|.\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. For every $f, g \in \mathcal{H}^{\odot n}$ the following multiplication formula holds $E(I_n(f)I_n(g)) = n!\langle f, g\rangle_{\mathcal{H}^{\otimes n}}$.

Let $f, g : [0, T] \to \mathbb{R}$ be Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$. Young proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (\frac{1}{2}, 1)$ and $F : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^1$, the integrals $\int_0^{\cdot} \frac{\partial F}{\partial f}(f_u, g_u)df_u$ and $\int_0^{\cdot} \frac{\partial F}{\partial g}(f_u, g_u)dg_u$ exist in the Young sense and the following change of variables formula holds:

$$F(f_t, g_t) = F(f_0, g_0) + \int_0^t \frac{\partial F}{\partial f}(f_u, g_u)df_u + \int_0^t \frac{\partial F}{\partial g}(f_u, g_u)dg_u,\ t \in [0, T]. \tag{2.8}$$

As a consequence, if $\frac{1+b}{2} > \frac{1}{2}$ and $(u_t, t \in [0, T])$ be a process with Hölder paths of order $\alpha > 1 - \frac{1+b}{2}$, the integral $\int_0^T u_s dB_s^{a,b}$ is well-defined as Young integral. Suppose that for any $t \in [0, T], u_t \in D^{1,2}(|\mathcal{H}|)$, and

$$\int_0^T \int_0^T |D_s u_t|(t \wedge s)^a (t \vee s - t \wedge s)^{b-1} ds dt < \infty \quad \text{a.s..}$$

Then by the same argument as in Alòs and Nualart [2], we have

$$\int_0^t u_s dB_s^{a,b} = \int_0^t u_s \delta B_s^{a,b} + b\int_0^t \int_0^t D_s u_r (r \wedge s)^a (r \vee s - r \wedge s)^{b-1} dr ds. \tag{2.9}$$

In particular, when φ is a non-random Hölder continuous function of order $\alpha > 1 - \frac{1+b}{2}$, we have

$$\int_0^t \varphi_s dB_s^{a,b} = \int_0^t \varphi_s \delta B_s^{a,b} = B^{a,b}(\varphi). \tag{2.10}$$

In addition, for all $\varphi, \psi \in |\mathcal{H}|$,

$$E\left(\int_0^T \varphi_s dB_s^{a,b} \int_0^T \psi_s dB_s^{a,b}\right) = b\int_0^T \int_0^T \varphi_u \psi_v (u \wedge v)^a (u \vee v - u \wedge v)^{b-1} du dv. \tag{2.11}$$

3 Asymptotic Behavior of the Least Squares Estimator

Throughout this paper we assume $a > -1, |b| < 1, |b| \leq a + 1$. We will study eq. (1.1) driven by a weighted fractional Brownian motion $B^{a,b}$ and $\alpha > 0$ is the unknown parameter to be estimated for discretely observed X. It is readily checked that we have the following explicit expression for X_t:

$$X_t = (T - t)^\alpha \int_0^t (T - s)^{-\alpha} dB_s^{a,b}, \qquad 0 \leq t < T,$$

where the integral can be understood as Young integral. In order to study the asymptotic behavior of the least squares estimator, let us introduce the following processes

$$A_t := \int_0^t (T-s)^{-\alpha} dB_s^{a,b}, \qquad 0 \le t < T.$$

Hence, we have

$$X_t = (T-t)^{\alpha} A_t, \qquad 0 \le t < T. \tag{3.1}$$

For simplicity, we assume that the notation $a_n \trianglerighteq b_n$ means that there exists positive constants $C = C_{H,\alpha} > 0$ (depending only on H, α and its value may differ from line to line) so that

$$\sup_{n\ge 1} |a_n|/|b_n| < C < \infty.$$

We first give the following lemmas.

Lemma 3.1 Let $\alpha > 0, -1 < a < 0, 0 < b < 1, b < a+1$. Then

$$\int_0^{T_n} \frac{X_s}{T-s} dB_s^{a,b} = \int_0^{T_n} \frac{X_s}{T-s} \delta^{a,b} B_s^{a,b} + \beta_n,$$

where

$$\beta_n = b \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} s^a (r-s)^{b-1} ds dr \tag{3.2}$$

and

$$\lim_{n\to\infty} \beta_n = bB(a,b)B(b,a+1)T^{a+b}.$$

Proof By (2.9), we have

$$\begin{aligned}
& \int_0^{T_n} \frac{X_s}{T-s} dB_s^{a,b} \\
= & \int_0^{T_n} \frac{X_s}{T-s} \delta^{a,b} B_s^{a,b} + b \int_0^{T_n} \int_0^{T_n} D_s^{a,b} \frac{X_r}{T-r} (r \wedge s)^a (r \vee s - r \wedge s)^{b-1} dr ds \\
= & \int_0^{T_n} \frac{X_s}{T-s} \delta^{a,b} B_s^{a,b} + b \int_0^{T_n} \int_0^{r} (T-r)^{\alpha-1} (T-s)^{-\alpha} s^a (r-s)^{b-1} ds dr \\
= & \int_0^{T_n} \frac{X_s}{T-s} \delta^{a,b} B_s^{a,b} + \beta_n.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} \beta_n &= b \lim_{n\to\infty} \int_0^{T_n} \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} s^a (r-s)^{b-1} ds dr \\
&= b \int_0^T \int_0^r (T-r)^{\alpha-1} (T-s)^{-\alpha} s^a (r-s)^{b-1} ds dr \\
&= b \int_0^T \int_s^T (T-r)^{\alpha-1} (T-s)^{-\alpha} s^a (r-s)^{b-1} dr ds \\
&= b \int_0^T s^{-\alpha} (T-s)^a ds \int_0^s r^{\alpha-1} (s-r)^{b-a} dr \\
&= bB(\alpha, b) B(b, a+1) T^{a+b}.
\end{aligned}$$

This completes the proof.

Lemma 3.2 Let $-1 < a < 0, 0 < b < 1, 2\alpha - 1 < b < a + 1$, then

$$E(\frac{X_t}{T-t})^2 \leq 2bB(1-\alpha, b)B(b-2\alpha+1, a+1)(T-t)^{2\alpha-2}T^{a+b-2\alpha+1}, 0 \leq t < T.$$

Proof In fact, we have

$$\begin{aligned}
EA_t^2 &= b\int_0^t\int_0^t (T-u)^{-\alpha}(T-v)^{-\alpha}(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv \\
&= 2b\int_0^t\int_0^u (T-u)^{-\alpha}(T-v)^{-\alpha}v^a(u-v)^{b-1}dvdu \\
&= 2b\int_0^t\int_v^t (T-u)^{-\alpha}(T-v)^{-\alpha}v^a(u-v)^{b-1}dudv \\
&= 2b\int_{T-t}^t\int_{T-t}^v u^{-\alpha}v^{-\alpha}(T-v)^a(v-u)^{b-1}dudv \\
&\leq 2b\int_0^t\int_0^v u^{-\alpha}v^{-\alpha}(T-v)^a(v-u)^{b-1}dudv \\
&= 2b\int_0^T v^{-\alpha}(T-v)^a dv\int_0^v u^{-\alpha}(v-u)^{b-1}du \\
&= 2bB(1-\alpha, b)B(b-2\alpha+1, a+1)T^{a+b-2\alpha+1}.
\end{aligned}$$

So we obtain that

$$E(\frac{X_t}{T-t})^2 \leq 2bB(1-\alpha, b)B(b-2\alpha+1, a+1)(T-t)^{2\alpha-2}T^{a+b-2\alpha+1}, 0 \leq t < T.$$

Lemma 3.3 Assume $-1 < a < 0, 0 < b < 1, 1-b < 2\alpha < 1+b \leq a+2$ and let $F_{T_n} = \displaystyle\int_0^{T_n} \frac{X_t}{T-t}\delta^{a,b}B_t^{a,b}$. Then

$$\lim_{n\to\infty} E(F_{T_n}^2) = \frac{b^2}{2}B(1-\alpha, b)B(b-2\alpha+1, a+1)B(\alpha, b)B(2\alpha+b-1, \alpha+1)T^{2a+2b-2\alpha}.$$

Proof By the isometry property of the double stochastic integral I_2, the variance of F_{T_n} is given by

$$E(F_{T_n}^2) = \frac{b^2}{2}I_{T_n},$$

where

$$\begin{aligned}
I_{T_n} = & \int_{[0,T_n]^4}(T-t_1)^{\alpha-1}(T-s_1)^{-\alpha}(T-t_2)^{\alpha-1}(T-s_2)^{-\alpha}(s_1\wedge s_2)^a \\
& (s_1\vee s_2 - s_1\wedge s_2)^{b-1}(t_1\wedge t_2)^a(t_1\vee t_2 - t_1\wedge t_2)^{b-1}ds_1ds_2dt_1dt_2.
\end{aligned}$$

Now, we study I_{T_n}, by setting

$$\begin{aligned}
I_1 &= \int_{[0,T_n]^2}(T-s_1)^{-\alpha}(T-s_2)^{-\alpha}(s_1\wedge s_2)^a(s_1\vee s_2 - s_1\wedge s_2)^{b-1}ds_1ds_2, \\
I_2 &= \int_{[0,T_n]^2}(T-t_1)^{\alpha-1}(T-t_2)^{\alpha-1}(t_1\wedge t_2)^a(t_1\vee t_2 - t_1\wedge t_2)^{b-1}dt_1dt_2.
\end{aligned}$$

We have $I_{T_n} = I_1 I_2$. In a similar way

$$\begin{aligned} I_1 &= \int_{[0,T_n]^2} (T-s_1)^{-\alpha}(T-s_2)^{-\alpha}(s_1\wedge s_2)^a(s_1\vee s_2 - s_1\wedge s_2)^{b-1}ds_1ds_2 \\ &\to B(\alpha,b)B(b+2\alpha-1,a+1)T^{a+b-1}, \quad n\to\infty \end{aligned}$$

and

$$I_2 \to B(1-\alpha,b)B(b-2\alpha+1,a+1)T^{a+b-2\alpha+1}, \quad n\to\infty.$$

Thus the proof is finished.

The following theorem give the consistency of the least squares estimator $\widehat{\alpha}_n$ of α.

Theorem 3.1 Let $\alpha > 1/2, -1 < a < 0, -a < b < 1, 1-b < 2\alpha < 1+b+a \le 2a+2$. If

$$\Delta_n \to 0,\ \ T_n = n\Delta_n \to T$$

as $n\to\infty$ and $T_n + \triangle_n = T$. Then we have

$$\widehat{\alpha}_n \xrightarrow{\mathrm{P}} \alpha, \qquad n\to\infty,$$

where $\xrightarrow{\mathrm{P}}$ means convergence in probability.

Proof By (1.3), we have

$$\widehat{\alpha}_n - \alpha = -\frac{\frac{\alpha}{n}\sum\limits_{i=1}^n M_i}{\frac{\alpha\Delta_n}{n}\sum\limits_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}.$$

Let $0<\varepsilon<1$, we obtain

$$\begin{aligned} P(|\widehat{\alpha}_n-\alpha|>\varepsilon) &= P\left(\left|\frac{\frac{\alpha}{n}\sum\limits_{i=1}^n M_i}{\frac{\alpha\Delta_n}{n}\sum\limits_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}\right| > \varepsilon\right) \\ &\le P\left(\left|\frac{\alpha}{n}\sum_{i=1}^n M_i\right| > \varepsilon(1-\varepsilon)\right) + P\left(\left|\frac{\alpha\triangle_n}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2} - 1\right| > \varepsilon\right) \\ &:= B_1(n)+B_2(n). \end{aligned}$$

First, we consider the term $B_1(n)$, we have

$$\begin{aligned} B_1(n) &= P\left(\left|\frac{\alpha}{n}\sum_{i=1}^n M_i\right| > \varepsilon(1-\varepsilon)\right) \\ &\le P\left(\left|\frac{\alpha}{n}\sum_{i=1}^n [M_i - \int_{t_{i-1}}^{t_i} X_{t_{i-1}}\delta^{a,b}B_t^{a,b}]\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha}{n}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t})\delta^{a,b}B_t^{a,b}\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &\quad + P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\frac{X_t}{T-t}\delta^{a,b}B_t^{a,b}\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right) \\ &:= B_{1,1}(n)+B_{1,2}(n)+B_{1,3}(n). \end{aligned}$$

For the term $B_{1,1}(n)$, using Lemma 3.2, we obtain

$$
\begin{aligned}
& \sum_{i=1}^{n} E\left|[M_i - \int_{t_{i-1}}^{t_i} X_{t_{i-1}} \delta^{a,b} B_t^{a,b}]\right| \\
\leq\ & \alpha \sum_{i=1}^{n} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\
\trianglerighteq\ & \sum_{i=1}^{n} (T-t_{i-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\
\leq\ & \triangle_n^{\alpha-1} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right)^2\right)^{1/2} dt \\
\leq\ & \triangle_n^{\alpha-1} \left[\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} dt + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_t}{T-t}\right)^2\right)^{1/2} dt\right] \\
\leq\ & \triangle_n^{\alpha-1} \left[\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2\right)^{1/2} dt + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(E\left(\frac{X_{t_i}}{T-t_i}\right)^2\right)^{1/2} dt\right] \\
\trianglerighteq\ & n\, \triangle_n^{2\alpha-1}.
\end{aligned}
$$

So we get

$$
\frac{\alpha}{n} \sum_{i=1}^{n} E\left|[M_i - \int_{t_{i-1}}^{t_i} X_{t_{i-1}} \delta^{a,b} B_t^{a,b}]\right| \trianglerighteq \triangle_n^{2\alpha-1}.
$$

Hence

$$
B_{1,1}(n) \trianglerighteq \frac{\triangle_n^{2\alpha-1}}{\varepsilon(1-\varepsilon)}.
$$

For the term $B_{1,2}(n)$, it follow the fact that for $0 \leq t < T$,

$$
\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t} = -[((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1})A_{t_{i-1}} + (T-t)^{\alpha-1}(A_t - A_{t_{i-1}})].
$$

We have

$$
\begin{aligned}
& E\left|\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{T-t_{i-1}} - \frac{X_t}{T-t}\right) \delta^{a,b} B_t^{a,b}\right| \\
\leq\ & E\left|\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} ((T-t)^{\alpha-1} - (T-t_{i-1})^{\alpha-1}) A_{t_{i-1}} \delta^{a,b} B_t^{a,b}\right| \\
& + E\left|\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (T-t)^{\alpha-1} (A_t - A_{t_{i-1}}) \delta^{a,b} B_t^{a,b}\right|.
\end{aligned}
$$

Using inequality (2.7) and $EA_t = 0, D_s^H A_t = (T-s)^{-\alpha} 1_{[0,t]}(s)$, we have

$$
\begin{aligned}
& E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1})A_{t_{i-1}}\delta^{a,b}B_t^{a,b}\right| \\
=& E\left|\int_0^{T_n}\sum_{i=1}^{n}((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1})A_{t_{i-1}}1_{(t_{i-1},t_i]}(t)\delta^{a,b}B_t^{a,b}\right| \\
\le& \left(E\left|\int_0^{T_n}\sum_{i=1}^{n}((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1})A_{t_{i-1}}1_{(t_{i-1},t_i]}(t)\delta^{a,b}B_t^{a,b}\right|^2\right)^{1/2} \\
\le& C_{a,b,\alpha}\left(\int_0^{T_n}\int_0^{T_n}\left|\sum_{i=1}^{n}((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1})D_s^{a,b}A_{t_{i-1}}1_{(t_{i-1},t_i]}(t)\right|^{\frac{2}{1+a+b}}dsdt\right)^{\frac{1+a+b}{2}} \\
=& C_{a,b,\alpha}\left(\int_0^{T_n}\int_0^{T_n}\sum_{i=1}^{n}\left|((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1})(T-s)^{-\alpha}\right|^{\frac{2}{1+a+b}}1_{(t_{i-1},t_i]}(t)1_{[0,t_{i-1})}(s)dsdt\right)^{\frac{1+a+b}{2}} \\
=& C_{a,b,\alpha}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1})^{\frac{2}{1+a+b}}dt\int_0^{t_{i-1}}(T-s)^{\frac{-2\alpha}{1+a+b}}ds\right)^{\frac{1+a+b}{2}} \\
\trianglerighteq& C_{a,b,\alpha}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\triangle_n^{\frac{2\alpha-2}{1+a+b}}dt\right)^{\frac{1+a+b}{2}} \le C_{a,b,\alpha}n\triangle_n^{\frac{2\alpha+a+b-1}{2}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{\alpha-1}(A_t-A_{t_{i-1}})\delta^{a,b}B_t^{a,b}\right| \\
=& E\left|\int_0^{T_n}\sum_{i=1}^{n}(T-t)^{\alpha-1}(A_t-A_{t_{i-1}})1_{(t_{i-1},t_i]}(t)\delta^{a,b}B_t^{a,b}\right| \\
\le& C_{a,b,\alpha}\left(\int_0^{T_n}\int_0^{T_n}\left|\sum_{i=1}^{n}(T-t)^{\alpha-1}D_s^{a,b}(A_t-A_{t_{i-1}})1_{(t_{i-1},t_i]}(t)\right|^{\frac{2}{1+a+b}}dsdt\right)^{\frac{1+a+b}{2}} \\
\le& C_{a,b,\alpha}\left(\int_0^{T_n}\int_0^{T_n}\sum_{i=1}^{n}\left|(T-t)^{\alpha-1}D_s^{a,b}(A_t-A_{t_{i-1}})\right|^{\frac{2}{1+a+b}}1_{(t_{i-1},t_i]}(t)dsdt\right)^{\frac{1+a+b}{2}} \\
=& C_{a,b,\alpha}\left(\int_0^{T_n}\int_0^{T_n}\sum_{i=1}^{n}\left((T-t)^{\alpha-1}(T-s)^{-\alpha}\right)^{\frac{2}{1+a+b}}1_{[t_{i-1},t]}(s)1_{(t_{i-1},t_i]}(t)dsdt\right)^{\frac{1+a+b}{2}} \\
=& C_{a,b,\alpha}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{\frac{2\alpha-2}{1+a+b}}dt\int_{t_{i-1}}^{t}(T-s)^{\frac{-2\alpha}{1+a+b}}ds\right)^{\frac{1+a+b}{2}} \\
\le& C_{a,b,\alpha}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{\frac{2\alpha-2}{1+a+b}}dt\int_{t_{i-1}}^{t_i}(T-t_n)^{\frac{-2\alpha}{1+b}}ds\right)^{\frac{1+a+b}{2}} \\
\le& C_{a,b,\alpha}(n\triangle_n^{\frac{2b+2a}{1+a+b}})^{\frac{1+a+b}{2}} \le C_{a,b,\alpha}n\triangle_n^{a+b}.
\end{aligned}
$$

So we get

$$E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}-\frac{X_t}{T-t}\right)\delta^{a,b}B_t^{a,b}\right| \trianglerighteq n\,\triangle_n^{a+b}.$$

Thus

$$\frac{\alpha}{n}E\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}-\frac{X_t}{T-t}\right)\delta^{a,b}B_t^{a,b}\right| \trianglerighteq \triangle_n^{a+b}.$$

Hence $B_{1,2}(n) \trianglerighteq \frac{\triangle_n^{a+b}}{\varepsilon(1-\varepsilon)}$. For the term $B_{1,3}(n)$, by setting $F_{T_n}=\int_0^{T_n}\frac{X_t}{T-t}\delta^{a,b}B_t^{a,b}$, by using Lemma 3.3, we get

$$\begin{aligned}B_{1,3}(n) &= P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\frac{X_t}{T-t}\delta^{a,b}B_t^{a,b}\right| > \frac{1}{3}\varepsilon(1-\varepsilon)\right)\\ &\le \left[\frac{3\alpha}{\varepsilon(1-\varepsilon)n}\right]^2 E(F_{T_n}^2) \trianglerighteq \frac{1}{\varepsilon^2(1-\varepsilon)^2n^2}.\end{aligned}$$

As consequence,

$$B_1(n) \trianglerighteq \frac{\triangle_n^{2\alpha-1}}{\varepsilon(1-\varepsilon)}+\frac{\triangle_n^{a+b}}{\varepsilon(1-\varepsilon)}+\frac{1}{\varepsilon^2(1-\varepsilon)^2n^2}. \tag{3.3}$$

Second, we estimate the term $B_2(n)$,

$$\begin{aligned}B_2(n) &= P\left(\left|\frac{\alpha\triangle_n}{n}\sum_{i=1}^{n}\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-1\right|>\varepsilon\right)\\ &\le P\left(\left|\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-\left(\frac{X_t}{T-t}\right)^2\right]dt\right|>\varepsilon/2\right)\\ &\quad + P\left(\left|\frac{\alpha}{n}\int_0^{T_n}\left(\frac{X_t}{T-t}\right)^2dt-1\right|>\varepsilon/2\right)\\ &:= B_{2,1}(n)+B_{2,2}(n).\end{aligned}$$

We first consider $B_{2,1}(n)$. Since

$$\begin{aligned}&E\left|\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-\left(\frac{X_t}{T-t}\right)^2\right]\right|dt\\ \le&\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}E\left|\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2-\left(\frac{X_t}{T-t}\right)^2\right|dt\\ \le&\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2+E\left(\frac{X_t}{T-t}\right)^2\right)dt\\ \le&\frac{\alpha}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(E\left(\frac{X_{t_{i-1}}}{T-t_{i-1}}\right)^2+E\left(\frac{X_{t_i}}{T-t_i}\right)^2\right)dt\\ \le&\frac{2\alpha}{n}\sum_{i=1}^{n}\triangle_n^{2\alpha-1}\trianglerighteq\triangle_n^{2\alpha-1}.\end{aligned}$$

By Markov inequality, we obtain

$$B_{2,1}(n) \trianglerighteq \frac{\triangle_n^{2\alpha-1}}{\varepsilon}.$$

Now, we estimate the term $B_{2,2}(n)$. Applying the change of variable formula (2.8), we get

$$\frac{\alpha}{n}\int_0^{T_n}\left(\frac{X_t}{T-t}\right)^2 dt - 1 = \frac{1}{n(\alpha-\frac{1}{2})}\left(\frac{X_{T_n}}{2\triangle_n} - \int_0^{T_n}\frac{X_t}{T-t}\delta B_t^{a,b} - \beta_n\right).$$

Hence

$$\begin{aligned} B_{2,2}(n) \le\ & P\left(\left|\frac{X_{T_n}}{T_n(2\alpha-1)}\right| > \varepsilon/6\right) + P\left(\left|\frac{1}{n(\alpha-\frac{1}{2})}\int_0^{T_n}\frac{X_t}{T-t}\delta B_t^H\right| > \varepsilon/6\right) \\ & + P\left(\left|\frac{\beta_n}{n(\alpha-\frac{1}{2})}\right| > \varepsilon/6\right). \end{aligned}$$

By Markov inequality and Lemma 3.2, we obtain

$$B_{2,2}(n) \trianglerighteq \frac{\triangle_n^{2\alpha}}{\varepsilon^2 T_n^2} + \frac{1}{\varepsilon n^2} + \frac{1}{\varepsilon n}.$$

Therefore

$$B_2(n) \trianglerighteq \frac{\triangle_n^{2\alpha-1}}{\varepsilon} + \frac{\triangle_n^{2\alpha}}{\varepsilon^2 T_n^2} + \frac{1}{\varepsilon n^2} + \frac{1}{\varepsilon n} \le \frac{\triangle_n^{2\alpha-1}}{\varepsilon} + \frac{\triangle_n^{2\alpha}}{\varepsilon^2 T_n^2} + \frac{1}{\varepsilon n}. \tag{3.4}$$

Combining (3.3) and (3.4), this completes the proof.

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离散时间观测下的α -赋权分数桥的参数估计

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摘要: 本文研究了赋权分数桥$dX_t = -\alpha\frac{X_t}{T-t}dt + dB_t^{a,b}, \quad 0 \le t < T$, 中未知参数$\alpha > 0$的参数估计问题, 其中$B^{a,b}$是参数为$a > -1, |b| < 1, |b| \le a+1$ 的赋权分数布朗运动. 假设对随机过程X_t进行离散观测$t_i = i\triangle_n, i = 0, \cdots, n$, 且$T_n = n\triangle_n$. 本文构造了$\alpha$的最小二乘估计$\widehat{\alpha}_n$, 证明了当$n \to \infty$时, $\widehat{\alpha}_n$ 依概率收敛到α.

关键词: 赋权分数桥; 最小二乘估计; α -赋权分数桥; 参数估计

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