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THE CENTRAL EXTENSION OF THE MODULAR LIE SUPERALGEBRA OF CARTAN TYPE $S(m,n;\underline{t})$

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Abstract: In this paper, we study the central extension of the finite-dimensional special Lie superalgebra $S(m,n;\underline{t})$, where $\mathbb{F}$ is an algebracially closed field of prime characteristic $p>2$. By computing the skew outer derivations of $S(m,n;\underline{t})$ to $S(m,n;\underline{t})^*$, we obtain the second cohomology group $H^2(S(m,n;\underline{t}),\mathbb{F})$ is trivial. As applications, we determine the central extension of $S(m,n;\underline{t})$ is trivial.

Keywords: modular Lie superalgebra of Cartan type; derivation; skew derivation; cohomology group

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1 Introduction

As a natural generalization of Lie algebras, Lie superalgebras become efficient tools for analyzing the properties of physical systems. In the last ten years, many important results of Lie superalgebras were obtained. Central extensions are often used in the structure theory and the representation theory of Lie superalgebras. If L is a Lie algebra or a Lie superalgebra, the structures of the extensions of L-modules are described by the 1-cohomology of L and the structures of the Lie (super)algebra exensions are related to the 2-cohomology (see [1–4]). Let L be a modular Lie superalgebra, i.e., a Lie superalgebra over an algebraically closed field $\mathbb{F}$ of prime characteristic $p>2$. Central extensions of L, or equivalently, its second cohomology group $H^2(L,\mathbb{F})$, can be conveniently described by means of derivations $\varphi : L \longrightarrow L^*$. If L is simple and does not possess any non-degenerate associative form, then $H^2(L,\mathbb{F})$ and $H^1(L,L^*)$ are isomorphic (see [4]). In 1997, Professor Zhang constructed four finite-dimensional simple modular Lie superalgebras of Cartan type, i.e., the finite dimensional Witt superalgebra $W(m,n;\underline{t})$, Special superalgebra $S(m,n;\underline{t})$, Hamiltonian superalgebra $H(m,n;\underline{t})$ and the contact superalgebra $K(m,n;\underline{t})$ (see [5]). The central extensions of $W(m,n;\underline{t})$, $H(m,n;\underline{t})$ and $K(m,n;\underline{t})$ were determined (see [4, 6, 7]).

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In this paper, we shall determine the central extension of the finite-dimensional special Lie superalgebra $S(m,n;\underline{t})$. Throughout this paper we always assume that $\mathbb{F}$ is an algebraically closed field and $\mathrm{char}\mathbb{F}=p>2$. Write $\mathbb{N}$ and $\mathbb{N}_0$ for the set of positive integers and the set of nonnegative integers, respectively.

2 Preliminary

Adopting the notation of [6, Section 1], we suppose that $L=L_{\bar{0}}\oplus L_{\bar{1}}$ is a finite-dimensional Lie superalgebra over $\mathbb{F}$ and L possesses a $\mathbb{Z}$-gradation: $L=\bigoplus\limits_{i=-r}^{q}L_i$. Then $L^*:=\mathrm{Hom}_{\mathbb{F}}(L,\mathbb{F})=\bigoplus\limits_{i=-q}^{r}(L^*)_i$ is a $\mathbb{Z}$-graded L-module by means of

$$(x\cdot f)(y)=-(-1)^{\mathrm{p}(x)\mathrm{p}(f)}f([x,y])\ \text{ for }\ x,y\in L,f\in L^*,$$

where $\mathrm{p}(x)$ denotes the parity of a $\mathbb{Z}_2$-homogeneous elements x. We assume throughout that the symbol $\mathrm{p}(x)$ implies that x is $\mathbb{Z}_2$-homogeneous.

Let $H\subset L_0\cap L_{\bar{0}}$ be a nilpotent subalgebra of $L_{\bar{0}}$ and let $\rho:H\longrightarrow \mathbf{gl}(V)$ be a finite-dimensional representation, where V is a H-module. Then we define $V_{(\alpha)}:=\{v\in V\mid \forall h\in H\ \exists n(h,v)\in\mathbb{N}:(\rho(h)-\alpha(h)\mathbf{id}_V)^{n(h,v)}(v)=0\}$. The mapping α is called a weight and $V_{(\alpha)}$ the weight space if $V_{(\alpha)}\neq 0$. Let

$$L=\bigoplus_{\alpha\in\Delta}L_{(\alpha)}\quad\text{and}\quad L^*=\bigoplus_{\beta\in\Gamma}(L^*)_{(\beta)}$$

be the weight space decompositions of L and L^* with respect to H, respectively. As $H\subset L_0\cap L_{\bar{0}}$, there exist subsets $\Delta_i\subset\Delta$ and $\Gamma_j\subset\Gamma$ such that

$$L_i=\bigoplus_{\alpha\in\Delta_i}L_i\cap L_{(\alpha)}\quad\text{and}\quad (L^*)_j=\bigoplus_{\beta\in\Gamma_j}(L^*)_j\cap(L^*)_{(\beta)}.$$

Proposition 2.1 [6, Proposition 1.1] Let $L^*=\bigoplus\limits_{\beta\in\Gamma}(L^*)_{(\beta)}$ be the weight space decomposition relative to H. Then the following statements hold:

(1) $\Gamma=-\Delta$ and there is an isomorphism $(L^*)_{(\beta)}\cong(L_{(-\beta)})^*$ of H-modules for all $\beta\in\Gamma$;

(2) $\Gamma_i=-\Delta_{-i}$ for $-q\leq i\leq r$.

Definition 2.2 A linear mapping $\varphi:L\longrightarrow L^*$ is called a derivation if

$$\varphi([x,y])=(-1)^{\mathrm{p}(\varphi)\mathrm{p}(x)}x\cdot\varphi(y)-(-1)^{\mathrm{p}(\varphi(x))\mathrm{p}(y)}y\cdot\varphi(x)\ \text{ for all }\ x,y\in L.$$

Let $\mathrm{Der}_{\mathbb{F}}(L,L^*)$ denote the space of derivations from L into L^* and $\mathrm{Inn}_{\mathbb{F}}(L,L^*)$ be the subspace of inner derivations. Recall that a derivation φ from L into L^* is called inner if there is some $f\in L^*$ such that $\varphi(x)=(-1)^{\mathrm{p}(f)\mathrm{p}(x)}x\cdot f$ for all $x\in L$.

Definition 2.3 A derivation $\varphi:L\longrightarrow L^*$ is said to be skew if

$$\varphi(x)(y)=-(-1)^{\mathrm{p}(x)\mathrm{p}(y)}\varphi(y)(x)\ \text{ for all }\ x,y\in L.$$

Let $U(L)$ denote the universal enveloping algebra of L and $L^- = \sum\limits_{i=-r}^{-1} L_i$.

Let $\mathfrak{S} : U(L) \longrightarrow U(L)$ denote the antipode map of $U(L)$, i.e., the antihomomorphism of $U(L)$ satisfying $\mathfrak{S}(x) = -x$ for $x \in L$, $\mathfrak{S}(uv) = (-1)^{\mathrm{p}(u)\mathrm{p}(v)}\mathfrak{S}(v)\mathfrak{S}(u)$ for $u, v \in U(L)$ and $\mathfrak{S}(1) = 1$. Observe that the $U(L)$-module structure of L^* induced by the representation $L \longrightarrow \mathfrak{gl}(L^*)$ fulfills

$$(u \cdot f)(x) = (-1)^{\mathrm{p}(u)\mathrm{p}(f)} f(S(u) \cdot x) \ \text{ for } \ u \in U(L) \ \text{ and } \ x \in L.$$

Let $\Phi_1 : H^1(L, L^*) \longrightarrow H^1(L^-, L^*)$ be the canonical map, which is induced by the restriction map $\mathrm{Der}_{\mathbb{F}}(L, L^*) \longrightarrow \mathrm{Der}_{\mathbb{F}}(L^-, L^*)$.

Lemma 2.4 [6, Lemma 1.5] Let φ: $L \longrightarrow L^*$ be a derivation and suppose that $e \in L$ such that $(\mathrm{ad} e)^{p^r} = 0$. Then $e^{p^r-1} \cdot \varphi(e) \in (L^*)^L$, where

$$(L^*)^L = \{f \in L^* \mid L \cdot f = 0\} = \{f \in L^* \mid f([L, L]) = 0\}.$$

Proposition 2.5 [6, Proposition 1.6] Let $V \subset L$ be a $\mathbb{Z}_2$-graded subspace such that $L = U(L^-)^+ \cdot V \oplus V$, where $U(L)^+$ denotes the two-sided ideal generated by L. Suppose $T \subset \mathbb{N}_0^n$ and $\{e_1, e_2, \cdots, e_n\}$ is a basis of L^- such that

(1) $\mathrm{ann}_{U(L^-)^+}(L) = \mathrm{span}_{\mathbb{F}}\{e^b \mid b \notin T\}$, where $b = (b_1, b_2, \cdots, b_n)$, $e^b := e_1^{b_1} e_2^{b_2} \cdots e_n^{b_n}$ and $\mathrm{ann}_{U(L^-)^+}(L) := \{u \in U(L^-)^+ \mid u \cdot L = 0\}$;

(2) there is a basis $\{v_1, v_2, \cdots, v_m\}$ of V such that $\{e^a \cdot v_j \mid a \in T,\ 1 \leq j \leq m\}$ is a basis of L over $\mathbb{F}$.

Then the following statements hold:

(1) if $\varphi : L \longrightarrow L^*$ is a derivation satisfying $\ker(\mathrm{ad} e_i) \subset \ker \varphi(e_i)$ for $1 \leq i \leq n$, then it defines an element of $\ker \Phi_1$;

(2) if there is $\mu \in \mathbb{N}_0^n$ such that $T = \{b \in \mathbb{N}_0^n \mid b \leq \mu\}$, then $\ker(\mathrm{ad} e_i) \subset \ker \varphi(e_i)$ if and only if $e_i^{\mu_i} \cdot \varphi(e_i) = 0$;

(3) if $\mu_i = p^{k_i} - 1$ for $1 \leq i \leq n$ and $L = [L, L]$, then the canonical mapping Φ_1 : $H^1(L, L^*) \longrightarrow H^1(L^-, L^*)$ is trivial.

We consider the subalgebra $L^+ = \sum\limits_{i=1}^{q} L_i$ as well as $M = M(L) := [L^+, L^+]$. Note that M is a graded subalgebra of L on which H operates. Hence, for $i > 1$, there is $\phi_i \subset \Delta_i$ such that $L_i = M_i + \bigoplus\limits_{\alpha \in \phi_i} L_{(\alpha)} \bigcap L_i$.

The proof of the following propositions is similar to [6, Propositions 2.3, 3.1 and 3.2].

Proposition 2.6 Suppose that $L = \bigoplus\limits_{i=-r}^{q} L_i$ is a finite-dimensional simple Lie superalgebras. Let $\varphi : L \longrightarrow L^*$ be a homogeneous skew derivation of degree l, where $-2q \leq l \leq -q-1$. If $-\Delta_q \not\subset \phi_{-(q+l)}$, then $\varphi = 0$.

Proposition 2.7 Suppose that $L = \bigoplus\limits_{i=-r}^{q} L_i$ is a finite-dimensional simple Lie superalgebras. Let $\varphi : L \longrightarrow L^*$ be a homogeneous derivation of degree l.

(1) if $l > r - q$ and φ defines an element of $\ker \Phi_1$, then φ is an inner derivation;

(2) if $l = r - q$, φ is skew and defines an element of $\ker \Phi_1$, then φ is an inner derivation.

Proposition 2.8 Suppose that $L = \bigoplus_{i=-r}^{q} L_i$ is a finite-dimensional simple Lie superalgebras. Let $\varphi : L \longrightarrow L^*$ be a skew derivation of degree $(l, 0)$ which defines an element of $\ker \Phi_1$. Then the following statements hold:

(1) if $-q < l \leq r - q - 1$, then φ is an inner derivation;

(2) if $l = -q$ and $\Delta_q \cap -\Delta_0 = \emptyset$, then φ is an inner derivation.

3 The Properties of $S(m, n; \underline{t})$

Fix $m, n \in \mathbb{N}\backslash\{1\}$. If $a = (a_1, \cdots, a_m) \in \mathbb{N}_0^m$, then let $|a| = \sum_{i=1}^{m} a_i$. Let $\mathcal{O}(m)$ be the divided power algebra with the $\mathbb{F}$-basis $\{x^{(a)} \mid a \in \mathbb{N}_0^m\}$. Fix $\underline{t} = (t_1, \cdots, t_m) \in \mathbb{N}^m$ and $\pi = (\pi_1, \cdots, \pi_m)$, where $\pi_i = p^{t_i} - 1$ for $i = 1, \cdots, m$. Let

$$\mathbb{A}(m, \underline{t}) = \{\ a = (a_1, \cdots, a_m) \in \mathbb{N}_0^m \mid a_i \leq \pi_i, i = 1, \cdots, m\},$$
$$\mathcal{O}(m, \underline{t}) = \mathrm{span}_{\mathbb{F}}\{x^{(a)} \in \mathcal{O}(m) \mid a \in \mathbb{A}(m, \underline{t})\}.$$

Then $\mathcal{O}(m, \underline{t})$ is a subalgebra of $\mathcal{O}(m)$. We write $x_i = x^{(\varepsilon_i)}$ for $i = 1, \cdots, m$, where $\varepsilon_i = (\delta_{i1}, \cdots, \delta_{im}) \in \mathbb{N}_0^m$. Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in n variables $x_{m+1}, \cdots, x_s$, where $s = m + n$. Let $\mathcal{O}(m, n; \underline{t}) = \mathcal{O}(m, \underline{t}) \otimes \Lambda(n)$. For $x \in \mathcal{O}(m, \underline{t})$ and $\xi \in \Lambda(n)$, we abbreviate $x \otimes \xi$ to $x\xi$. Write

$$x^{(a)} := x^{(a)} \otimes 1, \ \ x_i := x_i \otimes 1, \ \ x_j := 1 \otimes x_j$$

for $a \in \mathbb{A}(m, \underline{t})$, $i = 1, \cdots, m$ and $j = m + 1, \cdots, s$. Then we have the following formulas hold in $\mathcal{O}(m, n; \underline{t})$:

$$x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)}, \ \ x^{(a)}x_j = x_j x^{(a)}, \ \ x_i x_j = -x_j x_i$$

for $a, b \in \mathbb{A}(m, \underline{t})$ and $i, j = m + 1, \cdots, s$, where $\binom{a+b}{b} = \prod_{i=1}^{m} \binom{a_i+b_i}{b_i}$. Set

$$\mathbb{B}_k = \{\ (i_1, \cdots, i_k) \mid m + 1 \leq i_1 < \cdots < i_k \leq s\},$$

where $1 \leq k \leq n$. We put $\mathbb{B}_0=\emptyset$ and $\mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k$. If $u = (i_1, \cdots, i_k) \in \mathbb{B}_k$, then let $|u|=k$ and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k} \in \Lambda(n)$. We write $x^E := x_{m+1} x_{m+2} \cdots x_s$ and set $x^{\emptyset} = 1$. Then $\{x^{(a)} x^u \mid a \in \mathbb{A}(m, \underline{t}), u \in \mathbb{B}(n)\}$ is a basis of $\mathcal{O}(m, n; \underline{t})$. If $u \in \mathbb{B}_k$, then let $\mathrm{p}(x^{(a)} x^u) = \bar{k} \in \mathbb{Z}_2$. Then $\mathcal{O}(m, n; \underline{t})$ is an associative superalgebra. Put

$$Y_0 = \{1, 2, \cdots, m\}, \ Y_1 = \{m + 1, \cdots, s\}, \ Y = Y_0 \cup Y_1.$$

Let $D_1, D_2, \cdots, D_s$ be the linear transformations of $\mathcal{O}(m, n; \underline{t})$ such that

$$D_i(x^{(a)} x^u) = \begin{cases} x^{(a-\varepsilon_i)} x^u & \text{for } \ i \in Y_0, \\ x^{(a)} \partial_i(x^u) & \text{for } \ i \in Y_1, \end{cases}$$

where ∂_i is the derivation of $\Lambda(n)$ such that $\partial_i(x_j)=\delta_{ij}$ for $i,j\in Y_1$. Then $D_1,D_2,\cdots,D_s$ are derivations of the superalgebra $\mathcal{O}(m,n;\underline{t})$. Let $W(m,n,\underline{t})=\{\sum\limits_{i=1}^{s} f_iD_i \mid f_i\in\mathcal{O}(m,n;\underline{t})\}$. Then $W(m,n;\underline{t})$ is a finite-dimensional simple Lie superalgebra (see [5]). The following formula holds in $W(m,n;\underline{t})$:

$$[fD_i,gD_j]=fD_i(g)D_j-(-1)^{\mathrm{p}(fD_i)\mathrm{p}(gD_j)}gD_j(f)D_i \text{ for } f,g\in\mathcal{O}(m,n;\underline{t}),\ \ i,j\in Y.$$

For $i,j\in Y$, define

$$D_{ij}:\mathcal{O}(m,n;\underline{t})\longrightarrow W(m,n;\underline{t}),\quad D_{ij}(f)=f_iD_i+f_jD_j,$$

where

$$f_i=-(-1)^{\mathrm{p}(f)(\tau(i)+\tau(j))}D_j(f),\quad f_j=(-1)^{\tau(i)\tau(j)}D_i(f),$$
$$\tau(i):=\begin{cases}\bar{0}, & i\in Y_0,\\ \bar{1}, & i\in Y_1.\end{cases}$$

Let

$$S(m,n,\underline{t}):=\mathrm{span}_{\mathbb{F}}\{D_{ij}(f)\mid i,j\in Y,f\in\mathcal{O}(m,n;\underline{t})\}.$$

Then $S(m,n;\underline{t})$ is a finite-dimensional simple Lie superalgebra and does not possess any non-degenerate associative form (see [5, 8]). It is $\mathbb{Z}$-graded by means of $S(m,n;\underline{t})=\bigoplus\limits_{k=-1}^{q} S(m,n;\underline{t})_k$, where

$$S(m,n;\underline{t})_k=\mathrm{span}_{\mathbb{F}}\{D_{ij}(x^{(a)}x^u)\mid i,j\in Y,|a|+|u|=k+2\}.$$

The subalgebra $S(m,n;\underline{t})$ of $W(m,n;\underline{t})$ is called the special superalgebra.

For convenience, we first give the following formulas.

Lemma 3.1 [9, Lemma 3.10, p. 41] The following formulas hold in $S(m,n;\underline{t})$:

(1) $D_{ii}(f)=0$ for $i\in Y_0$; $D_{ii}(f)=-2D_i(f)D_i$ for $i\in Y_1$;

$$D_{ji}(f)=-(-1)^{\mathrm{p}(f)(\tau(i)+\tau(j))+\tau(i)\tau(j)}D_{ij}(f)$$

for $i,j\in Y$.

(2) $[D_k,D_{ij}(f)]=-(-1)^{\tau(k)\tau(i)}D_{ij}(D_k(f))$ for $k,i,j\in Y$.

(3) Let $i,j,k,l\in Y$, then

$$\begin{aligned}[D_{ij}(f),D_{kl}(g)] \;=\;& (-1)^{\tau(i)\tau(k)+\tau(j)(\tau(i)+\mathrm{p}(f))+(\tau(k)+\tau(l))\mathrm{p}(g)}D_{ik}(D_j(f)D_l(g))\\ &-(-1)^{\tau(j)\tau(k)+\tau(j)\mathrm{p}(f)+(\tau(k)+\tau(l))\mathrm{p}(g)}D_{jk}(D_i(f)D_l(g))\\ &-(-1)^{\tau(i)\tau(l)+\tau(j)(\tau(i)+\mathrm{p}(f))+\tau(k)\tau(l)}D_{il}(D_j(f)D_k(g))\\ &+(-1)^{\tau(j)\tau(l)+\tau(j)\mathrm{p}(f)+\tau(k)\tau(l)}D_{jl}(D_i(f)D_k(g)).\end{aligned}$$

Put $T=\sum\limits_{i=1}^{s}\mathbb{F}h_i$, where $h_i=x_iD_i$. Clearly, $S(m,n;\underline{t})_0=\mathrm{span}_{\mathbb{F}}\{A_{ij},x_iD_j\mid i,j\in Y,i\neq j\}$, where

$$A_{ij}=h_i-(-1)^{\tau(i)+\tau(j)}h_j.$$

Put $H=T\bigcap S(m,n;\underline{t})_0=\mathrm{span}_{\mathbb{F}}\{A_{ij}\mid i,j\in Y,i\neq j\}$. Then H is a nilpotent subalgebra of $S(m,n;\underline{t})_0\bigcap S(m,n;\underline{t})_{\bar{0}}$. Let E be a set. If $i\in E$, we put $\eta(i,E)=1$; if $i\notin E$, $\eta(i,E)=0$. Suppose $a\in\mathbb{N}_0^m$, $u\in\mathbb{B}(n)$. Define a linear map

$$a\pm u:H\longrightarrow\mathbb{F},(a\pm u)(h_i)=\eta(i,Y_0)a_i\pm\eta(i,\{u\}).$$

Suppose $b+u$ is a weight of $S(m,n;\underline{t})$ relative to H. Choose $a\in\mathbb{N}_0^m$, such that $0\leq a_i<p$ and $b_i\equiv a_i\pmod p, i=1,2,\cdots,m$. It is clear that $b+u=a+u$ in H^*.

Proposition 3.2 Suppose the weight space decomposition of $S(m,n;\underline{t})$ relative to H is $S(m,n;\underline{t})=\bigoplus\limits_{\alpha\in\Delta}S(m,n;\underline{t})_{(\alpha)}$. Let $D_{ij}(x^{(a)}x^u)\in S(m,n;\underline{t})$, then

$$D_{ij}(x^{(a)}x^u)\in S(m,n;\underline{t})_{(a+u-\eta(i,Y_0)\varepsilon_i-\eta(j,Y_0)\varepsilon_j-\eta(i,Y_1)(i)-\eta(j,Y_1)(j))}.$$

Proof For convenience, write

$$a+u-\eta(i,Y_0)\varepsilon_i-\eta(j,Y_0)\varepsilon_j-\eta(i,Y_1)(i)-\eta(j,Y_1)(j):=\beta(a,u,i,j).$$

Clearly, H is generated by $\{h_k\mid k\in Y\}$. It remains to show

$$[h_k,D_{ij}(x^{(a)}x^u)]=\beta(a,u,i,j)(h_k)D_{ij}(x^{(a)}x^u)\tag{3.1}$$

for any $k\in Y$. Without loss of generality, suppose $D_{ij}(x^{(a)}x^u)\neq 0$.

The proof now can be completed by considering the following cases:

Case 1 $k\in Y_0$.

Case 1.1 $i,j\in Y_0$.

Case 1.2 $i,j\in Y_1$.

Case 1.3 $i\in Y_0,\ j\in Y_1$ or $i\in Y_1,\ j\in Y_0$.

Case 2 $k\in Y_1$.

Case 2.1 $i,j\in Y_0$.

Case 2.2 $i,j\in Y_1$.

Case 2.3 $i\in Y_0,\ j\in Y_1$ or $i\in Y_1,\ j\in Y_0$.

We only deal with Cases 1.3 and 2.2. The other cases can be treated similarly.

Case 1.3 Suppose $i\in Y_0,j\in Y_1$. By a direct computation, we have $[h_k,D_{ij}(x^{(a)}x^u)]=(a_k-\delta_{ki})D_{ij}(x^{(a)}x^u)$. Since $\beta(a,u,i,j)(h_k)=(a+u-\varepsilon_i-(j))(h_k)=a_k-\delta_{ik}$, we obtain(3.1) holds. From (1) of Lemma 3.1 , it is easily seen the case of $i\in Y_1$ and $j\in Y_0$ also implies (3.1) holds.

Case 2.2 Suppose $k,i,j\in Y_1$. First, consider the case $k\notin\{i,j\}$. If $k\in\{u\}$, then

$$[h_k,D_{ij}(x^{(a)}x^u)]=D_{ij}(x^{(a)}x^u),\quad \beta(a,u,i,j)(h_k)=(a+u-(j)-(i))(h_k)=1.$$

If $k \notin \{u\}$, then

$$[h_k, D_{ij}(x^{(a)}x^u)] = \beta(a,u,i,j)(h_k) = 0.$$

Second, we consider the case $k = i$ and $k \neq j$. A direct computation shows that

$$[h_k, D_{ij}(x^{(a)}x^u)] = \beta(a,u,i,j)(h_k) = 0.$$

The case $k = j$ and $k \neq i$ can be treated similarly. Finally, we consider the case $k = i = j$. Since $D_{ij}(x^{(a)}x^u) \neq 0$, we known $i = j \in \{u\}$. Thus we obtain

$$[h_k, D_{ij}(x^{(a)}x^u)] = [h_k, D_{kk}(x^{(a)}x^u)] = -D_{kk}(x^{(a)}x^u), \quad \beta(a,u,i,j)(h_k) = -1.$$

Now we conclude that (3.1) holds.

4 The Second Cohomology Group $H^2(S(m,n;\underline{t}),\mathbb{F})$

Lemma 4.1 Let $\varphi : S(m,n;\underline{t}) \longrightarrow S(m,n;\underline{t})^*$ be a derivation. Then there exists $f \in S(m,n;\underline{t})^*$ such that $\varphi(x) = (-1)^{p(x)p(f)}x \cdot f$ for all $x \in S(m,n;\underline{t})_{-1}$.

Proof We shall apply Proposition 2.5 to complete the proof. Put $L := S(m,n;\underline{t})$. Then $\{D_1, D_2, \cdots, D_s\}$ is a basis of the subalgebra L^-. Consider $V = S(m,n;\underline{t})_q$ with a basis

$$\{D_{ij}(x^{(\pi)}x^E) \mid i,j \in Y\}.$$

The simplicity of L entails that $L = U(L^-)^+ \cdot V \bigoplus V$. For $a \in \mathbb{N}_0^s$, we put $D^a := D_1^{a_1}D_2^{a_2}\cdots D_s^{a_s}$, where $a_i = 0$ or 1 for $i \in Y_1$. Suppose that $a_{i_1} = a_{i_2} = \cdots = a_{i_k} = 0$, where $m+1 \leq i_1 < i_2 < \cdots < i_k \leq s$. Let $x^u = x_{i_1}x_{i_2}\cdots x_{i_k}$ and $b = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_m\varepsilon_m$. Then

$$D^a \cdot D_{ij}(x^{(\pi)}x^E) = \lambda D_{ij}(x^{(\pi-b)}x^u), \ \ \lambda \in \{1,-1\}.$$

Thus

$$\{D^a \cdot D_{ij}(x^{(\pi)}x^E) \mid a \in T, i,j \in Y\}$$

is a basis of L over $\mathbb{F}$, where

$$T := \{a \in \mathbb{N}_0^s \mid 0 \leq a_i \leq \pi_i \text{ for } i \in Y_0, \ \ 0 \leq a_i \leq 1 \ \text{ for } i \in Y_1\}.$$

Clearly $D^a \in \mathrm{ann}_{U(L^-)^+}(L)$ for $a \notin T$. Suppose that $y = \sum\limits_{a>0} \alpha(a)D^a \in \mathrm{ann}_{U(L^-)^+}(L)$. Choose $i,j \in Y$ such that $D_{ij}(x^{(\pi)}x^E) \neq 0$. Then we obtain

$$0 = y \cdot D_{ij}(x^{(\pi)}x^E) = \sum_{a \in T} \alpha(a)\lambda D_{ij}(x^{(\pi-b)}x^u),$$

where $\lambda \in \{1,-1\}$ and $b = a_1\varepsilon_1 + \cdots + a_m\varepsilon_m$. Thus $\alpha(a) = 0$ for all $a \in T$ and $y \in \mathrm{span}_{\mathbb{F}}\{D^a \mid a \notin T\}$. In analogy with the proof of (3) of [6, Proposition 1.6], we have $D_i^{\mu_i} \cdot \varphi(D_i) \in (L^*)^L = 0$. By (1) and (2) of Proposition 2.5, we obtain the desired result.

Lemma 4.2 Suppose that $h \geq 3$ and $M := M(S(m,n;\underline{t}))$. Then

$$S(m,n;\underline{t})_h = M_h + \sum_{i,j\in Y} \sum_{a_k\equiv 0(\mathrm{mod} p),\forall k\in Y_0} \mathbb{F}D_{ij}(x^{(a)}x^u) + \sum_{i,j\in Y} \mathbb{F}D_{ij}(x^{(b)}), \tag{4.1}$$

where $|a|+|u| = h+2, |b| = h+2$.

Proof It suffices to show that $S(m,n;\underline{t})_h$ is contained in the right-hand side of (4.1). Let $D_{ij}(x^{(a)}x^u) \neq 0$ be an element of $S(m,n;\underline{t})_h$, then $h = |a|+|u|-2 \geq 3$. Without loss of generality, we may suppose $|u| > 0$.

(1) $i,j \in Y_0$. If there is $k \in Y_0\backslash j$ such that $a_k \not\equiv 0 \pmod p$, then

$$[D_{ij}(x^{(a+\varepsilon_j-\varepsilon_k)}x^{u_1})\ , D_{jk}(x^{(2\varepsilon_k)}x^{u_2})] = a_k D_{ij}(x^{(a)}x^{u_1}x^{u_2}). \tag{4.2}$$

It follows that $D_{ij}(x^{(a)}x^u) \in M_h$ unless $a_j = \pi_j$.

If $a_j = \pi_j$, put $x^u = x^{u_1}x^{u_2}$ such that $|u_2| = 1$. Choose $l, r \in Y_1$ such that $l \notin \{u_1\}$ and $r \notin \{u_2\}$. By the identity

$$[D_{il}(x^{(a_j\varepsilon_j)}x_l x^{u_1}), D_{jr}(x^{(a-a_j\varepsilon_j)}x_r x^{u_2})] = D_{ij}(x^{(a)}x^{u_1}x^{u_2}), \tag{4.3}$$

we obtain $D_{ij}(x^{(a)}x^u) \in M_h$.

If for any $k \in Y_0\backslash j$, we have $a_k \equiv 0 \pmod p$ and $a_j \not\equiv 0 \pmod p$, then by

$$[D_{ji}(x^{(a+\varepsilon_i-\varepsilon_j)}x^{u_1})\ , D_{ij}(x^{(2\varepsilon_j)}x^{u_2})] = -a_j D_{ij}(x^{(a)}x^{u_1}x^{u_2}), \tag{4.4}$$

we obtain $D_{ij}(x^{(a)}x^u) \in M_h$ unless $a_i = \pi_i$. If $a_i = \pi_i$, then $a_i \equiv -1 \pmod p$, a contradiction with the assumption.

(2) $i \in Y_0,\ j \in Y_1$ or $i \in Y_1,\ j \in Y_0$. By (1) of Lemma 3.1 , we only consider the case $i \in Y_0$ and $j \in Y_1$. If there is $k \in Y_0\backslash i$ such that $a_k \not\equiv 0 \pmod p$, then put $x^u = x^{u_1}x^{u_2}$ with $|u_2| = 1$ and $j \notin \{u_1\}$. Choose $l \in Y_1\backslash\{u_2\}$. By the identity

$$[D_{kj}(x^{(a)}x^{u_1})\ , D_{il}(x^{(\varepsilon_k)}x^{u_2})] = \lambda a_k D_{ij}(x^{(a)}x^{u_1}x^{u_2}),\ \lambda \in \{-1,1\}, \tag{4.5}$$

we have $D_{ij}(x^{(a)}x^u) \in M_h$.

Suppose $a_k \equiv 0 \pmod p$ for any $k \in Y_0\backslash i$ and $a_i \not\equiv 0 \pmod p$. If $j \notin \{u\}$, put $x^u = x^{u_1}x^{u_2}$. Since $|u| > 0$, one may choose $r \in \{u\}$ such that $r \in \{u_2\}$. Take $l \in Y_1\backslash\{u_2\}$. By the identity

$$[D_{ir}(x^{(a-\varepsilon_i)}x_r x^{u_1})\ , D_{jl}(x^{(\varepsilon_i)}x_l x^{u_2})] = \lambda a_i D_{ij}(x^{(a)}x^{u_1}x^{u_2}),\ \lambda \in \{-1,1\}, \tag{4.6}$$

we obtain $D_{ij}(x^{(a)}x^u) \in M_h$. If $j \in \{u\}$ and $|u| \geq 2$, choose $r \in \{u\}\backslash j$. Put $x^u = x^{u_1}x^{u_2}$ such that $r \in \{u_2\}$ and $j \in \{u_1\}$. Take $l \in Y_1\backslash\{u_2\}$, then by (4.6), we also have $D_{ij}(x^{(a)}x^u) \in M_h$. Now only the case $j \in \{u\}$ and $|u| = 1$ needs to be considered. Note that $x^u = x_j$ and $|a| \geq 4$. If $a_i \neq 1$ and $a \neq a_i\varepsilon_i$, take $r \in Y_1\backslash j$, then by

$$[D_{ir}(x^{(a-a_i\varepsilon_i)}x_j x_r)\ , D_{jr}(x^{(a_i\varepsilon_i)}x_r)] = \lambda D_{ij}(x^{(a)}x_j),\ \lambda \in \{-1,1\}, \tag{4.7}$$

we have $D_{ij}(x^{(a)}x_j)\in M_h$.

If $a_i=1$, then

$$[D_{ij}(x^{(a-\varepsilon_i)}x_j)\,,D_{ij}(x^{(2\varepsilon_i)}x_j)]=\lambda a_iD_{ij}(x^{(a)}x_j),\ \lambda\in\{-1,1\}. \tag{4.8}$$

It follows that $D_{ij}(x^{(a)}x_j)\in M_h$.

If $a=a_i\varepsilon_i$, pick $k\in Y_0\backslash i$. Consequently,

$$[D_{ik}(x^{(\varepsilon_i+\varepsilon_k)}x_j)\,,D_{ji}(x^{(a)}x_j)]=a_iD_{ij}(x^{(a)}x_j)-D_{kj}(x^{(a+\varepsilon_k-\varepsilon_i)}x_j). \tag{4.9}$$

By the above proof, we have $D_{kj}(x^{(a+\varepsilon_k-\varepsilon_i)}x_j)\in M_h$. Thus $D_{ij}(x^{(a)}x_j)\in M_h$.

(3) $i,j\in Y_1$. It can be treated similarly.

Theorem 4.3 The central extension of $S(m,n;\underline{t})$ is trivial.

Proof It was shown in [4] that there is an isomorphism between $H^2(L,\mathbb{F})$ (the central extension of L) and the vector space of skew outer derivations from L into L^* if the modular Lie superalgebra L is simple and does not possess any non-degenerate associative form. Note that $S(m,n;\underline{t})$ satisfies these requirements. Without loss of generality, we may suppose that $\varphi:S(m,n;\underline{t})\longrightarrow S(m,n;\underline{t})^*$ is a homogeneous skew derivation of degree $(l,0)$ (see [10, Theorem 1.1]).

(i) $l\geq 1-q$. We apply Proposition 2.7 and Lemma 4.1 in order to see that φ is an inner derivation.

(ii) $l=-q$. Let $x^{(a)}x^uD_j\in W$. From Proposition 3.2, we have

$$D_{ij}(x^{(a)}x^u)\in S(m,n;\underline{t})_{(a+u-\eta(i,Y_0)\varepsilon_i-\eta(j,Y_0)\varepsilon_j-\eta(i,Y_1)(i)-\eta(j,Y_1)(j))}.$$

Clearly,

$$\begin{aligned}&\Delta_0=-\Delta_0=\{\varepsilon_k+\varepsilon_l-\varepsilon_i-\varepsilon_j\mid k,l,i,j\in Y_0\}\bigcup\{\pm(\varepsilon_k+(l)-\varepsilon_i-\varepsilon_j)\mid k,i,j\in Y_0,l\in Y_1\}\\&\bigcup\{\pm((k)+(l)-\varepsilon_i-\varepsilon_j)\mid i,j\in Y_0,k,l\in Y_1\}\bigcup\{(k)+(l)-(i)-(j)\mid k,l,i,j\in Y_1\}\\&\bigcup\{\pm(\varepsilon_k+(l)-(i)-(j))\mid k\in Y_0,l,i,j\in Y_1\}\bigcup\{\varepsilon_k+(l)-\varepsilon_i-(j)\mid k,i\in Y_0,l,j\in Y_1\}\end{aligned}$$

and

$$\Delta_q=\{\pi+E-\varepsilon_i-\varepsilon_j\,|\,i,j\in Y_0\}\bigcup\{\pi+E-(i)-(j)\,|\,i,j\in Y_1\}\bigcup\{\pi+E-\varepsilon_i-(j)\,|\,i\in Y_0,j\in Y_1\}.$$

Since $m,n\in\mathbb{N}\backslash\{1\}$, we obtain $-\Delta_0\bigcap\Delta_q=\emptyset$. Then φ is inner by virtue of Lemma 4.1 and Proposition 2.8.

(iii) $-2q\leq l\leq -q-1$. If $l\leq -q-3$, then $-(q+l)\geq 3$. Write $-(q+l):=h$. By Lemma 4.2, we have

$$\begin{aligned}&\phi_{-(q+l)}=\{a+u-\varepsilon_i-\varepsilon_j\mid |a|+|u|=h+2,\ \forall a_k\equiv 0 \pmod p),\ k\in Y_0,\ i,j\in Y_0\}\\&\bigcup\{a+u-(i)-(j)\mid |a|+|u|=h+2,\ \forall a_k\equiv 0 \pmod p),\ k\in Y_0,\ i,j\in Y_1\}\\&\bigcup\{a+u-\varepsilon_i-(j)\mid |a|+|u|=h+2,\ \forall a_k\equiv 0 \pmod p),\ k\in Y_0,\ i\in Y_0,j\in Y_1\}\\&\bigcup\{b-\varepsilon_i-\varepsilon_j\mid |b|=h+2,\ i,j\in Y_0\}\bigcup\{b-\varepsilon_i-(j)\mid |b|=h+2,\ i\in Y_0,\ j\in Y_1\}\\&\bigcup\{b-(i)-(j)\mid |b|=h+2,\ i,j\in Y_1\}.\end{aligned}$$

It is easily seen that $-\Delta_q \not\subset \phi_{-(q+l)}$, then $\varphi = 0$ by virtue of Proposition 2.6.

Now we assume that $l \in \{-q-2, -q-1\}$. Note that

$$\begin{aligned}\Delta_2 \;=\; & \{a+u-\varepsilon_i-\varepsilon_j \mid i,j \in Y_0,\ |a|+|u|=4\}\bigcup\{a+u-(i)-(j) \mid i,j \in Y_1,\ |a|+|u|=4\}\\ & \bigcup\{a+u-\varepsilon_i-(j) \mid i \in Y_0, j \in Y_1,\ |a|+|u|=4\}.\end{aligned}$$

Clearly $-\Delta_q \not\subset \Delta_2$, therefore, $-\Delta_q \not\subset \phi_2$. Similarly, $-\Delta_q \not\subset \Delta_1 = \phi_1$. Then Proposition 2.6 applies and $\varphi = 0$. Hence we conclude that the central extension of $S(m,n;\underline{t})$ is trivial.

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Cartan型模李超代数$S(m,n;\underline{t})$的中心扩张

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摘 要: 本文研究了特征为素数$p > 2$的有限维Special李超代数$S(m,n;\underline{t})$的中心扩张. 通过计算从$S(m,n;\underline{t})$到$S(m,n;\underline{t})^*$的斜外导子, 得到二阶上同调群$H^2(S(m,n;\underline{t}),\mathbb{F})$是平凡的. 应用此结果, 可得$S(m,n;\underline{t})$的中心扩张是平凡的.

关键词: Cartan型模李超代数; 导子; 斜导子; 上同调群

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