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ON SOME NONSMOOTH HOMOGENEOUS OPTIMIZATION PROBLEMS IN BANACH SPACES

ZHANG Juan[1], LI Shu-min[2]

(*1. Dpt. of Engineering, Oxbridge College, Kunming University of Science and Technology, Kunming 650106, China*)

(*2. Dpt. of Appl. Math., Kunming University of Science and Technology, Kunming 650404, China*)

Abstract: In this paper, we mainly consider the nonsmooth homogeneous optimization problem (HOP). By using the generalized Euler identity for Clarke's subdifferential, a sufficient condition for an optimal solution of (HOP) to be a KKT point is obtained. Moreover, we also give an equivalent characterization of KKT points (optimal solutions) of (HOP) and $\widehat{(\text{HOP})}$, which extend previous ones in [1]. Examples are also given to illustrate our results.

Keywords: Clarke subdifferential; KKT point; Euler identity; nonsmooth homogeneous optimization problem

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1 Introduction

Due to the wide applications in many aspects of applied mathematics, properties of homogeneous functions were studied by many authors, see, for instance, [1–5]. However, it is worth mentioning that there are many homogeneous functions which are not differentiable, and there exist few studies of nonsmooth homogeneous optimization problems defined by positively homogeneous and locally Lipschitzian functions in real Banach spaces.

In the work [1], the homogeneous optimization problems were extended to nonsmooth functions, but there was still much work left to do. Meantime, we note that some results in Section 4 there require the assumption $\sum\limits_{i=1}^{m}\bar{\lambda}_i\frac{(p-q_i)q_ib_i}{p}\neq 0$, i.e., $p\neq q_i$ for some $i\in\{i\in M:\bar{\lambda}_i\neq 0\}$. In order to avoid the weakness mentioned above, inspired by the technique used in [2], we will give a modification of the model there.

In this paper, by using the generalized Euler identity for Clarke's subdifferential, we

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Biography: Zhang Juan (1983–), female, born at Zoucheng, Shandong, lecturer, major in applied functional analysis.

obtain a sufficient condition for an optimal solution of (HOP) to be a KKT point:

$$\begin{array}{lll}(\mathrm{HOP}) & \text{minimize} & f(x),\\ & \text{subject to} & g_i(x) \leqslant b_i, \quad i=1,2,\cdots,m,\\ & & x \in \Omega.\end{array}$$

Moreover, the relationship between (HOP) and its embedding problem $(\widehat{\mathrm{HOP}})$ is also considered:

$$\begin{array}{lll}(\widehat{\mathrm{HOP}}) & \text{minimize} & F_\alpha(x,u)=(u+\alpha)f(x)+\dfrac{1}{2}(u+\alpha-1)^2,\\ & \text{subject to} & (u+\alpha)\big[g_i(x)-b_i(1-\dfrac{q_i}{p})\big] \leqslant \dfrac{b_i q_i}{p}, \quad i=1,2,\cdots,m,\\ & & x \in \Omega,\\ & & u \geqslant 0,\end{array}$$

where the functions involved in (HOP) and $(\widehat{\mathrm{HOP}})$ are all positively homogeneous.

The paper is organized as follows. In Section 2, we give some preliminaries and definitions. In Section 3, we give a sufficient condition for an optimal solution of the nonsmooth homogeneous optimization problem (HOP) to be a KKT point. In Section 4, the one-to-one correspondence of the KKT points (the optimal solutions) of (HOP) and $(\widehat{\mathrm{HOP}})$ is established.

2 Preliminaries

Let X be a real Banach space with the topological dual X^*, $\langle .,. \rangle$ be the duality paring between X and X^* and Ω be a closed cone of X. We now introduce the nonsmooth homogeneous optimization problem:

$$\begin{array}{lll}(\mathrm{HOP}) & \text{minimize} & f(x),\\ & \text{subject to} & g_i(x) \leqslant b_i, \quad i=1,2,\cdots,m,\\ & & x \in \Omega,\end{array}$$

where f, $g_i : X \to R\,(i=1,2,\cdots,m)$ are positively homogeneous functions with degree p, $q_i\,(i=1,2,\cdots,m)$, respectively, and each $b_i \in R\,(i=1,2,\cdots,m)$.

We denote by K the set of feasible solutions of (HOP), i.e.,

$$K := \{x \in \Omega : g_i(x) \leqslant b_i,\, i=1,2,\cdots,m\}.$$

The following definitions and lemmas will be useful in the next two sections.

Definition 2.1 (see [6]) A function $f : X \to R$ is said to be Lipschitzian of rank L near a given point $x \in X$, if there exists some $\delta > 0$ such that

$$|f(y)-f(z)| \leqslant L\|y-z\|, \ \forall\, y,\, z \in B(x;\delta).$$

We say that $f : X \to R$ is locally Lipschitzian on X if it is Lipschitzian near any point of X.

Definition 2.2 (see [6]) Let $f : X \to R$ be locally Lipschitzian on X. The generalized directional derivative of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined as follows:

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t is a positive scalar.

We call the set

$$\partial f(x) := \big\{\xi \in X^* : \langle \xi, v\rangle \leqslant f^\circ(x; v),\ \forall\, v \in X\big\}$$

the Clarke's subdifferential of f at x.

It is easy to verify that

$$f^\circ(x; v) = \max\big\{\langle \xi, v\rangle : \xi \in \partial f(x)\big\},\ \forall\, v \in X.$$

Lemma 2.1 (see [6]) Let $f_i\,(i = 1, 2, \cdots, n)$ be Lipschitzian near x, and let $\lambda_i\,(i = 1, 2, \cdots, n)$ be scalars. Then $f := \sum\limits_{i=1}^{n} \lambda_i f_i$ is Lipschitzian near x, and we have

$$\partial\Big(\sum_{i=1}^{n} \lambda_i f_i\Big)(x) \subset \sum_{i=1}^{n} \lambda_i \partial f_i(x).$$

Definition 2.3 (see [7]) Let X be a real Banach space, and C be a nonempty subset of X. The Clarke tangent cone to C at $x \in C$ is defined by

$$T_C(x) = \big\{v \in X : d_C^\circ(x; v) = 0\big\},$$

where $d_C(x) = \inf\limits_{z \in C} \|z - x\|$, and the Clarke normal cone to C at $x \in C$ is defined by

$$N_C(x) := \big\{\zeta \in X^* : \langle \zeta, v\rangle \leqslant 0,\ \forall\, v \in T_C(x)\big\}.$$

Now we give the definition of invexity which was taken from [8].

Definition 2.4 A function $f : \Omega \to R$ is said to be nonsmooth invex at $\bar{x} \in \Omega$, if for any $x \in \Omega$ and $\xi \in \partial f(\bar{x})$, there exists $\eta(x, \bar{x}) \in T_\Omega(\bar{x})$ such that

$$f(x) - f(\bar{x}) \geqslant \big\langle \xi, \eta(x, \bar{x})\big\rangle.$$

Let $g = (g_1, g_2, \cdots, g_m)$ be a vector-valued function from Ω to R^m, then g is said to be nonsmooth invex at $x \in \Omega$, if each $g_i\,(i = 1, 2, \cdots, m)$ is nonsmooth invex at $x \in \Omega$.

Recall that a function $f : X \to R$ is said to be positively homogeneous with degree $p\,(p > 0)$ provided that the equality $f(\lambda x) = \lambda^p f(x)$ holds for any $x \in X$ and $\lambda > 0$.

As is well known, when a p-homogeneous function $\varphi : R^n \to R$ is differentiable, there is an identity, the so-called Euler formula,

$$\big\langle \nabla\varphi(x), x\big\rangle = p\cdot\varphi(x), \quad \forall\, x\in R^n.$$

The above formula was extended to nonsmooth homogeneous and locally Lipschitzian function defined on a real Banach space as follows.

Lemma 2.2 (see [9]) Let X be a real Banach space and $f : X \to R$ be a p-positively homogeneous ($p > 0$) and locally Lipschitzian function. Then, for each $x \in X$ and $\xi \in \partial f(x)$, the following identity holds:

$$\langle \xi, x\rangle = p\cdot f(x). \tag{2.1}$$

We denote it simply as the following formula:

$$\big\langle \partial f(x), x\big\rangle = p\cdot f(x),\ \forall\, x\in X.$$

Definition 2.5 $\bar{x} \in K$ is said to be an optimal solution or a global minimum of (HOP) if $f(\bar{x}) \leqslant f(x)$ for all $x \in K$, or equivalently, there exists no $x \in K$ such that $f(x) < f(\bar{x})$.

Definition 2.6 $\bar{x} \in K$ is said to be a KKT (Karash-Kuhn-Tucker) point of (HOP), if there exists a L-KKT (Lagrange-KKT) multiplier $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m) \in R^m_+$ such that

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}), \tag{2.2}$$

$$\bar{\lambda}_i\big(g_i(\bar{x}) - b_i\big) = 0,\ i = 1, 2, \cdots, m. \tag{2.3}$$

Definition 2.7 The problem (HOP) is said to satisfy the Slater constraint qualification, if there exists $\tilde{x} \in \Omega$ such that $g_i(\tilde{x}) - b_i < 0,\ i = 1, 2, \cdots, m$.

For notational convenience, we denote $M = \{i : i = 1, 2, \cdots, m\}$ in the next two sections.

3 Optimality Conditions

In this section, we first give a fine result about the KKT points under appropriate assumptions.

Theorem 3.1 Let f, $g_i\,(i \in M)$ be locally Lipschitzian and positively homogeneous functions with degree p, $q_i\,(i \in M)$, respectively. If $\bar{x} \in K$ is a KKT point of (HOP), then

$$f(\bar{x}) = -\sum_{i=1}^{m} \bar{\lambda}_i \frac{q_i}{p} b_i,$$

where $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m) \in R^m_+$ is a L-KKT multiplier associated with $\bar{x}$.

Proof Let $\bar{x} \in K$ be a KKT point of (HOP) with associated L-KKT multiplier $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m) \in R^m_+$. It follows from (2.2) that there exist $\xi \in \partial f(\bar{x})$ and $\zeta_i \in \partial g_i(\bar{x})\,(i \in M)$ such that

$$\xi + \sum_{i=1}^{m} \bar{\lambda}_i \zeta_i = 0. \tag{3.1}$$

By the homogeneity of f and the generalized Euler identity (2.1), we have

$$\begin{aligned} p \cdot f(\bar{x}) &= \langle \xi, \bar{x} \rangle = -\sum_{i=1}^{m} \bar{\lambda}_i \langle \zeta_i, \bar{x} \rangle \\ &= -\sum_{i=1}^{m} \bar{\lambda}_i q_i g_i(\bar{x}) = -\sum_{i=1}^{m} \bar{\lambda}_i q_i b_i. \end{aligned}$$

Hence, $f(\bar{x}) = -\sum\limits_{i=1}^{m} \bar{\lambda}_i \frac{q_i}{p} b_i$. This completes the proof.

Now, we are ready to prove that an optimal solution of (HOP) is necessarily a KKT point under appropriate conditions.

For notational convenience, we denote $I = \{i \in M : \lambda_i \neq 0\}$, where $\lambda_i \, (i \in M)$ is a L-KKT multiplier associated with some fixed KKT point $\bar{x}$.

Theorem 3.2 Suppose that

(1) f and $g_i \, (i \in M)$ are all Lipschitzian near $\bar{x} \in \Omega$;

(2) f and $g_i \; (i \in I)$ are nonsmooth invex at $\bar{x} \in \Omega$ with respect to same $\eta \in T_\Omega(\bar{x})$;

(3) f and $g_i \; (i \in I)$ are positively homogeneous with degree p, $q_i \, (i \in I)$, respectively;

(4) the Slater constraint qualification is satisfied.

If $\bar{x}$ is an optimal solution of (HOP), then $\bar{x}$ is a KKT point of (HOP) and there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m) \in R_+^m$ such that

$$f(\bar{x}) = \min_{x \in \Omega} f(x) = -\sum_{i=1}^{m} \bar{\lambda}_i \frac{q_i}{p} b_i.$$

Proof Since $\bar{x}$ is an optimal solution of (HOP), we have by [7, Theorem 6.1.1], there exist $\tau \in R_+$, $\lambda = (\lambda_1, \cdots, \lambda_m) \in R_+^m$, $(\tau, \lambda) \neq 0$, such that

$$0 \in \tau \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_\Omega(\bar{x}), \tag{3.2}$$

$$\lambda_i \big(g_i(\bar{x}) - b_i\big) = 0, \;\; i \in M. \tag{3.3}$$

By (3.2), there exist $\xi \in \partial f(\bar{x})$, $\zeta_i \in \partial g_i(\bar{x}) \, (i \in M)$ such that

$$-\Big(\tau \xi + \sum_{i=1}^{m} \lambda_i \zeta_i\Big) \in N_\Omega(\bar{x}),$$

or equivalently

$$-\Big(\tau \xi + \sum_{i \in I} \lambda_i \zeta_i\Big) \in N_\Omega(\bar{x}).$$

Using the nonsmooth invexity property of f and $g_i \, (i \in I)$ at $\bar{x} \in \Omega$, we have, for any $x \in \Omega$ and a suitable vector $\eta \in T_\Omega(\bar{x})$,

$$\begin{aligned} 0 &\leqslant \tau \langle \xi, \eta \rangle + \sum_{i \in I} \lambda_i \langle \zeta_i, \eta \rangle \\ &\leqslant \tau \big(f(x) - f(\bar{x})\big) + \sum_{i \in I} \lambda_i \big(g_i(x) - g_i(\bar{x})\big). \end{aligned} \tag{3.4}$$

Now we claim $\tau \neq 0$. In fact, if possible $\tau = 0$, then $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m) \neq 0$, by (3.3) and (3.4),

$$\sum_{i\in I} \big(g_i(x) - b_i\big) \geqslant 0,\ \forall\, x \in \Omega. \tag{3.5}$$

By Slater constraint qualification, there exists $\tilde{x} \in \Omega$ such that

$$\sum_{i\in I} \big(g_i(\tilde{x}) - b_i\big) < 0,$$

which is a contradiction to (3.5).

According to (3.4) and $\tau \neq 0$, we obtain

$$\big(f(x) - f(\bar{x})\big) + \sum_{i=1}^{m} \bar{\lambda}_i \big(g_i(x) - g_i(\bar{x})\big) \geqslant 0,\ \forall x \in \Omega,$$

where $\bar{\lambda}_i = \frac{\lambda_i}{\tau} \geqslant 0,\ i \in M$.

So $\bar{x}$ is a minimum point of the following problem

$$\min_{x\in\Omega} \Big(f + \sum_{i=1}^{m} \bar{\lambda}_i g_i\Big)(x),$$

which gives that

$$0 \in \partial\Big(f + \sum_{i=1}^{m} \bar{\lambda}_i g_i\Big)(\bar{x}) \subset \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}). \tag{3.6}$$

In view of (3.3) and (3.6), $\bar{x}$ is necessarily a KKT point of (HOP). And by Theorem 3.1, we have

$$f(\bar{x}) = \min_{x\in\Omega} f(x) = -\sum_{i=1}^{m} \bar{\lambda}_i \frac{q_i}{p} b_i.$$

Let $X = R^n$, then we have the following result.

Corollary 3.1 Suppose that

(1) f and $g_i\,(i \in M)$ are all continuously differentiable near $\bar{x} \in \Omega$;

(2) f and $g_i\,(i \in I)$ are all convex on Ω;

(3) f and g_i $(i \in I)$ are positively homogeneous with degree p, q_i $(i \in I)$, respectively;

(4) the Slater constraint qualification is satisfied.

If $\bar{x}$ is an optimal solution of (HOP), then $\bar{x}$ is a KKT point of (HOP) and there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m) \in R_+^m$ such that

$$f(\bar{x}) = \min_{x\in\Omega} f(x) = -\sum_{i=1}^{m} \bar{\lambda}_i \frac{q_i}{p} b_i.$$

Remark 3.1 It is worth mentioning that our results above can be applied in mathematical Finance theory. Assume that there are only n kinds of risk assets in the market, denoted by $X_1, X_2, \cdots, X_n$, and there are only two moments, today (denoted by 0) and future (denoted by 1). The one-period returns of the risk assets, $R_1, R_2, \cdots, R_n$, are all random variable. $R = (R_1, R_2, \cdots, R_n)$ is a vector, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, $\sum\limits_{i=1}^{n} \omega_i = 1$, is a portfolio, where ω_i is the investment ratio on $X_i\,(i = 1, 2, \cdots, n)$. $E(R) = \big(E(R_1), E(R_2), \cdots, E(R_n)\big)^T$ is a vector, where $E(R_i)$ is the expected return on $X_i\,(i = 1, 2, \cdots, n)$. As is well known, Markowitz's Mean-Variance model can be written as follows (for more details, see [10] or [11]):

$$\begin{aligned} \min \quad & \frac{1}{2}\sigma_\omega^2 = \frac{1}{2}\omega^T \Sigma \omega, && (3.7) \\ \text{s.t.} \quad & l^T \omega = 1, && (3.8) \\ & E(R_\omega) = \omega^T E(R) = \bar{\mu}, && (3.9) \end{aligned}$$

where $l = (1, 1, \cdots, 1)^T$ is a n-dimensional vector, each component of which is 1, Σ is a reversible variance-covariance matrix. The constraint qualification (3.8), (3.9) shows that ω is a portfolio and the expected return of the portfolio is a constant $\bar{\mu}$.

It is obvious that model (3.7) is a special homogeneous optimization problem. For this problem, we take the functions f and $g_i\,(i = 1, 2)$ in the Theorem 3.2 as follows:

$$f(\omega) = \frac{1}{2}\omega^T \Sigma \omega,\ g_1(\omega) = l^T \omega,\ g_2(\omega) = E(R_\omega)$$

and

$$p = 2,\ q_i = 1\,(i = 1, 2),\ b_1 = 1,\ b_2 = \bar{\mu}.$$

Since the feasible set of the problem is a bounded closed set in R^n for each fixed $\bar{\mu}$, then problem (3.7) has an optimal solution $\bar{\omega}$, and it is not difficult to verify that the conditions of Corollary 3.1 are satisfied. Therefore

$$f(\bar{\omega}) = -\sum_{i=1}^{2} \lambda_i \frac{q_i}{p} b_i = -\frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 \bar{\mu}.$$

Let

$$L = \frac{1}{2}\omega^T \Sigma \omega + \lambda_1 (l^T \omega - 1) + \lambda_2 \big(\omega^T E(R) - \bar{\mu}\big),$$

then it is not difficult to obtain that

$$\lambda_1 = -\frac{1}{D}(B - \bar{\mu} A),\ \lambda_2 = -\frac{1}{D}(\bar{\mu} C - A),$$

where

$$C = l^T \Sigma^{-1} l, A = l^T \Sigma^{-1} E(R), B = E(R)^T \Sigma^{-1} E(R), D = BC - A^2.$$

Therefore

$$f(\bar{\omega}) = -\sum_{i=1}^{2} \lambda_i \frac{q_i}{p} b_i = -\frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 \bar{\mu} = \frac{C}{2D}(\mu - \frac{A}{C})^2 + \frac{1}{2C}.$$

4 Some Duality Results

In [1], the homogeneous optimization problem was extended to nonsmooth homogeneous functions, but there exist weak points in the model there. In fact, some results in section 4 there require the assumption $\sum\limits_{i=1}^{m}\bar{\lambda}_i\frac{(p-q_i)q_ib_i}{p}\neq 0$, i.e., $p\neq q_i$ for some $i\in\{i\in M:\bar{\lambda}_i\neq 0\}$.

Let $0<\alpha\leqslant 1$ be a fixed positive scalar. In order to avoid the weakness mentioned above, following the idea of Ref. [2], we consider here a modification of $(\widehat{\text{HOP}})$ as follows (we still denote it as $(\widehat{\text{HOP}})$ for convenience):

$$\begin{aligned}
(\widehat{\text{HOP}})\quad & \text{minimize} && F_\alpha(x,u)=(u+\alpha)f(x)+\frac{1}{2}(u+\alpha-1)^2,\\
& \text{subject to} && (u+\alpha)\Big[g_i(x)-b_i(1-\frac{q_i}{p})\Big]\leqslant\frac{b_iq_i}{p},\quad i\in M,\\
& && x\in\Omega,\\
& && u\geqslant 0,
\end{aligned}$$

where $f,\ g_i:X\to R\,(i\in M)$ are positively homogeneous functions with degree p, $q_i\,(i\in M)$, respectively, and $b_i\in R\,(i\in M)$.

We denote by H the set of feasible solutions of $(\widehat{\text{HOP}})$, i.e.,

$$H:=\Big\{(x,u)\in\Omega\times R:(u+\alpha)\big[g_i(x)-b_i(1-\frac{q_i}{p})\big]\leqslant\frac{b_iq_i}{p},\ u\geqslant 0,\ b_i\in R,\ i\in M\Big\}.$$

Definition 4.1 A point $(\bar{x},\bar{u})\in H$ is said to be a KKT point of $(\widehat{\text{HOP}})$ if there exists a L-KKT multiplier $(\bar{\lambda},\bar{\mu})\in R_+^m\times R_+$ such that

$$0\in(\bar{u}+\alpha)\Big[\partial f(\bar{x})+\sum_{i=1}^{m}\bar{\lambda}_i\partial g_i(\bar{x})\Big],\tag{4.1}$$

$$f(\bar{x})+(\bar{u}+\alpha-1)+\sum_{i=1}^{m}\bar{\lambda}_i\Big[g_i(\bar{x})-b_i(1-\frac{q_i}{p})\Big]=\bar{\mu},\tag{4.2}$$

$$\bar{\lambda}_i\Big\{(\bar{u}+\alpha)\big[g_i(\bar{x})-b_i(1-\frac{q_i}{p})\big]-\frac{q_ib_i}{p}\Big\}=0,\ i\in M,\tag{4.3}$$

$$\bar{\mu}\bar{u}=0.\tag{4.4}$$

Next, we are ready to prove that the KKT points of (HOP) have a one to one correspondence to the KKT points of $(\widehat{\text{HOP}})$.

Theorem 4.1 Let X be a real Banach space and Ω be a closed cone of X. Assume that $f:\Omega\to R$ is a locally Lipschitzian and positively homogeneous function with degree p, and $g_i:\Omega\to R\,(i\in M)$ are locally Lipschitzian and positively homogeneous functions with degree $q_i\,(i\in M)$. Then

(i) If $\bar{x}\in\Omega$ is a KKT point of (HOP) with associated L-KKT multiplier $\bar{\lambda}\in R_+^m$, then $(\bar{x},1-\alpha)$ is a KKT point of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(\bar{\lambda},0)\in R_+^m\times R_+$.

(ii) If $(\bar{x},\bar{u})\in H$ is a KKT point of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(\bar{\lambda},\bar{\mu})\in R_+^m\times R_+$, and suppose further that

$$\sum_{i=1}^{m}\bar{\lambda}_i(1-\frac{q_i}{p})\frac{q_i}{p}b_i<\bar{u}+\alpha,\tag{4.5}$$

then $\bar{u} = 1-\alpha$, $\bar{\mu} = 0$; hence, $\bar{x}$ is a KKT point of (HOP) and $\bar{\lambda} \in R_+^m$ is a L-KKT multiplier associated with $\bar{x}$.

Proof (i) Let $\bar{x} \in \Omega$ be a KKT point with associated L-KKT multiplier

$$\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m) \in R_+^m.$$

It follows from (2.2) that there exist $\xi \in \partial f(\bar{x})$ and $\zeta_i \in \partial g_i(\bar{x})$ such that

$$\xi + \sum_{i=1}^m \bar{\lambda}_i \zeta_i = 0.$$

Therefore

$$\langle \xi, \bar{x} \rangle + \sum_{i=1}^m \bar{\lambda}_i \langle \zeta_i, \bar{x} \rangle = 0.$$

By the generalized Euler identity (2.1), the above equality can be written as

$$pf(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i q_i g_i(\bar{x}) = 0.$$

Dividing both sides of the above equality by $p\,(> 0)$, we have

$$f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \frac{q_i}{p} g_i(\bar{x}) = 0. \tag{4.6}$$

Considering $\bar{\lambda}_i[g_i(\bar{x}) - b_i] = 0 (i \in M)$, we obtain

$$f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \big[g_i(\bar{x}) - b_i(1 - \frac{q_i}{p})\big] = 0. \tag{4.7}$$

Take $\bar{u} = 1-\alpha$, $\bar{\mu} = 0$, then (4.1)–(4.3) follow from (2.2), (2.3), (4.7), directly. Moreover, $\bar{u}\bar{\mu} = 0$ holds trivially, since $\bar{\mu} = 0$.

(ii) Suppose that $(\bar{x}, \bar{u})$, $(\bar{\lambda}, \bar{\mu})$ satisfies conditions (4.1)–(4.4). Now we show that $\bar{x}$, $\bar{\lambda}$ satisfies (2.2) and (2.3). Since $\bar{u} + \alpha > 0$, (4.1) reduces to

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial g_i(\bar{x}).$$

Thus (2.2) is satisfied. And, there exist $\xi \in \partial f(\bar{x})$, $\zeta_i \in \partial g_i(\bar{x})\,(i \in M)$ such that

$$\xi + \sum_{i=1}^m \bar{\lambda}_i \zeta_i = 0.$$

Therefore,

$$\langle \xi, \bar{x} \rangle + \sum_{i=1}^m \bar{\lambda}_i \langle \zeta_i, \bar{x} \rangle = 0,$$

i.e.,

$$f(\bar{x})+\sum_{i=1}^{m}\bar{\lambda}_i\frac{q_i}{p}g_i(\bar{x})=0.$$

From the above equality and (4.2), we have

$$\sum_{i=1}^{m}\bar{\lambda}_i\big(g_i(\bar{x})-b_i\big)(1-\frac{q_i}{p})=\bar{\mu}-(\bar{u}+\alpha-1), \tag{4.8}$$

and it follows from (4.3) that

$$\sum_{i=1}^{m}\bar{\lambda}_i(g_i(\bar{x})-b_i)(1-\frac{q_i}{p})=\sum_{i=1}^{m}\bar{\lambda}_i\frac{q_i}{p}(1-\frac{q_i}{p})\frac{1-\bar{u}-\alpha}{\bar{u}+\alpha}b_i. \tag{4.9}$$

From (4.8) and (4.9), we see that

$$(\bar{u}+\alpha-1)\Big[1-\frac{1}{\bar{u}+\alpha}\sum_{i=1}^{m}\bar{\lambda}_i\frac{q_i}{p}(1-\frac{q_i}{p})b_i\Big]=\bar{\mu}. \tag{4.10}$$

Since the condition (4.5) is satisfied, it gives that

$$1-\frac{1}{\bar{u}+\alpha}\sum_{i=1}^{m}\bar{\lambda}_i\frac{q_i}{p}(1-\frac{q_i}{p})b_i>0.$$

If $\bar{\mu}>0$, then it follows from (4.10) that $\bar{u}>1-\alpha\geqslant 0$, which contradicts (4.4). So, $\bar{\mu}=0$, $\bar{u}=1-\alpha$. Then (2.3) holds directly from (4.3), which completes the proof.

In particular, setting $\alpha=1$, we have the following results.

Corollary 4.1 Let X, Ω, f and $g_i(i\in M)$ be given as in Theorem 4.1, then

(i) If $\bar{x}\in\Omega$ is a KKT point of (HOP) with associated L-KKT multiplier $\bar{\lambda}\in R_+^m$, then $(\bar{x},0)$ is a KKT point of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(\bar{\lambda},0)\in R_+^m\times R_+$.

(ii) If $(\bar{x},\bar{u})\in H$ is KKT point of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(\bar{\lambda},\bar{\mu})\in R_+^m\times R_+$, and suppose further that

$$\sum_{i=1}^{m}\bar{\lambda}_i\frac{q_i}{p}(1-\frac{q_i}{p})b_i<\bar{u}+1, \tag{4.11}$$

then $\bar{u}=0$, $\bar{\mu}=0$; hence, $\bar{x}$ is a KKT point of (HOP) and $\bar{\lambda}\in R_+^m$ is a L-KKT multiplier associated with $\bar{x}$.

Remark 4.1 Problems similar to that of Theorem 4.1 were considered in [2] under the condition that all homogeneous function are differentiable on $\Omega\subset R^n$. When a function $\varphi:\Omega\to R^n$ is differentiable, the Clarke's subdifferential $\partial\varphi(x)$ becomes a singleton at any point $x\in\Omega$, and $\partial\varphi(x)=\{\nabla\varphi(x)\}$. So Theorem 4.1 is a generalization of [2, Theorem 2.3].

Moreover, based on the above results, the next theorem shows the one-to-one correspondence of the optimal solutions of (HOP) and $(\widehat{\text{HOP}})$.

Theorem 4.2 Let X, Ω, f and $g_i\,(i \in M)$ be given as in Theorem 4.1. Assume that the condition

$$(u+\alpha)^{\frac{q_i}{p}}\Big[\Big(\frac{1}{u+\alpha}-1\Big)\frac{q_i}{p}+1\Big] \leqslant 1 \tag{4.12}$$

is satisfied and $b_i > 0\,(i \in M)$, where $(\cdot, u)$ is any feasible point of H.

Then, $\bar{x}$ is an optimal solution to (HOP) if and only if $(\bar{x}, 1-\alpha)$ is an optimal solution to $(\widehat{\mathrm{HOP}})$. Both problems have the same optimal values.

Proof If $\bar{x}$ is an optimal point of (HOP), then $(\bar{x}, 1-\alpha)$ is a feasible point of $(\widehat{\mathrm{HOP}})$. Thus

$$\min_{x\in K} f(x) = f(\bar{x}) = F_\alpha(\bar{x}, 1-\alpha) \geqslant \min_{(x,u)\in H} F_\alpha(x,u). \tag{4.13}$$

Next we show that $\min\limits_{x\in K} f(x) = \min\limits_{(x,u)\in H} F_\alpha(x,u)$, on the contrary, if

$$\min_{x\in K} f(x) > \min_{(x,u)\in H} F_\alpha(x,u),$$

then there is a minimizing sequence $\big\{(y_n, u_n)\big\} \in H$, such that

$$(u_n+\alpha)f(y_n) + \frac{1}{2}(u_n+\alpha-1)^2 \downarrow \min_{(x,u)\in H} F_\alpha(x,u).$$

Therefore, for n sufficiently large, say $n \geqslant n_0$,

$$(u_n+\alpha)f(y_n) + \frac{1}{2}(u_n+\alpha-1)^2 < f(\bar{x}).$$

But let $x_n = (u_n+\alpha)^{\frac{1}{p}} y_n \in \Omega$, then, by (4.12) and $\big\{(y_n, u_n)\big\} \in H$, we have

$$\begin{aligned} g_i(x_n) &= g_i\big((u_n+\alpha)^{\frac{1}{p}} y_n\big) = (u_n+\alpha)^{\frac{q_i}{p}} g_i(y_n) \\ &\leqslant (u_n+\alpha)^{\frac{q_i}{p}}\Big[\frac{(1-u_n-\alpha)q_i}{(u_n+\alpha)p}+1\Big]b_i \leqslant b_i,\ i \in M, \end{aligned}$$

that is, $x_n \in K$ and

$$f(x_n) = (u_n+\alpha)f(y_n) \leqslant (u_n+\alpha)f(y_n) + \frac{1}{2}(u_n+\alpha-1)^2 < f(\bar{x}),$$

when n is sufficiently large. This contradicts the fact that $\bar{x}$ is an optimal solution of (HOP). Hence, $\min\limits_{x\in K} f(x) = \min\limits_{(x,u)\in H} F_\alpha(x,u)$.

Conversely, let $(\bar{x}, 1-\alpha)$ be an optimal solution to $(\widehat{\mathrm{HOP}})$. We now show that $\bar{x}$ must be an optimal solution of (HOP).

Let $\bar{y} = (\bar{u}+\alpha)^{\frac{1}{p}}\bar{x}$. We note that $\bar{y}$ is a feasible point of (HOP). Indeed, by the homogeneity of the function $g_i\,(i\in M)$ and (4.12), we know

$$g_i(\bar{y}) = g_i\big[(\bar{u}+\alpha)^{\frac{1}{p}}\bar{x}\big] = (\bar{u}+\alpha)^{\frac{q_i}{p}} g_i(\bar{x}) \leqslant (\bar{u}+\alpha)^{\frac{q_i}{p}}\Big[\frac{(1-\bar{u}-\alpha)q_i}{(\bar{u}+\alpha)p}+1\Big]b_i \leqslant b_i,\ i \in M.$$

Hence

$$\min_{x\in K} f(x) \leqslant f(\bar{y}) = (\bar{u}+\alpha)f(\bar{x}) \leqslant (\bar{u}+\alpha)f(\bar{x}) + \frac{1}{2}(\bar{u}+\alpha-1)^2 = \min_{(x,u)\in H} F_\alpha(x,u). \quad (4.14)$$

It follows from (4.13), (4.14) that

$$\min_{x\in K} f(x) = \min_{(x,u)\in H} F_\alpha(x,u).$$

Therefore

$$\min_{x\in K} f(x) = f(\bar{y}) = (\bar{u}+\alpha)f(\bar{x}) = (\bar{u}+\alpha)f(\bar{x}) + \frac{1}{2}(\bar{u}+\alpha-1)^2 = \min_{(x,u)\in H} F_\alpha(x,u),$$

then $\bar{u} = 1-\alpha$, and $\bar{y} = (\bar{u}+\alpha)^{\frac{1}{p}}\bar{x} = \bar{x}$ is an optimal point of (HOP).

Remark 4.2 It is not difficult to verify that inequality (4.12) holds trivially if one of the following holds:

(i) $p \leqslant q_i,\ i \in M$;

(ii) $u + \alpha = 1$.

That is, our results is an extension of that in [1, Theorem 4.4].

Now we present two examples of nonsmooth homogeneous optimization problem to illustrate the effectiveness of the results in Theorems 4.1 and 4.2.

Example 4.1

$$\begin{array}{lll} \text{(HOP)} & \text{minimize} & f(x), \\ & \text{subject to} & g(x) \leqslant 0, \\ & & x \in \Omega, \end{array}$$

where $f(x) = |x|$ is absolutely 1-homogeneous, $g(x) = \max\{0, x\}$ is positively 1-homogeneous, $b = b_1 = 0$, $\Omega = (-\infty, +\infty)$ is a closed cone of R.

Let $\alpha = \frac{1}{2}$, and the embedding problem of (HOP) is presented as follows:

$$\begin{array}{lll} (\widehat{\text{HOP}}) & \text{minimize} & (u+\frac{1}{2})f(x) + \frac{1}{2}(u-\frac{1}{2})^2, \\ & \text{subject to} & (u+\frac{1}{2})g(x) \leqslant 0, \\ & & u \geqslant 0, \\ & & x \in \Omega. \end{array}$$

(i) $0 \in \partial f(0) = [-1, 1]$, $0 \in \partial g(0) = [0, 1]$, take $\lambda_1 = 1$, then

$$0 \in \partial f(0) + 1 \cdot \partial g(0),$$

and $1 \cdot g(0) = 0$, i.e., 0 is a KKT point of (HOP).

Now we show that $(0, \frac{1}{2})$ is a KKT point of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(1, 0)$. In fact, $0 \in \partial f(0) + \partial g(0) = (\frac{1}{2} + \frac{1}{2})\big[\partial f(0) + 1 \cdot \partial g(0)\big]$ is obvious and equations (4.2)–(4.4) hold trivially.

(ii) Conversely, it is easy to check $(0, \frac{1}{2})$ is a KKT point of $(\widehat{\text{HOP}})$ with associated $\text{L}-\text{KKT}$ multiplier $(1,0)$, and $\bar{u} = 1 - \frac{1}{2} = \frac{1}{2}$, $\bar{\mu} = 0$, $\bar{x} = 0$ is a KKT point of (HOP) with associated $\text{L}-\text{KKT}$ multiplier $\bar{\lambda} = 1$.

Furthermore, graphically, we can see that 0 is an optimal solution of (HOP) if and only if $(0, \frac{1}{2})$ is an optimal solution of $(\widehat{\text{HOP}})$.

Example 4.2

$$
\begin{aligned}
(\text{HOP}) \quad & \text{minimize} \quad && f(x), \\
& \text{subject to} \quad && g(x) \leqslant 0, \\
& && x \in \Omega,
\end{aligned}
$$

where $f : R^2 \to R$, $f(x) = \|x\|$ is absolutely 1-homogeneous, $g(x) = x_1^2 + x_2^2$ is positively 2-homogeneous, $\Omega = \big\{(x_1, x_2) : x_2 \geqslant |x_1|\big\}$ is a closed cone of R^2.

Let $\alpha = \frac{1}{2}$, then the embedding problem of (HOP) is presented as follows:

$$
\begin{aligned}
(\widehat{\text{HOP}}) \quad & \text{minimize} \quad && (u + \frac{1}{2}) f(x) + \frac{1}{2}(u - \frac{1}{2})^2, \\
& \text{subject to} \quad && (u + \frac{1}{2}) g(x) \leqslant 0, \\
& && u \geqslant 0, \\
& && x \in \Omega.
\end{aligned}
$$

(i) $\partial f(0,0) = \{\zeta \in R^2 : \zeta_i \in [-1,1],\ i = 1,2\}$, $\partial g(0,0) = \{(0,0)\}$. Take $\bar{\lambda} = 1$, it follows that $(0,0) \in \partial f(0,0) + 1 \cdot \partial g(0,0)$ and $1 \cdot \big(g(0,0) - 0\big) = 0$, i.e., $(0,0)$ is a KKT point of (HOP). Now we show that $\big((0,0), \frac{1}{2}\big)$ is a KKT point of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(1,0)$. In fact, $(0,0) \in \partial f(0,0) + 1 \cdot \partial g(0,0) = (\frac{1}{2} + \frac{1}{2})\big[\partial f(0,0) + 1 \cdot \partial g(0,0)\big]$ is obvious and (4.2)–(4.4) hold trivially.

(ii) Conversely, it is easy to verify that $\big((0,0), \frac{1}{2}\big)$ is a KKT of $(\widehat{\text{HOP}})$ with associated L-KKT multiplier $(1,0)$. And $\bar{u} = 1 - \frac{1}{2} = \frac{1}{2}$, $\bar{\mu} = 0$, then $\bar{x} = (0,0)$ is a KKT of (HOP) with associated L-KKT multiplier $\bar{\lambda} = 1$.

Furthermore, graphically, we can see that $(0,0)$ is an optimal solution of (HOP) if and only if $\big((0,0), \frac{1}{2}\big)$ is an optimal solution of $(\widehat{\text{HOP}})$.

Remark 4.3 So and first, the above two examples verify that even when $p \neq q_i\,(i \in M)$, f and $g_i\,(i \in M)$ involved are nonsmooth, the one-to-one correspondence of KKT points (optimal solutions) of (HOP) and $(\widehat{\text{HOP}})$ is still true, that is, our results is a true extension of those in [1, 2]. For more details of the duality problems, see [12, 13] and references therein.

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Banach空间中的一类非光滑齐次优化问题

张 娟[1], 李庶民[2]

(1. 昆明理工大学津桥学院工学系, 云南 昆明 650106)

(2. 昆明理工大学数学系, 云南 昆明 650404)

摘要: 本文主要研究了一类非光滑齐次优化问题(HOP). 通过运用Clarke 次微分的广义欧拉恒等式获得了使得(HOP)问题的最优解成为KKT 点的充分条件并给出了(HOP)问题与$\widehat{(\mathrm{HOP})}$问题的KKT 点及最优解之间的等价刻画. 本文的结果是文[1]中已有结果的推广. 文中还举例说明了结果的正确性.

关键词: Clarke 次微分; KKT 点; 欧拉恒等式; 非光滑齐次优化问题

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