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CHERN CONNECTION AND FUNDAMENTAL EQUATIONS ON FINSLER SUBMANIFOLDS

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Abstract: In this paper, we study the induced Chern connection on Finsler submanifolds. By using moving frame method, we built the fundamental equations with respect to the induced Chern connection D on Finsler submanifolds. Moreover, we also obtain the relation between D and the Chern connection ∇ of the induced Finsler metric, which enrich some known results in relevant literatures.

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1 Introduction

As the study on Riemann submanifolds, the study on Finsler submanifolds is also important and valuable. It is well known that the Finsler submanifolds have the induced Chern connection D and the Chern connection ∇ of the induced Finsler metric. In the Riemann case $D = \nabla$, called Levi-Civita connection. With the help of the Levi-Civita connection, Gauss, Codazzi and Ricci equations are established, which play an important role in studying Riemann submanifolds. Therefore, to study the same problems on Finsler submanifolds is also important and necessary. However, to our knowledge, there were not many researches on this topic (see [1, 2, 6, 7]).

In [1], the author built Gauss and Codazzi equations by the Chern connection ∇ of the induced Finsler metric. The main purpose of this paper is to study Finsler submanifolds via the induced Chern connection D. In general $D \neq \nabla$, so the relevant fundamental equations have a bit difference from each other. Naturally, we care for: what are the relations D and ∇? To answer this question, we give the following theorems.

Theorem 1.1 Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, $\widetilde{\nabla}$ is the Chern connection of $\widetilde{M}$. Then $\nabla = D$ if and only if

$$\widetilde{A}(X, Y, \widetilde{\nabla}_Z e_n) = A(X, Y, \nabla_Z e_n),$$

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where $\widetilde{A}$ and A are the Cartan tensors of $\widetilde{F}$ and F, respectively, $X, Y, Z \in \pi^*TM$ and e_n is the distinguished field.

Theorem 1.2 Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. If M is a weakly totally geodesic submanifold of $\widetilde{M}$, then $D = \nabla$.

The paper is organized into 4 sections. After introducing some basic concepts of Finsler geometry in Section 2, we build Gauss, Codazzi and Ricci equations with respect to the induced Chern connection D on Finsler submanifold in Section 3. In Section 4, we discuss some relations between D and ∇.

2 Preliminaries

Let M be an n-dimensional smooth manifold. A Finsler metric on M is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties:

(i) F is smooth on $TM\backslash 0$;

(ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;

(iii) the induced quadratic form g is positive-definite, where

$$g := g_{ij} dx^i \otimes dx^j, \ \ g_{ij} = \frac{1}{2}[F^2]_{y^i y^j},$$

here and from now on, we will use the following convention of index ranges unless otherwise stated:

$$1 \le i, j \cdots \le n; \ 1 \le \lambda, \mu \cdots \le n-1; \ 1 \le a, b \cdots \le n+p; \ \ n+1 \le \alpha, \beta \cdots \le n+p.$$

The projection $\pi : TM \longrightarrow M$ gives rise to the pull-back bundle π^*TM and its dual π^*T^*M over $TM\backslash 0$. In π^*T^*M there is a global section $\omega = [F]_{y^i} dx^i$, called the Hilbert form, whose dual is $\ell = \ell^i \frac{\partial}{\partial x^i}, \ell^i = \frac{y^i}{F}$, called the distinguished field.

Let (M^n, F) and $(\widetilde{M}^{n+p}, \widetilde{F})$ be the two Finsler manifolds. For an immersion f : $(M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$, if $F(x, y) = \widetilde{F}(x, df(y))$ for all $(x, y) \in TM \setminus 0$, then f is called an isomertric immersion. It is clear that

$$g_{ij}(x, y) = \tilde{g}_{ab}(\tilde{x}, \tilde{y}) f_i^a f_j^b, \ A_{ijk} = \widetilde{A}_{abc} f_i^a f_j^b f_k^c,$$

where

$$\tilde{x}^a = f^a, \ \tilde{y}^a = f_i^a y^i, \ f_i^a = \frac{\partial f^a}{\partial x^i},$$

g (resp. $\tilde{g}$), A (resp. $\tilde{A}$) are the fundamental tensor and the Cartan tensor of M (resp. $\widetilde{M}$), respectively.

The map f admits a lift $\tilde{f} : TM \longrightarrow T\widetilde{M}$ defined by

$$\tilde{f}(x, y) = (\tilde{x}, \tilde{y}), \ \tilde{x} = f(x), \ \tilde{y} = f_* y.$$

Let $(\pi^*TM)^\perp$ be the orthogonal complement of π^*TM in $\pi^*(f^{-1}T\widetilde{M})$ with respect to $\tilde{g}$. Then

$$\pi^*(f^{-1}T\widetilde{M}) = \pi^*TM \oplus (\pi^*TM)^\perp,$$

where $(\pi^*TM)^\perp$ is called the normal bundle of f.

3 Fundamental Equations on Finsler Submanifolds

Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Take a $\tilde{g}-$orthonormal frame field $\{e_a\}$ of $\pi^*(T\widetilde{M})$ and let $\{\theta^a\}$ be a local dual coframe such that $\{e_i\}$ is a frame field of $\pi^*(TM)$ and e_n is the distinguish field (See [1]). Denote by $\{\theta^a_b\}$ the 1-form of the Chern connection $\widetilde{\nabla}$. Set $\omega^i = f^*\theta^i$ and $\omega^i_j = f^*\theta^i_j$. Then ω^i is the local dual coframe of $\{e_i\}$ and ω^i_j is the 1-form of the induced connection D. It is not difficult to conclude that

$$A(e_i, e_j, e_n) = 0,\ \widetilde{A}(e_a, e_b, e_n) = 0,\ \forall i, j, a, b.$$

The Gauss and Weingarten formulas are written by

$$\widetilde{\nabla}_X Y = D_X Y + B(X, Y),\ \widetilde{\nabla}_X \xi = -W_\xi X + \nabla^\perp_X \xi,\ \forall X, Y \in \pi^*TM,\ \xi \in (\pi^*TM)^\perp,$$

where B is the second fundamental form of M, W_ξ is called Weingarten transformation, $\nabla^\perp$ is called normal connection on $(\pi^*TM)^\perp$. By simple arguments, we get

Proposition 3.1 *D, B, W and $\nabla^\perp$ have the following properties:*

(1) D determines a linear torsion-free connection on $\pi^*(TM)$.

(2) $B : \pi^*TM \otimes \pi^*TM \longrightarrow (\pi^*TM)^\perp$ is a symmetric bilinear map.

(3) $W_\xi : \pi^*TM \longrightarrow \pi^*TM$ is a linear map and $W : \pi^*TM \otimes (\pi^*TM)^\perp \longrightarrow \pi^*TM$ is a bilinear map.

(4) $\nabla^\perp$ determines a linear connection on $(\pi^*TM)^\perp$.

Let $B(e_i, e_j) = B^\alpha_{ij} e_\alpha, W_{e_\alpha} e_i = W^\alpha_{ij} e_j$. Then we have

$$W^\alpha_{ij} = B^\alpha_{ij} + 2\widetilde{A}_{aj\alpha}\omega^a_n(e_i).$$

The structure equations of $\widetilde{M}$ are given by

$$\begin{cases} d\theta^a = -\theta^a_b \wedge \theta^b, \\ \theta^a_b + \theta^b_a = -2\widetilde{A}_{abc}\theta^c_n, \\ d\theta^a_b = -\theta^a_c \wedge \theta^c_b + \frac{1}{2}\widetilde{R}^a_{bcd}\theta^c \wedge \theta^d + \widetilde{P}^a_{bcd}\theta^c \wedge \theta^d_n, \end{cases} \tag{3.1}$$

where $\widetilde{R}^a_{bcd}$ and $\widetilde{P}^a_{bcd}$ are called the first Chern curvature tensors and the second Chern curvature tensors, respectively. Restricting them to M yields

$$\begin{cases} d\omega^i = -\omega^i_j \wedge \omega^j, \omega^\alpha = 0, \\ \omega^i_j + \omega^j_i = -2\widetilde{A}_{ijc}\omega^c_n, \\ d\omega^i_j = -\omega^i_a \wedge \omega^a_j + \frac{1}{2}\widetilde{R}^i_{jkl}\omega^k \wedge \omega^l + \widetilde{P}^i_{jkc}\omega^k \wedge \omega^c_n. \end{cases} \tag{3.2}$$

Exterior differentiating $\omega^\alpha = 0$ gives

$$0 = d\omega^\alpha = -\omega_i^\alpha \wedge \omega^i.$$

By Cartan lemma, we have

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha. \tag{3.3}$$

On the other hand, from Gauss formula one gets

$$B_{ij}^\alpha = \tilde{g}(B(e_i, e_j), e_\alpha) = \tilde{g}(\widetilde{\nabla}_{e_i} e_j, e_\alpha) = \omega_j^\alpha(e_i).$$

So

$$h_{ij}^\alpha = B_{ij}^\alpha. \tag{3.4}$$

The curvature 2-forms of the induced Chern connection D are

$$d\omega_j^i + \omega_k^i \wedge \omega_j^k := \Omega_j^i = \frac{1}{2} R_{jkl}^i \omega^k \wedge \omega^l + P_{jk\lambda}^i \omega^k \wedge \omega_n^\lambda,$$

where R_{jkl}^i and $P_{jk\lambda}^i$ are the curvature tensors with respect to the induced Chern connection D. By using the above formula and (3.2)–(3.4), we have

$$\begin{aligned}
\frac{1}{2} R_{jkl}^i \omega^k \wedge \omega^l + P_{jkl}^i \omega^k \wedge \omega_n^l =& -\omega_\alpha^i \wedge \omega_j^\alpha + \frac{1}{2} \widetilde{R}_{jkl}^i \omega^k \wedge \omega^l + \widetilde{P}_{jkc}^i \omega^k \wedge \omega_n^c \\
=& (\omega_i^\alpha + 2\widetilde{A}_{i\alpha c} \omega_n^c) \wedge B_{jl}^\alpha \omega^l + \frac{1}{2} \widetilde{R}_{jkl}^i \omega^k \wedge \omega^l \\
& + \widetilde{P}_{jk\lambda}^i \omega^k \wedge \omega_n^\lambda + \widetilde{P}_{jk\alpha}^i B_{nl}^\alpha \omega^k \wedge \omega^l \\
=& \{\frac{1}{2} \widetilde{R}_{jkl}^i + \widetilde{P}_{jk\alpha}^i B_{nl}^\alpha + (B_{ik}^\alpha + 2\widetilde{A}_{i\alpha\beta} B_{nk}^\beta) B_{jl}^\alpha\} \omega^k \wedge \omega^l + \widetilde{P}_{jk\lambda}^i \omega^k \wedge \omega_n^\lambda.
\end{aligned}$$

Therefore, we get the following result.

Theorem 3.2 (the Gauss equations) Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Then we have

$$\left\{ \begin{aligned}
R_{jkl}^i =& \ \widetilde{R}_{jkl}^i + \widetilde{P}_{jk\alpha}^i B_{nl}^\alpha - \widetilde{P}_{jl\alpha}^i B_{nk}^\alpha + B_{ik}^\alpha B_{jl}^\alpha \\
& -B_{il}^\alpha B_{jk}^\alpha + 2\widetilde{A}_{i\alpha\beta}(B_{nk}^\beta B_{jl}^\alpha - B_{nl}^\beta B_{jk}^\alpha), \\
P_{jk\lambda}^i =& \ \widetilde{P}_{jk\lambda}^i - 2B_{jk}^\alpha \widetilde{A}_{i\alpha\lambda}.
\end{aligned} \right. \tag{3.5}$$

Exterior differentiating $\omega_i^\alpha = B_{ij}^\alpha \omega^j$, we have

$$\begin{aligned}
d\omega_i^\alpha =& d(B_{ij}^\alpha \omega^j) = dB_{ij}^\alpha \omega^j + B_{ij}^\alpha d\omega^j \\
=& (B_{ij|k}^\alpha \omega^k + B_{ij;\lambda}^\alpha \omega_n^\lambda + B_{lj}^\alpha \omega_i^l + B_{li}^\alpha \omega_j^l \\
& - B_{ij}^\beta \omega_\beta^\alpha) \wedge \omega^j - B_{ij}^\alpha \omega_k^j \wedge \omega^k,
\end{aligned} \tag{3.6}$$

where "$|$" denotes the horizontal covariant derivative with respect to D and "$;$" denotes the vertical derivative.

On the other hand, from (3.2)–(3.4) one obtains

$$\begin{aligned} d\omega_i^\alpha =& -\omega_a^\alpha \wedge \omega_i^a + \frac{1}{2}\widetilde{R}_{ikl}^\alpha \omega^k \wedge \omega^l + \widetilde{P}_{ikc}^\alpha \omega^k \wedge \omega_n^c \\ =& -\omega_\beta^\alpha \wedge \omega_i^\beta - B_{kl}^\alpha \omega^l \wedge \omega_i^k + \frac{1}{2}\widetilde{R}_{ikl}^\alpha \omega^k \wedge \omega^l \\ & + \widetilde{P}_{ik\beta}^\alpha B_{nl}^\beta \omega^k \wedge \omega^l + \widetilde{P}_{ik\lambda}^\alpha \omega^k \wedge \omega_n^\lambda. \end{aligned} \tag{3.7}$$

Substituting (3.6) into (3.7), we get the following result.

Theorem 3.3 (the Codazzi equations) Let $f:(M^n,F)\longrightarrow(\widetilde{M}^{n+p},\widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Then we have

$$\left\{ \begin{array}{ll} B_{ij|k}^\alpha - B_{ik|j}^\alpha & = -\widetilde{R}_{ijk}^\alpha + \widetilde{P}_{ik\beta}^\alpha B_{nj}^\beta - \widetilde{P}_{ij\beta}^\alpha B_{nk}^\beta, \\ \quad B_{ij;\lambda}^\alpha & = -\widetilde{P}_{ik\lambda}^\alpha. \end{array} \right. \tag{3.8}$$

Set

$$d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma := \Omega_\beta^{\perp\alpha} = \frac{1}{2}R_{\beta kl}^{\perp\alpha}\omega^k \wedge \omega^l + P_{\beta kc}^{\perp\alpha}\omega^k \wedge \omega_n^c + Q_{\beta cd}^{\perp\alpha}\omega_n^c \wedge \omega_n^d,$$

where $R_{\beta kl}^{\perp\alpha}, P_{\beta kc}^{\perp\alpha}, Q_{\beta cd}^{\perp\alpha}$ are the normal curvature tensors. Then

$$\begin{aligned} d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma =& \left\{ \frac{1}{2}R_{\beta kl}^{\perp\alpha} + Q_{\beta\gamma\delta}^{\perp\alpha} B_{nk}^\gamma B_{nl}^\delta \right\} \omega^k \wedge \omega^l + Q_{\beta\lambda\mu}^{\perp\alpha}\omega_n^\lambda \wedge \omega_n^\mu \\ & + \{P_{\beta k\lambda}^{\perp\alpha} + Q_{\beta\gamma\lambda}^{\perp\alpha}B_{nk}^\gamma - Q_{\beta\lambda\gamma}^{\perp\alpha}B_{nk}^\gamma\}\omega^k \wedge \omega_n^\lambda. \end{aligned} \tag{3.9}$$

On the other hand, from (3.1)–(3.4) we have

$$\begin{aligned} d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma =& -\omega_i^\alpha \wedge \omega_\beta^i + \frac{1}{2}\widetilde{R}_{\beta kl}^\alpha \omega^k \wedge \omega^l + \widetilde{P}_{\beta kc}^\alpha \omega^k \wedge \omega_n^c \\ =& B_{ik}^\alpha \omega^k \wedge (B_{il}^\beta \omega^l + 2\widetilde{A}_{i\beta\lambda}\omega_n^\lambda + 2\widetilde{A}_{i\beta\gamma}B_{ik}^\gamma \omega^l) \\ & + \frac{1}{2}\widetilde{R}_{\beta kl}^\alpha \omega^k \wedge \omega^l + \widetilde{P}_{\beta kc}^\alpha \omega^k \wedge \omega_n^c \\ =& \{B_{ik}^\alpha B_{il}^\beta + 2B_{ik}^\alpha \widetilde{A}_{i\beta\gamma}B_{nl}^\gamma + \frac{1}{2}\widetilde{R}_{\beta kl}^\alpha + \widetilde{P}_{\beta k\gamma}^\alpha B_{nl}^\gamma\}\omega^k \wedge \omega^l \\ & + \{2B_{ik}^\alpha \widetilde{A}_{i\beta\lambda} + \widetilde{P}_{\beta k\lambda}^\alpha\}\omega^k \wedge \omega_n^\lambda. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we can state the following theorem.

Theorem 3.4 (the Ricci equations) Let $f:(M^n,F)\longrightarrow(\widetilde{M}^{n+p},\widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Then we have

$$\left\{ \begin{array}{rcl} R_{\beta kl}^{\perp\alpha} + Q_{\beta\gamma\delta}^{\perp\alpha}(B_{nk}^\gamma B_{nl}^\delta - B_{nl}^\gamma B_{nk}^\delta) = & & -\widetilde{R}_{\beta kl}^\alpha + \widetilde{P}_{\beta k\gamma}^\alpha B_{nl}^\gamma - \widetilde{P}_{\beta l\gamma}^\alpha B_{nk}^\gamma \\ & & +(B_{ik}^\alpha B_{il}^\beta - B_{il}^\alpha B_{ik}^\beta) + 2\widetilde{A}_{i\beta\gamma}(B_{ik}^\alpha B_{nl}^\gamma - B_{il}^\alpha B_{nk}^\gamma), \\ P_{\beta k\lambda}^{\perp\alpha} + Q_{\beta\gamma\lambda}^{\perp\alpha}B_{nk}^\gamma - Q_{\beta\lambda\gamma}^{\perp\alpha}B_{nk}^\gamma = & & 2B_{ik}^\alpha \widetilde{A}_{i\beta\lambda} + \widetilde{P}_{\beta k\lambda}^\alpha, \\ Q_{\beta\lambda\mu}^{\perp\alpha} = & & 0. \end{array} \right.$$

Definition 3.1 [2] Let $f:(M^n,F)\longrightarrow(\widetilde{M}^{n+p},\widetilde{F})$ be an isometric immersion. Write $H := \frac{1}{n}\mathrm{tr}B = \frac{1}{n}\sum\limits_{i\alpha} B_{ii}^\alpha e_\alpha$, H is called the mean curvature vector field, M is called to be minimal (or totally geodesic) if H (or B) vanishes identically, M is called to have flat normal bundle if $\Omega^{\perp\alpha}_{\beta} = 0$.

Definition 3.2 [1] A submanifold (M^n, F) of $(\widetilde{M}^{n+p})$ is said to be weakly totally geodesic if $B(e_n, e_n) = 0$.

From Theorem 3.4, one obtains

Proposition 3.5 The totally geodesic submanifold in Minkowski space has flat normal bundle.

In the end of this section, we give the Gauss equations on the flag curvature.

Theorem 3.6 Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Then we have

$$K(e_n; e_i) = \widetilde{K}(e_n; e_i) + \widetilde{L}_{ii\alpha} B^{\alpha}_{nn} + B^{\alpha}_{ii} B^{\alpha}_{nn} - (B^{\alpha}_{in})^2,$$

where $\widetilde{L}_{abc} = -\dot{\widetilde{A}}_{abc}$ is Landsberg curvature.

Proof Setting $j = l = n$ in $(3.5)_1$, we obtain

$$\begin{aligned} R^i_{nkn} &= \widetilde{R}^i_{nkn} + \widetilde{P}^i_{nk\alpha} B^{\alpha}_{nn} - \widetilde{P}^i_{nn\alpha} B^{\alpha}_{nk} + B^{\alpha}_{ik} B^{\alpha}_{nn} \\ &\quad - B^{\alpha}_{in} B^{\alpha}_{nk} + 2\widetilde{A}_{i\alpha\beta}(B^{\beta}_{nk} B^{\alpha}_{nn} - B^{\beta}_{nn} B^{\alpha}_{nk}) \\ &= \widetilde{R}^i_{nkn} + \widetilde{L}_{ik\alpha} B^{\alpha}_{nn} + B^{\alpha}_{ik} B^{\alpha}_{nn} - B^{\alpha}_{in} B^{\alpha}_{nk}, \end{aligned}$$

where we have used $\widetilde{P}^i_{nk\alpha} = \widetilde{L}_{ik\alpha}$. So the Gauss equations on the flag curvature can be derived. This finishes the proof.

4 The Relationship between D and ∇

Lemma 4.1 [1] Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold and $\widetilde{\nabla}$ be the Chern connection of $\widetilde{M}$. If

$$\begin{aligned} \widetilde{\nabla}_X Y =& \nabla_X Y + B(X, Y) + \sum_i \{\widetilde{A}(X, Y, \widetilde{\nabla}_{e_i} e_n - \nabla_{e_i} e_n) \\ & - \widetilde{A}(X, e_i, \widetilde{\nabla}_Y e_n - \nabla_Y e_n) - \widetilde{A}(e_i, Y, \widetilde{\nabla}_X e_n - \nabla_X e_n)\} e_i, \end{aligned} \tag{4.1}$$

where $X, Y \in \Gamma(\pi^* TM)$, $B(X, Y) \in \Gamma(\pi^* TM)^{\perp}$, then ∇ is the Chern connection of M.

In order to illustrate the relationship between ∇ and D, we give the following:

Theorem 4.2 Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. $\widetilde{\nabla}$ is the Chern connection of $\widetilde{M}$. Then $\nabla = D$ if and only if

$$\widetilde{A}(X, Y, \widetilde{\nabla}_Z e_n) = A(X, Y, \nabla_Z e_n),$$

where $\widetilde{A}$ and A are the Cartan tensors of $\widetilde{F}$ and F, respectively, $X, Y, Z \in \pi^* TM$.

Proof Recall that the Gauss formula is

$$\widetilde{\nabla}_X Y = D_X Y + B(X, Y). \tag{4.2}$$

If $\nabla = D$, then from (4.1) and (4.2) one gets

$$\widetilde{A}(X,Y,\widetilde{\nabla}_Z e_n - \nabla_Z e_n) - \widetilde{A}(X,Z,\widetilde{\nabla}_Y e_n - \nabla_Y e_n) - \widetilde{A}(Z,Y,\widetilde{\nabla}_X e_n - \nabla_X e_n) = 0.$$

Setting $X = Z$, we have

$$\widetilde{A}(X,X,\widetilde{\nabla}_Y e_n) = A(X,X,\nabla_Y e_n). \tag{4.3}$$

Substituting $X = U + V$ into (4.3) yields

$$\widetilde{A}(U,V,\widetilde{\nabla}_Y e_n) = A(U,V,\nabla_Y e_n).$$

The sufficient condition is evident from (4.1) and (4.2). This proves Theorem 4.2.

Theorem 4.3 Let $f : (M^n, F) \longrightarrow (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. If M is a weakly totally geodesic submanifold of $\widetilde{M}$, then $\nabla = D$.

Proof If M is a weakly totally geodesic submanifold, then[1]

$$B^{\alpha}_{nn} = 0, \forall \alpha.$$

So we have

$$B^{\alpha}_{nn|i}\omega^i + B^{\alpha}_{nn;\lambda}\omega^{\lambda}_n = dB^{\alpha}_{nn} - 2B^{\alpha}_{n\lambda}\omega^{\lambda}_n + B^{\beta}_{nn}\theta^{\alpha}_{\beta} = -2B^{\alpha}_{n\lambda}\omega^{\lambda}_n.$$

From which one gets

$$B^{\alpha}_{nn;\lambda} = -2B^{\alpha}_{n\lambda}.$$

By using $\widetilde{P}^{\alpha}_{nn\gamma} = 0$ and $(3.8)_2$, we obtain

$$B^{\alpha}_{ni} = 0, \forall \alpha, i. \tag{4.4}$$

Therefore, from (4.1) and (4.4), we get

$$\begin{aligned}
\widetilde{A}(X,Y,\widetilde{\nabla}_Z e_n) =& \widetilde{A}(X,Y,\nabla_Z e_n) + \widetilde{A}(X,Y,B(Z,e_n)) + \widetilde{A}(X,Y,\widetilde{\nabla}_Z e_n - \nabla_Z e_n) \\
& - \widetilde{A}(Z,Y,\widetilde{\nabla}_X e_n - \nabla_X e_n) - \widetilde{A}(X,Z,\widetilde{\nabla}_Y e_n - \nabla_Y e_n) \\
=& A(X,Y,\nabla_Z e_n) + \widetilde{A}(X,Y,B^{\alpha}_{ni}\omega^i(Z)) + \widetilde{A}(X,Y,B(Z,e_n)) \\
& - \widetilde{A}(X,Y,e_\lambda)\widetilde{A}(Z,e_\lambda,B(e_n,e_n)) + \widetilde{A}(X,Z,e_\lambda)\widetilde{A}(Y,e_\lambda,B(e_n,e_n)) \\
& - \widetilde{A}(X,Z,B(Y,e_n)) - \widetilde{A}(Y,Z,B(X,e_n)) + \widetilde{A}(Y,Z,e_\lambda)\widetilde{A}(X,e_\lambda,B(e_n,e_n)) \\
=& A(X,Y,\nabla_Z e_n),
\end{aligned}$$

which implies $\nabla = D$ by Theorem 4.2.

From Theorem 3.6, Theorem 4.3 and (4.4), we also obtain the following result which is the main theorem in [1].

Corollary 4.4 If M^n is a weakly totally geodesic submanifold of $\widetilde{M}^{n+p}$, then flag curvature of M^n equals flag curvature of $\widetilde{M}^{n+p}$.

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Finsler 子流形中的陈联络与基本方程

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摘要: 该文研究了Finsler 子流形中诱导的两种陈联络. 通过利用活动标架法, 利用诱导的陈联络D建立了Finsler 子流形的基本方程, 并给出了D 与诱导度量的陈联络∇ 之间的关系. 这些研究完善和充实了已有文献的相关结果.

关键词: Finsler 度量; 陈联络; 基本方程

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