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# THE DARBOUX TRANSFORMATION OF THE DERIVATIVE MANAKOV EQUATION AND ITS EXPLICIT SOLUTION

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**Abstract:** In this paper, we present a new Darboux transformation for the derivative Manakov equation. By applying the Darboux transformation, we obtain new soliton solutions of the derivative Manakov soliton equation. Finally, the figures of the soliton solution are obtained by choosing the suitable parameters.

**Keywords:** spectral problem; Darboux transformation; Darboux matrix; explicit solution

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## 1 Introduction

The soliton equation is one of the most prominent subjects in the field of nonlinear science, like nonlinear optics, the theory of deep water waves, and plasma physics. It was well known that there were many ways to obtain explicit solutions of soliton equations, such as the inverse scattering transformation (IST) [1, 2], the Hirota technique [3, 4], the Darboux transformation (DT) [5, 6], and so on [7–11]. Some interesting explicit solutions were found, the most important ones among which are pure-soliton solutions, finite-band solutions and polar expansion solutions. Among the various approaches, DT was known to be powerful in finding solutions of soliton equations from a trivial seed [12–18].

In this paper, we consider the derivative Manakov soliton equation [19]

$$\begin{cases} u_{1t} = \frac{1}{3}u_{1xx} + \frac{2}{3}[(u_1v_1 + u_2v_2)u_1]_x, \\ u_{2t} = \frac{1}{3}u_{2xx} + \frac{2}{3}[(u_1v_1 + u_2v_2)u_2]_x, \\ v_{1t} = -\frac{1}{3}v_{1xx} - \frac{2}{3}[(u_1v_1 + u_2v_2)v_1]_x, \\ v_{2t} = -\frac{1}{3}v_{2xx} - \frac{2}{3}[(u_1v_1 + u_2v_2)v_2]_x. \end{cases}$$

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The paper is arranged as follows: we start from a coupled equation which is related to a 3×3 spectral problem and give a basic DT [12–18]. In Section 3, we get from $u_1 = u_2 = v_1 = v_2 = 0$ the soliton solutions of the couple soliton equations, then we obtain the explicit solutions of the coupled equation. Finally, the figures of the soliton solution are obtained by choosing the suitable parameters.

## 2 Darboux Transformation

In this section, we shall construct a DT of the derivative Manakov equation

$$\left\{\begin{array}{l} u_{1t} = \frac{1}{3}u_{1xx} + \frac{2}{3}[(u_1v_1 + u_2v_2)u_1]_x, \\ u_{2t} = \frac{1}{3}u_{2xx} + \frac{2}{3}[(u_1v_1 + u_2v_2)u_2]_x, \\ v_{1t} = -\frac{1}{3}v_{1xx} - \frac{2}{3}[(u_1v_1 + u_2v_2)v_1]_x, \\ v_{2t} = -\frac{1}{3}v_{2xx} - \frac{2}{3}[(u_1v_1 + u_2v_2)v_2]_x. \end{array}\right. \tag{2.1}$$

This equation has a Lax pair, the spectral problem

$$\phi_x = U\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.2}$$

$$U = \begin{pmatrix} 2\lambda & v_1 & v_2 \\ \lambda u_1 & -\lambda & 0 \\ \lambda u_2 & 0 & -\lambda \end{pmatrix},$$

and the auxiliary problem

$$\phi_t = V\phi, \quad V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \tag{2.3}$$

with

$$\begin{aligned} &V_{11} = -2\lambda^2 + \frac{1}{3}(u_1v_1 + u_2v_2)\lambda, \quad V_{12} = -v_1\lambda - \frac{1}{3}v_{1x} + \frac{2}{9}(u_1v_1 + u_2v_2)v_1, \\ &V_{13} = -v_2\lambda - \frac{1}{3}v_{2x} + \frac{2}{9}(u_1v_1 + u_2v_2)v_2, \quad V_{21} = -u_1\lambda^2 + [\frac{1}{3}u_{1x} + \frac{2}{9}(u_1v_1 + u_2v_2)u_1]\lambda, \\ &V_{22} = \lambda^2 - \frac{1}{3}u_1v_1\lambda, \quad V_{23} = -\frac{1}{3}u_1v_2\lambda, \quad V_{31} = -u_2\lambda^2 + [\frac{1}{3}u_{2x} + \frac{2}{9}(u_1v_1 + u_2v_2)u_2]\lambda, \\ &V_{32} = -\frac{1}{3}u_2v_1\lambda, \quad V_{33} = \lambda^2 - \frac{1}{3}u_2v_2\lambda, \end{aligned}$$

where $u_1, u_2, v_1$ and $v_2$ are four potentials, and $\lambda$ is a spectral parameter.

In fact, a direct calculation shows that the zero-curvature equation $U_t - V_x + [U, V] = 0$, implies the derivative Manakov equation (2.1).

We assume that there is a matrix $T$ satisfying

$$\bar{\phi} = T\phi, \tag{2.4}$$

where $T$ is determined by

$$T_x + TU = \bar{U}T, \ \ T_t + TV = \bar{V}T. \tag{2.5}$$

It is easy to see that the Lax pair (2.2) and (2.3) are transformed to

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \qquad \bar{\phi}_t = \bar{V}\bar{\phi}. \tag{2.6}$$

Let

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}, \tag{2.7}$$

where $T_{ij} = t_{ij}^{(1)}\lambda + t_{ij}^{(0)}$, $t_{ij}^{(k)}(i,j=1,2,3;k=0,1)$ are functions of $x$ and $t$.

From (2.6), we can get

$$T = \begin{pmatrix} t_{11}^{(1)}\lambda + t_{11}^{(0)} & t_{12}^{(0)} & t_{13}^{(0)} \\ t_{21}^{(1)}\lambda & t_{22}^{(1)}\lambda + t_{22}^{(0)} & t_{23}^{(1)}\lambda \\ t_{31}^{(1)}\lambda & t_{32}^{(1)}\lambda & t_{33}^{(1)}\lambda + t_{33}^{(0)} \end{pmatrix}. \tag{2.8}$$

Let

$$\begin{aligned} &\varphi(\lambda_j) = (\varphi_1(\lambda_j), \varphi_2(\lambda_j), \varphi_3(\lambda_j))^T, \\ &\psi(\lambda_j) = (\psi_1(\lambda_j), \psi_2(\lambda_j), \psi_3(\lambda_j))^T, \\ &\chi(\lambda_j) = (\chi_1(\lambda_j), \chi_2(\lambda_j), \chi_3(\lambda_j))^T \end{aligned} \tag{2.9}$$

be three basic solutions of (2.2), from (2.4), we know that there exist constant $\gamma_j^{(1)}, \gamma_j^{(2)}$ satisfy

$$\begin{cases} T_{11}\varphi_1 + T_{12}\varphi_2 + T_{13}\varphi_3 + \gamma_j^{(1)}(T_{11}\psi_1 + T_{12}\psi_2 + T_{13}\psi_3) + \gamma_j^{(2)}(T_{11}\chi_1 + T_{12}\chi_2 + T_{13}\chi_3) = 0, \\ T_{21}\varphi_1 + T_{22}\varphi_2 + T_{23}\varphi_3 + \gamma_j^{(1)}(T_{21}\psi_1 + T_{22}\psi_2 + T_{23}\psi_3) + \gamma_j^{(2)}(T_{21}\chi_1 + T_{22}\chi_2 + T_{23}\chi_3) = 0, \\ T_{31}\varphi_1 + T_{32}\varphi_2 + T_{33}\varphi_3 + \gamma_j^{(1)}(T_{31}\psi_1 + T_{32}\psi_2 + T_{33}\psi_3) + \gamma_j^{(2)}(T_{31}\chi_1 + T_{32}\chi_2 + T_{33}\chi_3) = 0, \end{cases} \tag{2.10}$$

further, (2.10) can be written as a linear algebraic system

$$\begin{cases} T_{11} + \alpha_j^{(1)}T_{12} + \alpha_j^{(2)}T_{13} = 0, \\ T_{21} + \alpha_j^{(1)}T_{22} + \alpha_j^{(2)}T_{23} = 0, \\ T_{31} + \alpha_j^{(1)}T_{32} + \alpha_j^{(2)}T_{33} = 0, \end{cases} \tag{2.11}$$

where

$$\begin{cases} \alpha_j^{(1)} = \dfrac{\varphi_2(\lambda_j) + \gamma_j^{(1)}\psi_2(\lambda_j) + \gamma_j^{(2)}\chi_2(\lambda_j)}{\varphi_1(\lambda_j) + \gamma_j^{(1)}\psi_1(\lambda_j) + \gamma_j^{(2)}\chi_1(\lambda_j)}, \\ \alpha_j^{(2)} = \dfrac{\varphi_3(\lambda_j) + \gamma_j^{(1)}\psi_3(\lambda_j) + \gamma_j^{(2)}\chi_3(\lambda_j)}{\varphi_1(\lambda_j) + \gamma_j^{(1)}\psi_1(\lambda_j) + \gamma_j^{(2)}\chi_1(\lambda_j)}, \end{cases} \qquad j = 1,2,3. \tag{2.12}$$

Then we have

$$\det T(\lambda) = \mu(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3). \tag{2.13}$$

By using above fact, we can prove the following proposition:

**Proposition 1** The matrix $\bar{U}$ determined by (2.6) has the same form as $U$, that is

$$\bar{U} = \begin{pmatrix} 2\lambda & \bar{v}_1 & \bar{v}_2 \\ \lambda\bar{u}_1 & -\lambda & 0 \\ \lambda\bar{u}_2 & 0 & -\lambda \end{pmatrix}, \tag{2.14}$$

where the transformations between $u_1, u_2, v_1, v_2$ and $\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2$ are given by

$$\begin{aligned} \bar{u}_1 &= \frac{1}{t_{11}^{(1)}}(3t_{21}^{(1)} + u_1 t_{22}^{(1)} + u_2 t_{23}^{(1)}), \\ \bar{u}_2 &= \frac{1}{t_{11}^{(1)}}(3t_{31}^{(1)} + u_1 t_{32}^{(1)} + u_2 t_{33}^{(1)}), \\ \bar{v}_1 &= \frac{(v_1 t_{11}^{(1)} - 3t_{12}^{(0)})t_{33}^{(1)} - (v_2 t_{11}^{(1)} - 3t_{13}^{(0)})t_{32}^{(1)}}{t_{22}^{(1)}t_{33}^{(1)} - t_{23}^{(1)}t_{32}^{(1)}}, \\ \bar{v}_2 &= \frac{(v_1 t_{11}^{(1)} - 3t_{12}^{(0)})t_{23}^{(1)} - (v_2 t_{11}^{(1)} - 3t_{13}^{(0)})t_{22}^{(1)}}{t_{23}^{(1)}t_{32}^{(1)} - t_{22}^{(1)}t_{33}^{(1)}}. \end{aligned} \tag{2.15}$$

**Proof** Let $T^{-1} = T^*/\det T$ and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) & f_{13}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) & f_{23}(\lambda) \\ f_{31}(\lambda) & f_{32}(\lambda) & f_{33}(\lambda) \end{pmatrix}. \tag{2.16}$$

It is easy to see that $f_{sl}(\lambda)(s, l = 1, 2, 3)$ are third-order or fourth-order polynomial in $\lambda$. From (2.2) and (2.11), we find that

$$\begin{aligned} &\alpha_{jx}^{(1)} = u_1 - v_2(\alpha_j^{(1)})^2 - u_2\alpha_j^{(1)}\alpha_j^{(2)} + 3\lambda_j\alpha_j^{(1)}, \\ &\alpha_{jx}^{(2)} = v_1 - u_2(\alpha_j^{(2)})^2 - v_2\alpha_j^{(1)}\alpha_j^{(2)} + 3\lambda_j\alpha_j^{(2)}, \\ &T_{11} = -\alpha_j^{(1)}T_{12} - \alpha_j^{(2)}T_{13}, \qquad\qquad j = 1, 2, 3. \\ &T_{21} = -\alpha_j^{(1)}T_{22} - \alpha_j^{(2)}T_{23}, \\ &T_{31} = -\alpha_j^{(1)}T_{32} - \alpha_j^{(2)}T_{33}, \end{aligned} \tag{2.17}$$

By using (2.13) and (2.17), we can prove that all $\lambda_j (1 \le j \le 3)$ are roots of $f_{sl}(s, l = 1, 2, 3)$. Again noting (2.11), then we can conclude that

$$\det T \mid f_{sl}, \quad s, l = 1, 2, 3, \tag{2.18}$$

which together with (2.16) gives

$$(T_x + TU)T^* = (\det T)P(\lambda) \tag{2.19}$$

with

$$P(\lambda) = \begin{pmatrix} p_{11}^{(1)}\lambda + p_{11}^{(0)} & p_{12}^{(0)} & p_{13}^{(0)} \\ p_{21}^{(1)}\lambda + p_{21}^{(0)} & p_{22}^{(1)}\lambda + p_{22}^{(0)} & p_{23}^{(1)}\lambda + p_{23}^{(0)} \\ p_{31}^{(1)}\lambda + p_{31}^{(0)} & p_{32}^{(1)}\lambda + p_{32}^{(0)} & p_{33}^{(1)}\lambda + p_{33}^{(0)} \end{pmatrix},$$

where $p_{kj}^{(l)}(k,j=1,2,3;l=0,1.)$ are undetermined functions independent of $\lambda$. Now eq. (2.19) can be written in the form

$$T_x+TU=P(\lambda)T. \tag{2.20}$$

By comparing the coefficients of $\lambda^2$ in (2.20), we obtain

$$\begin{aligned}
&p_{11}^{(1)}=2,\quad p_{22}^{(1)}=p_{33}^{(1)}=-1,\quad p_{23}^{(1)}=p_{23}^{(1)}=0,\\
&p_{21}^{(1)}=\frac{1}{t_{11}^{(1)}}(3t_{21}^{(1)}+u_1t_{22}^{(1)}+u_2t_{23}^{(1)})=\bar{u}_1,\\
&p_{31}^{(1)}=\frac{1}{t_{11}^{(1)}}(3t_{31}^{(1)}+u_1t_{32}^{(1)}+u_2t_{33}^{(1)})=\bar{u}_2.
\end{aligned} \tag{2.21}$$

On the other hand, equating the coefficients $\lambda^1,\lambda^0$ in (2.21) leads to

$$\begin{aligned}
&p_{11}^{(0)}=p_{21}^{(0)}=p_{22}^{(0)}=p_{23}^{(0)}=p_{31}^{(0)}=p_{32}^{(0)}=p_{33}^{(0)}=0,\\
&p_{21}^{(1)}t_{11}^{(0)}=t_{21x}^{(1)}+u_1t_{22}^{(0)},\quad p_{21}^{(1)}t_{12}^{(0)}=t_{22x}^{(1)}+v_1t_{21}^{(1)},\\
&p_{21}^{(1)}t_{13}^{(0)}=t_{23x}^{(1)}+v_2t_{21}^{(1)},\quad p_{31}^{(1)}t_{11}^{(0)}=t_{31x}^{(1)}+u_2t_{33}^{(0)},\\
&p_{31}^{(1)}t_{12}^{(0)}=t_{32x}^{(1)}+v_1t_{31}^{(1)},\quad p_{31}^{(1)}t_{13}^{(0)}=t_{33x}^{(1)}+v_2t_{31}^{(1)},\\
&p_{12}^{(0)}t_{22}^{(0)}=t_{12x}^{(0)}+v_1t_{11}^{(0)},\quad p_{13}^{(0)}t_{33}^{(0)}=t_{13x}^{(0)}+v_2t_{11}^{(0)},\\
&p_{12}^{(0)}=\frac{(v_1t_{11}^{(1)}-3t_{12}^{(0)})t_{33}^{(1)}-(v_2t_{11}^{(1)}-3t_{13}^{(0)})t_{32}^{(1)}}{t_{22}^{(1)}t_{33}^{(1)}-t_{23}^{(1)}t_{32}^{(1)}}=\bar{v}_1,\\
&p_{13}^{(0)}=\frac{(v_1t_{11}^{(1)}-3t_{12}^{(0)})t_{23}^{(1)}-(v_2t_{11}^{(1)}-3t_{13}^{(0)})t_{22}^{(1)}}{t_{23}^{(1)}t_{32}^{(1)}-t_{22}^{(1)}t_{33}^{(1)}}=\bar{v}_2.
\end{aligned} \tag{2.22}$$

From (2.5) and (2.20), we see that $\bar{U}=P(\lambda)$. The proof is completed.

**Proposition 2** Under DT (2.4), the matrix $\bar{V}$ in (2.5) has the same form as V, that is

$$\bar{V}=\begin{pmatrix} \bar{V}_{11} & \bar{V}_{12} & \bar{V}_{13}\\ \bar{V}_{21} & \bar{V}_{22} & \bar{V}_{23}\\ \bar{V}_{31} & \bar{V}_{32} & \bar{V}_{33}\end{pmatrix} \tag{2.23}$$

in which

$$\begin{aligned}
&\bar{V}_{11}=-2\lambda^2+\frac{1}{3}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\lambda,\\
&\bar{V}_{12}=-\bar{v}_1\lambda-\frac{1}{3}\bar{v}_{1x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{v}_1,\\
&\bar{V}_{13}=-\bar{v}_2\lambda-\frac{1}{3}\bar{v}_{2x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{v}_2,\\
&\bar{V}_{21}=-\bar{u}_1\lambda^2+[\frac{1}{3}\bar{u}_{1x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{u}_1]\lambda,\\
&\bar{V}_{22}=\lambda^2-\frac{1}{3}\bar{u}_1\bar{v}_1\lambda,\quad \bar{V}_{23}=-\frac{1}{3}\bar{u}_1\bar{v}_2\lambda,\\
&\bar{V}_{31}=-\bar{u}_2\lambda^2+[\frac{1}{3}\bar{u}_{2x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{u}_2]\lambda,\\
&\bar{V}_{32}=-\frac{1}{3}\bar{u}_2\bar{v}_1\lambda,\quad \bar{V}_{33}=\lambda^2-\frac{1}{3}\bar{u}_2\bar{v}_2\lambda.
\end{aligned}$$

The old potentials $u_1, u_2, v_1$ and $v_2$ are mapped into new ones $\bar{u}_1, \bar{u}_2, \bar{v}_1$ and $\bar{v}_2$ according to the same DT (2.4) and (2.15).

**Proof** In a way similar to Proposition 1, we denote $T^{-1} = T^*/\det T$ and

$$(T_t + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) & g_{13}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) & g_{23}(\lambda) \\ g_{31}(\lambda) & g_{32}(\lambda) & g_{33}(\lambda) \end{pmatrix}. \tag{2.24}$$

Direct calculation shows that $g_{sl}(\lambda)(s, l = 1, 2, 3)$ are fourth-order or fifth-order polynomial in $\lambda$, respectively, with the help of (2.3) and (2.11), we find that

$$\left.\begin{aligned}
\alpha_{jt}^{(1)} &= -u_1\lambda_j^2 + [\frac{1}{3}u_{1x} + \frac{2}{9}(u_1v_1 + u_2v_2)u_1]\lambda_j + 3\lambda_j^2\alpha_j^{(1)} - \frac{1}{3}u_1v_2\lambda_j\alpha_j^{(2)} \\
&\quad -\frac{1}{3}(2u_1v_1 + u_2v_2)\lambda_j\alpha_j^{(1)} + [v_1\lambda_j + \frac{1}{3}v_{1x} - \frac{2}{9}(u_1v_1 + u_2v_2)v_1](\alpha_j^1)^2 \\
&\quad +[v_2\lambda_j + \frac{1}{3}v_{2x} - \frac{2}{9}(u_1v_1 + u_2v_2)v_2]\alpha_j^{(1)}\alpha_j^{(2)}, \\
\alpha_{jt}^{(2)} &= -u_2\lambda_j^2 + [\frac{1}{3}u_{2x} + \frac{2}{9}(u_1v_1 + u_2v_2)u_2]\lambda_j + 3\lambda_j^2\alpha_j^{(2)} - \frac{1}{3}u_2v_1\lambda_j\alpha_j^{(1)} \\
&\quad -\frac{1}{3}(u_1v_1 + 2u_2v_2)\lambda_j\alpha_j^{(2)} + [v_2\lambda_j + \frac{1}{3}v_{2x} - \frac{2}{9}(u_1v_1 + u_2v_2)v_2](\alpha_j^2)^2 \\
&\quad +[v_1\lambda_j + \frac{1}{3}v_{1x} - \frac{2}{9}(u_1v_1 + u_2v_2)v_1]\alpha_j^{(1)}\alpha_j^{(2)}, \\
T_{11t} &= -\alpha_{jt}^{(1)}T_{12} - \alpha_j^{(1)}T_{12t} - \alpha_{jt}^{(2)}T_{13} - \alpha_j^{(2)}T_{13t}, \\
T_{21t} &= -\alpha_{jt}^{(1)}T_{22} - \alpha_j^{(1)}T_{22t} - \alpha_{jt}^{(2)}T_{23} - \alpha_j^{(2)}T_{23t}, \\
T_{31t} &= -\alpha_{jt}^{(1)}T_{32} - \alpha_j^{(1)}T_{32t} - \alpha_{jt}^{(2)}T_{33} - \alpha_j^{(2)}T_{33t}.
\end{aligned}\right\} \quad j = 1, 2, 3. \tag{2.25}$$

We can verify by (2.13) and (2.25) that $\lambda_j(1 \le j \le 3)$ are also roots of $g_{sl}$ $(s, l = 1, 2, 3)$. Therefore, we have

$$\det T \mid g_{sl}, \quad s, l = 1, 2, 3, \tag{2.26}$$

and thus

$$(T_t + TV)T^* = (\det T)Q(\lambda) \tag{2.27}$$

with

$$Q(\lambda) = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix},$$

where

$$\begin{aligned}
&Q_{11} = q_{11}^{(2)}\lambda^2 + q_{11}^{(1)}\lambda + q_{11}^{(0)}, \quad Q_{12} = q_{12}^{(1)}\lambda + q_{12}^{(0)}, \quad Q_{13} = q_{13}^{(1)}\lambda + q_{13}^{(0)}, \\
&Q_{21} = q_{21}^{(2)}\lambda^2 + q_{21}^{(1)}\lambda + q_{21}^{(0)}, \quad Q_{22} = q_{22}^{(2)}\lambda^2 + q_{22}^{(1)}\lambda + q_{22}^{(0)}, \quad Q_{23} = q_{23}^{(2)}\lambda^2 + q_{23}^{(1)}\lambda + q_{23}^{(0)}, \\
&Q_{31} = q_{31}^{(2)}\lambda^2 + q_{31}^{(1)}\lambda + q_{31}^{(0)}, \quad Q_{32} = q_{32}^{(2)}\lambda^2 + q_{32}^{(1)}\lambda + q_{32}^{(0)}, \quad Q_{33} = q_{33}^{(2)}\lambda^2 + q_{33}^{(1)}\lambda + q_{33}^{(0)},
\end{aligned}$$

that is

$$T_t + TV = Q(\lambda)T. \tag{2.28}$$

Comparing the coefficients of $\lambda^3$ in (2.28), leads to

$$\begin{aligned}
&q_{11}^{(2)}=-2,\ \ q_{22}^{(2)}=q_{33}^{(2)}=0,\ \ q_{23}^{(2)}=q_{32}^{(2)}=0,\\
&q_{21}^{(2)}=-\frac{1}{t_{11}^{(1)}}(3t_{21}^{(1)}+u_1t_{22}^{(1)}+u_2t_{23}^{(1)})=-\bar{u}_1,\\
&q_{31}^{(2)}=-\frac{1}{t_{11}^{(1)}}(3t_{31}^{(1)}+u_1t_{32}^{(1)}+u_2t_{33}^{(1)})=-\bar{u}_2.
\end{aligned}\tag{2.29}$$

On the other hand, equating the coefficients $\lambda^2, \lambda^1, \lambda^0$ in (2.27) and (2.11), we can obtain

$$\begin{aligned}
q_{11}^{(1)}&=\frac{1}{3}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2),\\
q_{12}^{(1)}&=-\frac{(v_1t_{11}^{(1)}-3t_{12}^{(0)})t_{33}^{(1)}-(v_2t_{11}^{(1)}-3t_{13}^{(0)})t_{32}^{(1)}}{t_{22}^{(1)}t_{33}^{(1)}-t_{23}^{(1)}t_{32}^{(1)}}=-\bar{v}_1,\\
q_{13}^{(1)}&=-\frac{(v_1t_{11}^{(1)}-3t_{12}^{(0)})t_{23}^{(1)}-(v_2t_{11}^{(1)}-3t_{13}^{(0)})t_{22}^{(1)}}{t_{23}^{(1)}t_{32}^{(1)}-t_{22}^{(1)}t_{33}^{(1)}}=-\bar{v}_2,\\
q_{21}^{(1)}&=\frac{1}{3}\bar{u}_{1x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{u}_1,\ \ q_{22}^{(1)}=-\frac{1}{3}\bar{u}_1\bar{v}_1,\ \ q_{23}^{(1)}=-\frac{1}{3}\bar{u}_1\bar{v}_2,\\
q_{31}^{(1)}&=\frac{1}{3}\bar{u}_{2x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{u}_2,\ \ q_{32}^{(1)}=-\frac{1}{3}\bar{u}_2\bar{v}_1,\ \ q_{33}^{(1)}=-\frac{1}{3}\bar{u}_2\bar{v}_2,\\
q_{11}^{(0)}&=q_{21}^{(0)}=q_{22}^{(0)}=q_{23}^{(0)}=q_{31}^{(0)}=q_{32}^{(0)}=q_{33}^{(0)}=0,\\
q_{12}^{(0)}&=-\frac{1}{3}\bar{v}_{1x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{v}_1,\ \ q_{13}^{(0)}=-\frac{1}{3}\bar{v}_{2x}+\frac{2}{9}(\bar{u}_1\bar{v}_1+\bar{u}_2\bar{v}_2)\bar{v}_2.
\end{aligned}\tag{2.30}$$

Then the proof is completed.

According to Propositions 1 and 2, transformation (2.4) and (2.15) transform the Lax pairs (2.2) and (2.3) into another Lax pairs of the same type (2.6). Therefore both of the Lax pairs lead to the same eq. (2.1) Then we call the transformation $(u_1, u_2, v_1, v_2,) \longrightarrow (\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2,)$ a DT of eq. (2.1).We get the following assertion.

**Theorem** The solution $(u_1, u_2, v_1, v_2,)$ of the Derivative manakov equation are mapped into their new solutions $(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2,)$ under the DT (2.4) and (2.15), where $t_{ij}^k(i, j = 1, 2, 3, k = 0, 1.)$ are given by the linear algebraic system (2.11).

## 3 The Soliton Solutions

In this section, we apply the DT of the Derivative manakov equation and give its soliton solution.

Substituting $u_1 = u_2 = v_1 = v_2 = 0$ into the Lax pairs (2.2) and (2.3), then get three basic solutions

$$\varphi(\lambda_j)=\begin{pmatrix} e^{2\lambda_jx-2\lambda_j^2t}\\ 0\\ 0\end{pmatrix},\ \psi(\lambda_j)=\begin{pmatrix} 0\\ e^{-\lambda_jx+\lambda_j^2t}\\ 0\end{pmatrix},\ \chi(\lambda_j)=\begin{pmatrix} 0\\ 0\\ e^{-\lambda_jx+\lambda_j^2t}\end{pmatrix}\tag{3.1}$$

with

$$\begin{aligned}
\alpha_j^{(1)}&=\frac{\gamma_j^{(1)}e^{\lambda_jx-\lambda_j{}^2t}}{e^{2\lambda_jx-2\lambda_j^2t}}=e^{-3\lambda_jx+3\lambda_j^2t+\beta_j^{(1)}},\\
\alpha_j^{(2)}&=\frac{\gamma_j^{(2)}e^{\lambda_jx-\lambda_j{}^2t}}{e^{2\lambda_jx-2\lambda_j^2t}}=e^{-3\lambda_jx+3\lambda_j^2t+\beta_j^{(2)}},
\end{aligned}\tag{3.2}$$

where

$$\gamma_j^{(1)} = e^{\beta_j^{(1)}}, \gamma_j^{(2)} = e^{\beta_j^{(2)}}, j = 1, 2, 3.$$

By the linear algebraic system (2.11), using Cramer Rule to solve to get

$$\begin{aligned} t_{12}^{(0)} &= \frac{\Delta_{12}}{\Delta_1}, & t_{13}^{(0)} &= \frac{\Delta_{13}}{\Delta_1}, & t_{11}^{(1)} &= \frac{\Delta_{11}}{\Delta_1}, \\ t_{21}^{(1)} &= \frac{\Delta_{21}}{\Delta_2}, & t_{22}^{(1)} &= \frac{\Delta_{22}}{\Delta_2}, & t_{23}^{(1)} &= \frac{\Delta_{23}}{\Delta_2}, \\ t_{31}^{(1)} &= \frac{\Delta_{31}}{\Delta_2}, & t_{32}^{(1)} &= \frac{\Delta_{32}}{\Delta_2}, & t_{33}^{(1)} &= \frac{\Delta_{33}}{\Delta_2}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
&\Delta_1 = K_1\lambda_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + K_2\lambda_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + K_3\lambda_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t},\\
&\Delta_{11} = -t_{11}^{(0)}[K_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + K_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + K_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t}],\\
&\Delta_{12} = -t_{11}^{(0)}[\gamma_1 e^{-3\lambda_1 x+3\lambda_1^2 t} + \gamma_2 e^{-3\lambda_2 x+3\lambda_2^2 t} + \gamma_3 e^{-3\lambda_3 x+3\lambda_3^2 t}],\\
&\Delta_{13} = -t_{11}^{(0)}[\sigma_1 e^{-3\lambda_1 x+3\lambda_1^2 t} + \sigma_2 e^{-3\lambda_2 x+3\lambda_2^2 t} + \sigma_3 e^{-3\lambda_3 x+3\lambda_3^2 t}],\\
&\Delta_2 = \lambda_1\lambda_2\lambda_3[K_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + K_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + K_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t}],\\
&\Delta_{21} = -t_{22}^{(0)} e^{-3(\lambda_1+\lambda_2+\lambda_3)x+3(\lambda_1^2+\lambda_2^2+\lambda_3^2)t}[K_1\lambda_2\lambda_3 e^{\beta_1^{(1)}} + K_2\lambda_1\lambda_3 e^{\beta_2^{(1)}} + K_3\lambda_1\lambda_2 e^{\beta_3^{(1)}}],\\
&\Delta_{22} = -t_{22}^{(0)}[\eta_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + \eta_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + \eta_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t}],\\
&\Delta_{23} = -t_{22}^{(0)}[\xi_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + \xi_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + \xi_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t}],\\
&\Delta_{31} = -t_{33}^{(0)} e^{-3(\lambda_1+\lambda_2+\lambda_3)x+3(\lambda_1^2+\lambda_2^2+\lambda_3^2)t}[K_1\lambda_2\lambda_3 e^{\beta_1^{(2)}} + K_2\lambda_1\lambda_3 e^{\beta_2^{(2)}} + K_3\lambda_1\lambda_2 e^{\beta_3^{(2)}}],\\
&\Delta_{32} = -t_{33}^{(0)}[\rho_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + \rho_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + \rho_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t}],\\
&\Delta_{33} = -t_{33}^{(0)}[\tau_1 e^{-3(\lambda_2+\lambda_3)x+3(\lambda_2^2+\lambda_3^2)t} + \tau_2 e^{-3(\lambda_1+\lambda_3)x+3(\lambda_1^2+\lambda_3^2)t} + \tau_3 e^{-3(\lambda_1+\lambda_2)x+3(\lambda_1^2+\lambda_2^2)t}],\\
&K_1 = e^{\beta_2^{(1)}+\beta_3^{(2)}} - e^{\beta_3^{(1)}+\beta_2^{(2)}}, \quad K_2 = e^{\beta_3^{(1)}+\beta_1^{(2)}} - e^{\beta_1^{(1)}+\beta_3^{(2)}}, \quad K_3 = e^{\beta_1^{(1)}+\beta_2^{(2)}} - e^{\beta_2^{(1)}+\beta_1^{(2)}};\\
&\gamma_1 = (\lambda_2-\lambda_3)e^{\beta_1^{(2)}}, \quad \gamma_2 = (\lambda_3-\lambda_1)e^{\beta_2^{(2)}}, \quad \gamma_3 = (\lambda_1-\lambda_2)e^{\beta_3^{(2)}};\\
&\sigma_1 = (\lambda_3-\lambda_2)e^{\beta_1^{(1)}}, \quad \sigma_2 = (\lambda_1-\lambda_3)e^{\beta_2^{(1)}}, \quad \sigma_3 = (\lambda_2-\lambda_1)e^{\beta_3^{(1)}};\\
&\eta_1 = \lambda_1(\lambda_3 e^{\beta_2^{(1)}+\beta_3^{(2)}} - \lambda_2 e^{\beta_3^{(1)}+\beta_2^{(2)}}), \quad \eta_2 = \lambda_2(\lambda_1 e^{\beta_3^{(1)}+\beta_1^{(2)}} - \lambda_3 e^{\beta_1^{(1)}+\beta_3^{(2)}}),\\
&\eta_3 = \lambda_3(\lambda_2 e^{\beta_1^{(1)}+\beta_2^{(2)}} - \lambda_1 e^{\beta_2^{(1)}+\beta_1^{(2)}}); \quad \tau_1 = \lambda_1(\lambda_2 e^{\beta_2^{(1)}+\beta_3^{(2)}} - \lambda_3 e^{\beta_3^{(1)}+\beta_2^{(2)}}),\\
&\tau_2 = \lambda_2(\lambda_3 e^{\beta_3^{(1)}+\beta_1^{(2)}} - \lambda_1 e^{\beta_1^{(1)}+\beta_3^{(2)}}), \quad \tau_3 = \lambda_3(\lambda_1 e^{\beta_1^{(1)}+\beta_2^{(2)}} - \lambda_2 e^{\beta_2^{(1)}+\beta_1^{(2)}});\\
&\xi_1 = \lambda_1(\lambda_2-\lambda_3)e^{\beta_2^{(1)}+\beta_3^{(1)}}, \quad \xi_2 = \lambda_2(\lambda_3-\lambda_1)e^{\beta_1^{(1)}+\beta_3^{(1)}}, \quad \xi_3 = \lambda_3(\lambda_1-\lambda_2)e^{\beta_1^{(1)}+\beta_2^{(1)}};\\
&\rho_1 = \lambda_1(\lambda_3-\lambda_2)e^{\beta_2^{(2)}+\beta_3^{(2)}}, \quad \rho_2 = \lambda_2(\lambda_1-\lambda_3)e^{\beta_1^{(2)}+\beta_3^{(2)}}, \quad \rho_3 = \lambda_3(\lambda_2-\lambda_1)e^{\beta_1^{(2)}+\beta_2^{(2)}}.
\end{aligned}$$

Thus, we use Darboux transformation (2.15), from a trivial solution of eq. (2.1) to get

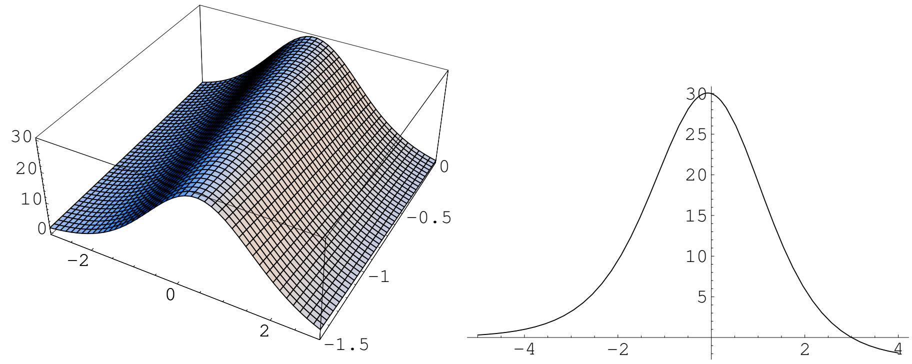


Fig. 1: Soliton solution $\bar{u}_1[1]$ with $\lambda_1 = 0.01, \lambda_2 = 0.5, \lambda_3 = -0.4, \beta_1^{(1)} = 1, \beta_2^{(1)} = 3, t = 0, \beta_3^{(1)} = -1, \beta_1^{(2)} = 1, \beta_2^{(2)} = 1, \beta_3^{(2)} = -1, t_{11}^{(0)} = 1, t_{22}^{(0)} = -1, t_{33}^{(0)} = 1.5$

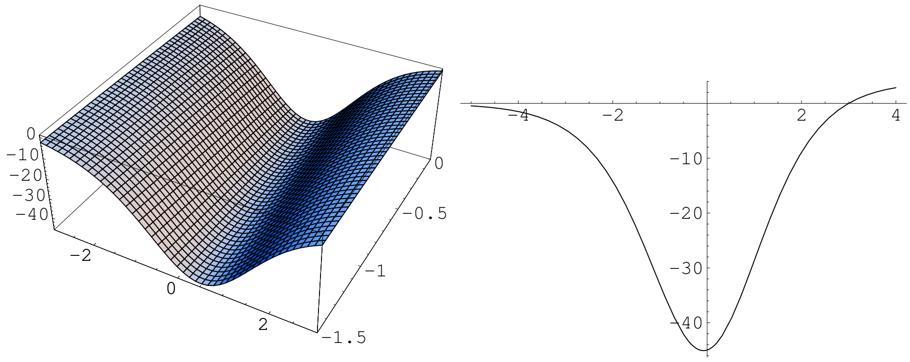


Fig. 2: Soliton solution $\bar{u}_2[1]$ with $\lambda_1 = 0.01, \lambda_2 = 0.5, \lambda_3 = -0.4, \beta_1^{(1)} = 1, \beta_2^{(1)} = 3, t = 0, \beta_3^{(1)} = -1, \beta_1^{(2)} = 1, \beta_2^{(2)} = 1, \beta_3^{(2)} = -1, t_{11}^{(0)} = 1, t_{22}^{(0)} = -1, t_{33}^{(0)} = 1.5$

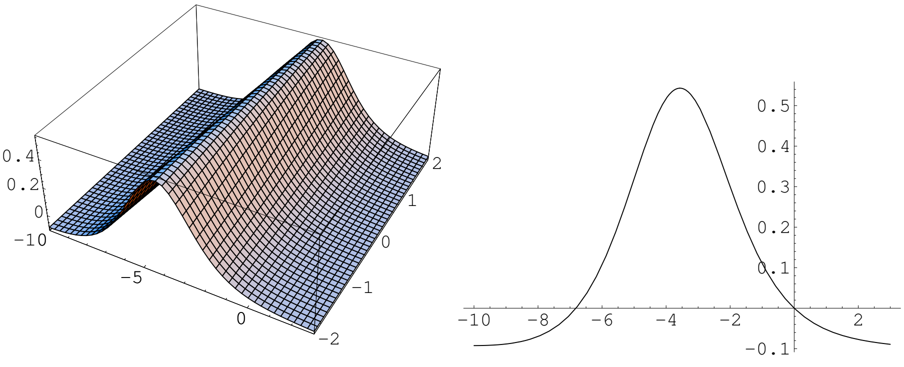


Fig. 3: Soliton solution $\bar{v}_1[1]$ with $\lambda_1 = 0.02, \lambda_2 = 0.3, \lambda_3 = -0.00, \beta_1^{(1)} = 1, \beta_2^{(1)} = 0, t = 0, \beta_3^{(1)} = 0, \beta_1^{(2)} = 1, \beta_2^{(2)} = 1, \beta_3^{(2)} = 1, t_{11}^{(0)} = 1, t_{22}^{(0)} = 1, t_{33}^{(0)} = 1$

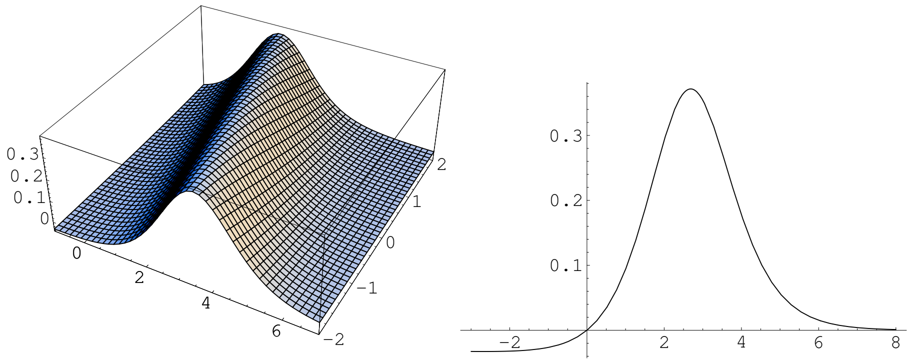


Fig. 4: Soliton solution $\bar{v}_2[1]$ with $\lambda_1 = 0.5, \lambda_2 = 0.012, \lambda_3 = -0.45, \beta_1^{(1)} = 1, \beta_2^{(1)} = 0, t = 0, \beta_3^{(1)} = 0, \beta_1^{(2)} = 1, \beta_2^{(2)} = 1, \beta_3^{(2)} = 1, t_{11}^{(0)} = 1, t_{22}^{(0)} = 1, t_{33}^{(0)} = 1$

a non-trivial solution of eq. (2.1)

$$\begin{cases} \bar{u}_1[1] = \dfrac{3t_{21}^{(1)}}{t_{11}^{(1)}} = \dfrac{3\Delta_1\Delta_{21}}{\Delta_2\Delta_{11}}, \\ \bar{u}_2[1] = \dfrac{3t_{31}^{(1)}}{t_{11}^{(1)}} = \dfrac{3\Delta_1\Delta_{31}}{\Delta_2\Delta_{11}}, \\ \bar{v}_1[1] = \dfrac{3(t_{13}^{(0)}t_{32}^{(1)} - t_{12}^{(0)}t_{33}^{(1)})}{t_{22}^{(1)}t_{33}^{(1)} - t_{23}^{(1)}t_{32}^{(1)}} = \dfrac{3\Delta_2(\Delta_{13}\Delta_{32} - \Delta_{12}\Delta_{33})}{\Delta_1(\Delta_{22}\Delta_{33} - \Delta_{23}\Delta_{32})}, \\ \bar{v}_2[1] = \dfrac{3(t_{13}^{(0)}t_{22}^{(1)} - t_{12}^{(0)}t_{23}^{(1)})}{t_{23}^{(1)}t_{32}^{(1)} - t_{22}^{(1)}t_{33}^{(1)}} = \dfrac{3\Delta_2(\Delta_{12}\Delta_{23} - \Delta_{13}\Delta_{22})}{\Delta_1(\Delta_{22}\Delta_{33} - \Delta_{23}\Delta_{32})}. \end{cases} \tag{3.4}$$

When parameters is suitable chosen, we can obtain the plots of $\bar{u}_1[1], \bar{u}_2[1], \bar{v}_1[1], \bar{v}_2[1]$ (see Figs. 1, 2, 3, 4).

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# 导数Manakov方程的Darboux变换及其精确解

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**摘 要**: 本文给出了导数Manakov方程新的Darboux变换. 利用此Darboux变换得到了导数Manakov方程的精确解. 最后, 通过选择适当的参数, 作出了孤子解的图形.

**关 键 词**: 谱问题; 达布变换; 达布阵; 精确解

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