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AN EXTENSION OF THE ALMOST SURE LIMIT THEOREM FOR THE MAXIMA OF SMOOTH STATIONARY GAUSSIAN PROCESS

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Abstract: In this paper, we study the limit behavior for the maxima of continuous mean square differentiable stationary Gaussian process. Using a different weight function from that in Tan (2013), we obtain an almost sure limit theorem for the maxima of continuous mean square differentiable stationary Gaussian process under some mild conditions, which expands the corresponding results in Tan (2013).

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1 Introduction

The class of stationary Gaussian processes is one of the most widely used families of stochastic processes for modeling the problems in many branches of natural and social sciences. The asymptotic properties of stationary Gaussian process recently received increasing attention. The limit behavior of stationary Gaussian sequences was well established, see Csáki and Gonchigdanzan [1] and Dudziński [2]. Kratz and Rootzen [3] studied the convergence for extremes of mean square differentiable stationary Gaussian processes and given the bounds for the convergent rate of the distribution of the maximum. Piterbarg [4] studied the joint distribution of maxima of a stationary Gaussian process on a continuous time and uniform discrete time points, proved them are asymptotically complete dependent and asymptotically independent under approximate restrictions. Tan and Hashorva [5] extended this result. Tan [6] obtained an almost sure limit theorem (ASLT) for the maxima of stationary Gaussian processes under some mild conditions.

The ASLT was first introduced independently by Brosamler [7] and Schatte [8] for partial sum. Lacey and Philipp [9] proved the ASLT for partial sum used a different method from Brosamler [7] and Schatte [8]. Zhang [10] obtained an ASLT for the maximum of Gaussian

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sequence under some conditions related to correlation. Fahrner and Stadtmüller [11] and Cheng et al. [12] independently proved the ALST under some certain conditions for the maxima of independent and identically distributed random variable sequences. Furthermore, Zhang [13] studied the ASLT for the maxima of independent random sequence.

Let $\{X(t), t \geq 0\}$ be a continuous mean square differentiable stationary Gaussian process with covariance function $r(t) \hat{=} \mathrm{E}X(s)X(t+s)$ satisfying the following condition

$$r(t) = 1 - \frac{\lambda}{2}|t|^2 + o(|t|^2), t \to 0, \tag{1.1}$$

where $\lambda = -r''(0)$. Next, set $M(T) = \max\{X(t), 0 \leq t \leq T\}$ and let $N_u(T)$ be the number of upcrossings of the level u by $\{X(t), 0 \leq t \leq T\}$, so that by Rice's formula (see, Lindgren and Leadbetter [14])

$$\mu = \mu(u) = \mathrm{E}N_u(1) = \frac{1}{2\pi}\lambda^{1/2}\mathrm{e}^{-u^2/2}, \tag{1.2}$$

when $\mathrm{E}N_u(T) = T\mu(u_T) \to \tau$ for some constant $\tau > 0$, then

$$\mathrm{P}\left(M(T) \leq u\right) \to \mathrm{e}^{-\tau},\ T \to \infty$$

and

$$\mathrm{P}\left(a_T\left(M(T) - b_T\right) \leq x\right) \to \exp(-\mathrm{e}^{-x}),\ T \to \infty. \tag{1.3}$$

Here the normalizing constants are defined for all large T by

$$a_T = \sqrt{2\ln T}, \quad b_T = a_T + a_T^{-1}\ln\left(\frac{\lambda^{1/2}}{2\pi}\right). \tag{1.4}$$

Tan [6] obtained the ASLT for the maximum $M(T)$ of the continuous mean square differentiable stationary Gaussian process $\{X(t), t \geq 0\}$ with weight function $1/t$, which is as follow:

Theorem 1.1 Let $\{X(t), t \geq 0\}$ be a continuous mean square differentiable stationary Gaussian process with covariance function $r(\cdot)$ satisfying (1.1) and

$$r''(t) - r''(0) \leq ct^2, t \geq 0 \tag{1.5}$$

for some constant $c > 0$ and

$$r(t)(\ln t)(\ln\ln t)^{3(1+\varepsilon)} = O(1)$$

for some constant $\varepsilon > 0$. Then

(i) if $T\mu(u_T) \to \tau$ for $0 < \tau < \infty$,

$$\lim_{T\to\infty}\frac{1}{\ln T}\int_1^T \frac{1}{t} I\left(\max_{1\leq s\leq t} X(s) \leq u_t\right)\mathrm{d}t = \mathrm{e}^{-\tau} \quad \text{a.s..}$$

(ii) if a_T, b_T are defined as in (1.4),

$$\lim_{T\to\infty}\frac{1}{\ln T}\int_1^T \frac{1}{t} I\left(a_T\left(\max_{1\leq s\leq t} X(s) - b_T\right) \leq x\right)\mathrm{d}t = \exp\left(-\mathrm{e}^{-x}\right) \quad \text{a.s..}$$

This result is a continuous version of the ASLT for the maximum of stationary Gaussian sequences in [1].

In this paper, we try to expand the ASLT for the maxima of continuous mean square differentiable stationary Gaussian process $\{X(t), t \geq 0\}$ by a different weight function from that in Tan [6]. The rest of the paper is organized as follows. The main result is listed in Section 2. Some preliminary lemmas and the proof of the main result is given in Section 3. The proofs of Lemma 3.1 and Lemma 3.2 are collected in Appendix.

2 Main Result

Theorem 2.1 Let $\{X(t), t \geq 0\}$ be a continuous mean square differentiable stationary Gaussian process with covariance function $r(\cdot)$ satisfying (1.1), (1.5) and

$$r(t)(\ln t)^{3\beta+\varepsilon} = O(1). \tag{2.1}$$

Suppose $0 < \beta < \frac{1}{2}$ and set

$$w_t = \frac{\exp\left((\ln t)^{\beta}\right)}{t}, \quad W_T = \int_1^T w_t \mathrm{d}t. \tag{2.2}$$

(i) If $T\mu(u_T) \to \tau$ for $0 < \tau < \infty$, then

$$\lim_{T\to\infty} \frac{1}{W_T} \int_1^T w_t I\left(\max_{1\leq s\leq t} X(s) \leq u_t\right) \mathrm{d}t = \mathrm{e}^{-\tau} \quad \text{a.s..} \tag{2.3}$$

(ii) If a_T, b_T are defined as in (1.4), then

$$\lim_{T\to\infty} \frac{1}{W_T} \int_1^T w_t I\left(a_T\left(\max_{1\leq s\leq t} X(s) - b_T\right) \leq x\right) \mathrm{d}t = \exp\left(-\mathrm{e}^{-x}\right) \quad \text{a.s..} \tag{2.4}$$

Remark 2.1 Theorem 2.1 remains valid if we replace the function of weight w_t by w_t^* such that $0 \leq w_t^* \leq w_t$, $\displaystyle\int_1^\infty w_t^* \mathrm{d}t = \infty$.

Remark 2.2 The lower limit of integral in (2.3), (2.4) and Remark 2.1 can be replaced by any positive constant.

3 Proof

The following lemmas will be useful in the proof of Theorem 2.1.

Lemma 3.1 Let $\{\xi(t), t \geq 0\}$ be a real-valued random process with continuous and bounded sample paths, if w_t, W_T are defined as in (2.2), and

$$\mathrm{Var}\left(\int_1^T w_t \xi(t) \mathrm{d}t\right) \ll (W_T)^2 (\ln W_T)^{-(1+\varepsilon)} \tag{3.1}$$

for some $\varepsilon > 0$, here $f(T) \ll g(T)$ denotes that there exists a constant $c > 0$ such that $f(T) \leq cg(T)$ for sufficiently large T. The symbol c stands for a generic positive constant which may differ from one place to another. Then, we have

$$\lim_{T\to\infty}\frac{1}{W_T}\int_1^T w_t\left(\xi(t)-\mathrm{E}\xi(t)\right)\mathrm{d}t=0\quad \text{a.s.}. \tag{3.2}$$

Proof The proof can refer to Appendix.

Lemma 3.2 Suppose $\{X(t), t\geq 0\}$ is a continuous mean square differentiable stationary Gaussian process with covariance function $r(\cdot)$ satisfying conditions (1.1), (1.5) and (2.1). Let $q=u_t^{-1}(\ln t)^{-\beta(1+\varepsilon)}$, we have

$$\sup_{s\in(0,t)}\frac{s}{q}\sum_{\delta\leq iq\leq t}|r(iq)|\exp\left(-\frac{u_s^2+u_t^2}{2\left(1+|r(iq)|\right)}\right)\ll(\ln t)^{-\beta(1+\varepsilon)}$$

for some constant $\delta>0$.

Proof The proof can refer to Appendix.

Lemma 3.3 (see Tan [6]) Let $\{X(t), t\geq 0\}$ be a stationary Gaussian process with covariance function $r(\cdot)$ satisfying the conditions (1.1), and $T\mu(u_T)\to\tau, 0<\tau<\infty$. For large enough s and t, $s<t$, we have

$$\mathrm{E}\left|I\left(M\left([1,t]\right)\leq u_t\right)-I\left(M\left([s,t]\right)\leq u_t\right)\right|\ll\frac{s}{t}.$$

Lemma 3.4 Let $\{X(t), t\geq 0\}$ be a stationary Gaussian process with covariance function $r(\cdot)$ satisfying (1.1), (1.5), (2.1) and $T\mu(u_T)\to\tau$, for $0<\tau<\infty$. Set $q=u_t^{-1}(\ln t)^{-\beta(1+\varepsilon)}$. For large enough s and t, $s<t$, we have

$$\left|\mathrm{Cov}\left(I\left(M\left([1,s]\right)\right)\leq u_s, I\left(M\left([s,t]\right)\right)\leq u_t\right)\right|\ll s^{-1}(\ln s)^{-1/2}+(\ln t)^{-\beta(1+\varepsilon)}.$$

Proof Using Lemma 3.2 and Lemma 3.3, the proof of Lemma 3.4 is similar to that of Lemma 3.5 of Tan [6].

Proof of Theorem 2.1 **Case (i)** Let

$$\eta(t)=I\left(M\left([1,t]\right)\leq u_t\right)-\mathrm{P}\left(M\left([1,t]\right)\leq u_t\right).$$

Notice that $\eta(t)$ is a real-valued random process with continuous and bounded sample paths and $\mathrm{Var}\left(\eta(t)\right)<1$. First, we estimate $\mathrm{Var}\left(\int_1^T w_t\eta(t)\mathrm{d}t\right)$. Clearly

$$\mathrm{Var}\left(\int_1^T w_t\eta(t)\mathrm{d}t\right)\leq\mathrm{E}\left(\int_1^T w_t\eta(t)\mathrm{d}t\right)^2=2\iint_{1\leq s<\leq T}w_sw_t\mathrm{E}\left(\eta(s)\eta(t)\right)\mathrm{d}t\mathrm{d}s.$$

Note that by Lemmas 3.3 and 3.4, for $s<t$, we have

$$\begin{aligned}
&\left|\mathrm{E}\left(\eta(s)\eta(t)\right)\right|\\
=\;&\left|\mathrm{Cov}\left(\left(I\left(M\left([1,s]\right)\leq u_s\right)\right), I\left(M\left([1,t]\right)\leq u_t\right)\right)\right|\\
\leq\;&\left|\mathrm{Cov}\left(\left(I\left(M\left([1,s]\right)\leq u_s\right)\right),\left[I\left(M\left([1,t]\right)\leq u_t\right)-I\left(M\left([s,t]\right)\leq u_t\right)\right]\right)\right|\\
&+\left|\mathrm{Cov}\left(\left(I\left(M\left([1,s]\right)\leq u_s\right)\right), I\left(M\left([s,t]\right)\leq u_t\right)\right)\right|\\
\ll\;&\mathrm{E}\left|I\left(M\left([1,t]\right)\leq u_t\right)-I\left(M\left([s,t]\right)\leq u_t\right)\right|\\
&+\left|\mathrm{Cov}\left(\left(I\left(M\left([1,s]\right)\leq u_s\right)\right), I\left(M\left([s,t]\right)\leq u_t\right)\right)\right|\\
\ll\;&\tfrac{s}{t}+s^{-1}(\ln s)^{-1/2}+(\ln t)^{-\beta(1+\varepsilon)}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\mathrm{Var}\left(\int_1^T w_t\eta(t)\mathrm{d}t\right)\\
\ll\ & \iint_{1\le s<t\le T} w_s w_t \frac{s}{t}\mathrm{d}t\mathrm{d}s+\iint_{1\le s<t\le T}\frac{\mathrm{e}^{\ln^\beta s}\mathrm{e}^{\ln^\beta t}}{s^2(\ln s)^{1/2}t}\mathrm{d}t\mathrm{d}s+\iint_{1\le s<t\le T}\frac{\mathrm{e}^{\ln^\beta s}\mathrm{e}^{\ln^\beta t}}{st(\ln t)^{\beta(1+\varepsilon)}}\mathrm{d}t\mathrm{d}s\\
\hat{=}\ & S_{T,1}+S_{T,2}+S_{T,3}.
\end{aligned}\tag{3.3}$$

For the second and the first terms, we have

$$\begin{aligned}
S_{T,2} &= \iint_{1\le s<t\le T}\frac{\mathrm{e}^{\ln^\beta s}\mathrm{e}^{\ln^\beta t}}{s^2(\ln s)^{1/2}t}\mathrm{d}t\mathrm{d}s=\int_1^T\frac{\mathrm{e}^{\ln^\beta t}}{t}\left(\int_1^T\frac{\mathrm{e}^{\ln^\beta s}}{s^2(\ln s)^{1/2}}\mathrm{d}s\right)\mathrm{d}t\\
&\ll \int_1^T\frac{\mathrm{e}^{\ln^\beta t}}{t}\mathrm{d}t\ll W_T\ll\frac{W_T^2}{(\ln W_T)^{1+\varepsilon}}
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
S_{T,1} &= \iint_{1\le s<t\le T} w_s w_t\frac{s}{t}\mathrm{d}t\mathrm{d}s\\
&\le \iint_{1\le s<t\le T,\frac{s}{t}\le(\ln W_T)^{-2}} w_s w_t\frac{s}{t}\mathrm{d}t\mathrm{d}s+\iint_{1\le s<t\le T,\frac{s}{t}>(\ln W_T)^{-2}} w_s w_t\mathrm{d}t\mathrm{d}s\\
&\hat{=} S_{T,1}^{(1)}+S_{T,1}^{(2)},
\end{aligned}$$

here

$$\begin{aligned}
S_{T,1}^{(1)} &= \iint_{1\le s<t\le T,\frac{s}{t}\le(\ln W_T)^{-2}} w_s w_t\frac{s}{t}\mathrm{d}t\mathrm{d}s\\
&\ll \iint_{1\le s<t\le T,\le(\ln W_T)^{-2}} w_s w_t\frac{1}{(\ln W_T)^2}\mathrm{d}t\mathrm{d}s\\
&\ll \frac{W_T^2}{(\ln W_T)^2}\ll\frac{W_T^2}{(\ln W_T)^{1+\varepsilon}}.
\end{aligned}$$

By Wu and Chen [15], we gain the elementary calculation

$$W_T\sim\frac{1}{\beta}(\ln T)^{1-\beta}\exp(\ln T)^\beta,\ \ \ln W_T\sim(\ln T)^\beta,\ \ \ln\ln W_T\sim\ln\ln T.\tag{3.5}$$

From $0<\beta<\frac{1}{2}$, we know that $\frac{1-2\beta}{2\beta}>0$. Set $\varepsilon\hat{=}\frac{1-2\beta}{2\beta}$, then $\frac{1}{2\beta}=1+\varepsilon$. Thus

$$\begin{aligned}
S_{T,1}^{(2)} &= \iint_{1\le s<t\le T,\frac{s}{t}>(\ln W_T)^{-2}} w_s w_t\mathrm{d}t\mathrm{d}s\ll\int_1^T w_s\int_s^{s(\ln W_T)^2}\frac{\mathrm{e}^{\ln^\beta T}}{t}\mathrm{d}t\mathrm{d}s\\
&\ll \frac{W_T}{(\ln W_T)^{\frac{1-\beta}{\beta}}}\int_1^T w_s\ln\ln W_T\mathrm{d}s=\frac{W_T^2\ln\ln W_T}{(\ln W_T)^{\frac{1-\beta}{\beta}}}=\frac{W_T^2}{(\ln W_T)^{\frac{1}{2\beta}}}\cdot\frac{\ln\ln W_T}{(\ln W_T)^{\frac{1-2\beta}{2\beta}}}\\
&\ll \frac{W_T^2}{(\ln W_T)^{\frac{1}{2\beta}}}=\frac{W_T^2}{(\ln W_T)^{1+\varepsilon}}.
\end{aligned}$$

So we obtain

$$S_{T,1}=\iint_{1\le s<t\le T} w_s w_t\frac{s}{t}\mathrm{d}t\mathrm{d}s\ll W_T^2(\ln W_T)^{-(1+\varepsilon)}.\tag{3.6}$$

It remains only to estimate the term $S_{T,3}$ in (3.3). Using (3.5), we get

$$\begin{aligned}
S_{T,3} &= \iint_{1\le s<t\le T} \frac{\mathrm{e}^{\ln^\beta s}\mathrm{e}^{\ln^\beta t}}{st\,(\ln t)^{\beta(1+\varepsilon)}}\mathrm{d}t\mathrm{d}s = \int_1^T \frac{\mathrm{e}^{\ln^\beta t}}{t\,(\ln t)^{\beta(1+\varepsilon)}}\int_1^t \frac{\mathrm{e}^{\ln^\beta s}}{s}\mathrm{d}s\mathrm{d}t \\
&\ll \int_1^T \frac{\mathrm{e}^{\ln^\beta t}}{t\,(\ln t)^{\beta(1+\varepsilon)}}(\ln t)^{1-\beta}\,\mathrm{e}^{\ln^\beta t}\mathrm{d}t = \int_0^{\ln T} y^{1-2\beta-\beta\varepsilon}\mathrm{e}^{2y^\beta}\mathrm{d}y \\
&\ll \int_0^{\ln T}\left(\frac{1}{2-3\beta-\beta\varepsilon}\cdot\frac{1}{y^\beta}\cdot y^{1-2\beta-\beta\varepsilon}\cdot\mathrm{e}^{2y^\beta}+2\beta\cdot y^{1-2\beta-\beta\varepsilon}\cdot\mathrm{e}^{2y^\beta}\right)\mathrm{d}y \qquad (3.7)\\
&= \int_0^{\ln T}\mathrm{d}\left(y^{2-3\beta-\beta\varepsilon}\mathrm{e}^{2y^\beta}\right) \ll (\ln T)^{2-3\beta-\beta\varepsilon}\,\mathrm{e}^{2(\ln T)^\beta} \\
&\ll \frac{W_T^2}{(\ln W_T)^{1+\varepsilon}}.
\end{aligned}$$

Thus, we can conclude from (3.3), (3.4), (3.6), (3.7) that

$$\mathrm{Var}\left(\int_1^T w_t\eta(t)\mathrm{d}t\right) \ll \frac{W_T^2}{(\ln W_T)^{1+\varepsilon}}.$$

Next, note that $r(T)(\ln T)^{1+3\beta(1+\varepsilon)} = O(1)$ implies $r(T)(\ln T) = o(1)$. From (1.3) we have

$$\lim_{t\to\infty} P\left(M[1,t]\le u_t\right) = \lim_{t\to\infty} P\left(M[0,t]\le u_t\right) = \mathrm{e}^{-\tau}.$$

Clearly, we can gain

$$\lim_{T\to\infty}\frac{1}{\ln T}\int_1^T w_t P\left(M\left([1,t]\right)\le u_t\right)\mathrm{d}t = \mathrm{e}^{-\tau}. \qquad (3.8)$$

Now, the result of the theorem follows by Lemma 3.1 and (3.8).

Case (ii) Case (ii) is a special of Case (i).

4 Appendix

Proof of Lemma 3.1

Set

$$\begin{aligned}
&\frac{1}{W_T}\int_1^T w_t\left(\xi(t)-\mathrm{E}\xi(t)\right)\mathrm{d}t \\
=&\frac{W_{[T]}}{W_T}\cdot\frac{1}{W_{[T]}}\sum_{k=2}^{[T]}\int_{k-1}^k w_t\left(\xi(t)-\mathrm{E}\xi(t)\right)\mathrm{d}t + \frac{1}{W_T}\int_{[T]}^T w_t\left(\xi(t)-\mathrm{E}\xi(t)\right)\mathrm{d}t \qquad (4.1)\\
\hat{=}&\frac{W_{[T]}}{W_T}\mu_{[T]}+\mu'_{[T]}.
\end{aligned}$$

Clearly as $T\to\infty$,

$$\mu'_{[T]} = \frac{1}{W_T}\int_{[T]}^T w_t\left(\xi(t)-\mathrm{E}\xi(t)\right)\mathrm{d}t \to 0 \quad \text{a.s.}. \qquad (4.2)$$

Now, we prove as $T \to \infty$ that

$$\mu_{[T]} = \frac{1}{W_{[T]}} \sum_{k=2}^{[T]} \int_{k-1}^{k} w_t \left(\xi_t - \mathrm{E}\xi_t\right) \mathrm{d}t \to 0 \quad \text{a.s..}$$

Let $[T]_k = \inf\left\{[T], W_{[T]} > \exp\left(k^{1-\eta}\right)\right\}$ for some $0 < \eta < \frac{\varepsilon}{1+\varepsilon}$, then $W_{[T]_k} \geq \exp\left(k^{1-\eta}\right)$ and $W_{[T]_k - 1} < \exp\left(k^{1-\eta}\right)$. By (3.5), we get

$$1 \leq \frac{W_{[T]_k}}{\exp\left(k^{1-\eta}\right)} \sim \frac{W_{[T]_k - 1}}{\exp\left(k^{1-\eta}\right)} < 1,$$

that is

$$W_{[T]_k} \sim \exp\left(k^{1-\eta}\right).$$

We have

$$\begin{aligned}
\sum_{k=3}^{\infty} E\left(\mu^2_{[T]_k}\right) &= \sum_{k=3}^{\infty} \left(\frac{1}{W^2_{[T]}}\right) \mathrm{var}\left(\sum_{k=2}^{[T]} \int_{k-1}^{k} w_t \left(\xi(t) - E\left(\xi(t)\right)\right) \mathrm{d}t\right) \\
&= \sum_{k=3}^{\infty} \left(\frac{1}{W^2_{[T]}}\right) \mathrm{var}\left(\int_1^{[T]} w_t \left(\xi(t)\right) \mathrm{d}t\right) \\
&\ll \sum_{k=3}^{\infty} \left(\frac{1}{W^2_{[T]}}\right) \cdot W^2_{[T]} \cdot \left(\ln W_{[T]}\right)^{-(1+\varepsilon)} \\
&\sim \sum_{k=3}^{\infty} \frac{1}{k^{(1-\eta)(1+\varepsilon)}}.
\end{aligned}$$

Since $\eta < \frac{\varepsilon}{1+\varepsilon}$ implies $1-\eta > \frac{1}{1+\varepsilon}$ and $(1-\eta)(1+\varepsilon) > 1$, thus for sufficiently large k, we get

$$\sum_{k=3}^{\infty} \frac{1}{k^{(1-\eta)(1+\varepsilon)}} < \infty.$$

This implies

$$\sum_{k=3}^{\infty} \mu^2_{[T]_k} < \infty \quad \text{a.s..}$$

Obviously for any given $[T]$ there is an integer k such that $[T]_k < [T] \leq [T]_{k+1}$, we have as $T \to \infty$,

$$\begin{aligned}
\mu_{[T]} &= \frac{1}{W_{[T]}} \sum_{j=2}^{[T]} \int_{j-1}^{j} w_t \left(\xi(t) - \mathrm{E}\xi(t)\right) \mathrm{d}t \\
&\leq \frac{1}{W_{[T]_k}} \left| \sum_{j=2}^{[T]_k} \int_{j-1}^{j} w_t \left(\xi(t) - \mathrm{E}\xi(t)\right) \mathrm{d}t \right| + \frac{1}{W_{[T]_k}} \sum_{j=[T]_k+1}^{[T]_{k+1}} \int_{j-1}^{j} w_t \left(\xi(t) - \mathrm{E}\xi(t)\right) \mathrm{d}t \\
&\leq \left|\mu_{[T]_k}\right| + \frac{1}{W_{[T]_k}} \left|W_{[T]_{k+1}} - W_{[T]_k + 1}\right| \leq \left|\mu_{[T]_k}\right| + \left|\frac{W_{[T]_{k+1}}}{W_{[T]_k}} - 1\right| \to 0 \quad \text{a.s..}
\end{aligned} \tag{4.3}$$

From $\frac{W_{[T]_{k+1}}}{W_{[T]_k}} \sim \frac{\exp\left((k+1)^{1-\eta}\right)}{\exp\left(k^{1-\eta}\right)} = \exp\left(k^{1-\eta}((1+\frac{1}{k})^{1-\eta} - 1)\right) \sim \exp\left((1-\eta)\, k^{-\eta}\right) \to 1$ a.s., (4.3) holds.

Now, the result of Lemma 3.1 follows by (4.1), (4.2) and (4.3).

Proof of Lemma 3.2 Let $\upsilon(\delta)=\sup_{\delta<iq}\{r(iq)\}$. By assumption (1.1) and $\{X(t),t\geq 0\}$ is a stationary Gaussian process, we have $\upsilon(\delta)=\sup_{\delta<iq}\{r(iq)\}<1$. Further, let α satisfy $0<\alpha<\frac{1-\upsilon(\delta)}{1+\upsilon(\delta)}$ for all sufficiently large t. We split the sum in (3.1) at t^α as

$$\begin{aligned}
&\sup_{s\in(0,t)}\tfrac{s}{q}\sum_{\delta\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2(1+|r(iq)|)}\right)\\
&=\sup_{s\in(0,t)}\tfrac{s}{q}\sum_{\delta\leq iq\leq t^\alpha}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2(1+|r(iq)|)}\right)+\sup_{s\in(0,t)}\tfrac{s}{q}\sum_{t^\alpha\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2(1+|r(iq)|)}\right)\\
&\hat{=}B_{t,1}+B_{t,2}.
\end{aligned}\tag{4.4}$$

Using the facts $u_t^2\sim 2\ln t$ and $q=u_t^{-1}(\ln t)^{-\beta(1+\varepsilon)}$ we have

$$\begin{aligned}
B_{t,1}&=\sup_{s\in(0,t)}\tfrac{s}{q}\sum_{\delta\leq iq\leq t^\alpha}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2(1+|r(iq)|)}\right)\leq\tfrac{st^\alpha}{q^2}\exp\left(-\tfrac{u_s^2+u_t^2}{2(1+|r(iq)|)}\right)\\
&\ll\tfrac{1}{q^2}t^{\alpha-\frac{1}{1+\upsilon(\delta)}}s^{1-\frac{1}{1+\upsilon(\delta)}}\ll t^{1+\alpha-\frac{2}{1+\upsilon(\delta)}}(\ln t)^2(\ln t)^{2\beta(1+\varepsilon)}.
\end{aligned}$$

Since $\alpha<\frac{1-\upsilon(\delta)}{1+\upsilon(\delta)}$, we get as $t\to\infty$ that

$$B_{t,1}<t^{1+\alpha-\frac{2}{1+\upsilon(\delta)}}\to 0\tag{4.5}$$

uniformly for $s\in(0,t]$. Notice that $r(t)(\ln t)^{1+3\beta(1+\varepsilon)}=O(1)$ and $u_t^2\sim 2\ln t$, we get

$$r(iq)\ll\frac{1}{(\ln(iq))^{1+3\beta(1+\varepsilon)}}<\frac{1}{(\ln(t^\alpha))^{1+3\beta(1+\varepsilon)}}\sim\frac{1}{(\ln t)^{1+3\beta(1+\varepsilon)}}$$

and as $t\to\infty$, we have

$$u_t^2|r(iq)|\ll\ln t\cdot\frac{1}{(\ln t)^{1+3\beta(1+\varepsilon)}}=\frac{1}{(\ln t)^{3\beta(1+\varepsilon)}}\to 0.$$

Consequently

$$\begin{aligned}
B_2&=\tfrac{s}{q}\sum_{t^\alpha\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2(1+|r(iq)|)}\right)\\
&=\tfrac{s}{q}\sum_{t^\alpha\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2}\right)\exp\left(\tfrac{(u_s^2+u_t^2)|r(iq)|}{2(1+|r(iq)|)}\right)\\
&\ll\tfrac{s}{q}\sum_{t^\alpha\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2}\right)\exp\tfrac{(u_s^2+u_t^2)|r(iq)|}{2}\\
&\ll\tfrac{s}{q}\sum_{t^\alpha\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2}\right)\exp\left(u_t^2|r(iq)|\right)\\
&\ll\tfrac{s}{q}\sum_{t^\alpha\leq iq\leq t}|r(iq)|\exp\left(-\tfrac{u_s^2+u_t^2}{2}\right)\\
&\leq\tfrac{st}{q^2}t^{-1}s^{-1}(\ln t)^{-1-3\beta(1+\varepsilon)}O(1)\\
&\ll(\ln t)^{-\beta(1+\varepsilon)}.
\end{aligned}\tag{4.6}$$

The result of Lemma 3.2 follows by (4.4), (4.5) and (4.6).

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光滑平稳高斯过程极值几乎处处极限定理的一个推广

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摘要: 本文研究了连续均方可微的平稳高斯过程的极限性态. 通过选择一个不同于 Tan (2013) 的权重函数, 在较弱的条件下得到了连续均方可微平稳高斯过程极值的一个几乎必然仅限定理, 推广了 Tan (2013) 的结论.

关键词: 平稳高斯过程; 几乎处处; 极限定理; 极值; 权重函数

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