ON TL-FUZZY IDEALS IN LATTICES

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Abstract: In this paper, we study TL-fuzzy ideals in lattices. By a TL-fuzzy ideal generated by an L-fuzzy subset, we prove that the lattice of TM-fuzzy ideals in a modular lattice is a complete modular lattice. Moreover, using the projection and the cut shadow of an L-fuzzy set, we obtain necessary and sufficient conditions for a TL-fuzzy ideal of a Cartesian product of lattices to be a T-product of TL-fuzzy ideals of lattices. Our results generalize and develop the fuzzy ideal theory in lattices.

Keywords: left continuous t-norm; TL-fuzzy ideal; modular lattice; T-product

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1 Introduction

An important notion in fuzzy set theory is that of triangular norms: t-norms are used to define a generalized intersection. By a fuzzy subset $\mu$ in a given universe $X$ we understand a mapping $\mu: X \rightarrow [0, 1]$, the membership degrees $\mu(x)$ can in a natural way be understood as the truth value (in fuzzy logic) of the statement “$x$ belongs to $\mu$”. In the same way, the intersection $\mu \cap \nu$ of fuzzy sets $\mu, \nu$ can be viewed as having the membership degree $(\mu \cap \nu)(x)$ corresponding to the truth degree of the statement “$x$ belongs to $\mu$” and “$x$ belongs to $\nu$”. Here AND refers to a suitably defined conjunction connective, defined according to the different possibilities which one has to determine the membership degrees $(\mu \cap \nu)(x)$. For example, AND can be understood as taking the minimum or as taking the (usual, i.e., algebraic) product, or more generally, it also can be understood as a $t$-norm. Accordingly, the $t$-norms were considered as the candidates for generalized conjunction connectives of the background many-valued logic. The fuzzy logic based on $t$-norms, especially left continuous $t$-norms, was developed significantly by Héjek, Esteva et al. (see [1–4]). Also, there are many applications of triangular norms in several fields of mathematics and artificial intelligence [5].

In 1971, Rosenfeld [6] used the concept of fuzzy sets to formulate the notion of fuzzy groups. Since then, many other fuzzy algebraic concepts of fuzzy groups have been developed. Anthony and Sherwood [7] redefined fuzzy subgroups, which we call $T$-fuzzy subgroups.

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in this note, in terms of $t$-norm $T$ which replaced the minimum operation and they [7, 8] characterized basic properties of $T$-fuzzy subgroups. In [9], Hu studied $T$-fuzzy groups with thresholds. Chon [10] characterized necessary and sufficient conditions whereby a fuzzy subgroup of a Cartesian product of groups is the product of fuzzy subgroups under minimum operation. He pointed out that finding necessary and sufficient conditions for $T$-fuzzy subgroups under a $t$-norm is still an open problem. In 2011, Yamak et al.[11] solved this open problem and identified necessary and sufficient conditions for $TL$-subgroups of a Cartesian product of groups, which can be represented as a $T$-product of $TL$-subgroups under $t$-norm operation. In the same paper, they also pointed out that the same problem could be studied in other algebraic structures such as rings and lattices.

In [6], the idea of a least fuzzy subgroupoid containing a given fuzzy set was also introduced. Consequently, Rosenfeld constructed the lattice of all fuzzy subgroupoids of a given group. In a recent paper [12], Jahan established that the lattice of all fuzzy ideals of a ring is modular. In fact, the proof of modularity is heavily based on the property of the unit interval that it is a dense chain. However, modularity of the lattices of $L$-normal subgroups of a group and $L$-ideals of a ring remains an open question. In 2011, Jahan [13] answered the question of modularity of the lattice of $L$-ideals of a ring.

With the development of theories of fuzzy algebra, Swamy [14] discussed the correspondence relation between fuzzy ideals and fuzzy congruences in a distributive lattice. In 2008, Kogu et al.[15] studied the notion of fuzzy prime ideal and highlighted the difference between fuzzy prime ideal and prime fuzzy ideal of a lattice. However, not much attention was paid to the studies of the lattices of fuzzy ideals of a lattice and the modularity of them.

The present work has been started as a continuation of these studies. In this paper, we will discuss modularity of the lattices of fuzzy ideals in a lattice. Moreover, we will explore necessary and sufficient conditions for a fuzzy ideal of a Cartesian product of lattices to be a $T$-product of fuzzy ideals of lattices under a left continuous $t$-norm $T$ on a complete lattice $L$.

2 Preliminaries

In this section, we recall some notions and definitions that will be used in the sequel.

Let $(L, \wedge, \vee, \leq, 0, 1)$ denote a complete lattice with the top and bottom elements 1 and 0, respectively.

**Definition 2.1** [5] A binary operation $T$ on $L$ is called a $t$-norm if it satisfies the following conditions: for any $a, b, c \in L$,

- (T1) $aT1 = a$;
- (T2) $aTb = bTa$;
- (T3) $(aTb)Tc = aT(bTc)$;
- (T4) if $b \leq c$, then $aTb \leq aTc$.

Because of associative and commutative properties, for any $a_1, a_2, \cdots, a_n \in L\ (n \geq 1)$, $a_1Ta_2T\cdots Ta_n$ is well defined and its value is irrelevant to the order of $a_1, a_2, \cdots, a_n$. We
write $T^n a_i = a_1 T a_2 T \cdots T a_n$. If $a T (\vee_{i \in I} b_i) = \vee_{i \in I} (a T b_i)$, for $a, b_i \in L$, where $I$ is the set of natural numbers, then $T$ is called a left continuous $t$-norm, see [2].

In what follows, let $L = [0, 1]$, there are some examples of the most popular $t$-norms: for any $x, y \in [0, 1]$,

(1) the Łukasiewicz $t$-norm: $x T_L y = \max\{x + y - 1, 0\}$;

(2) the algebraic product: $x T_p y = xy$;

(3) the standard min operation: $x T_M y = \min\{x, y\}$.

Throughout this paper, unless otherwise stated, $(L, \wedge, \vee, \leq, 0, 1)$ always represents a given complete lattice with a left continuous $t$-norm $T$.

An $L$-fuzzy subset of $X$ is a mapping from $X$ to $L$. The family of all $L$-subsets of $X$ is denoted by $LF[X]$, (see [16]). When $L = [0, 1]$, the $L$-subsets of $X$ are known as fuzzy subsets of $X$ (see [17]). Let $\mu, \nu \in LF[X]$ be given, $\mu$ is said to be included in $\nu$ and written as $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$.

The following are the most popular operators on $L$-fuzzy sets: for all $\mu, \nu \in LF[X]$, $x \in X$, $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$, $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$.

3 TL-Fuzzy Ideals

In this section, we shall introduce the notion of $TL$-fuzzy ideals in lattices and give some properties of them that will be used in the sequel.

**Definition 3.1** Let $(X, \wedge, \vee, \leq)$ be a lattice and $\mu$ be an $L$-fuzzy subset of $X$. Then $\mu$ is called a $TL$-fuzzy ideal of $X$ if it satisfies the following conditions: for all $x, y \in X$,

(i) $\mu(x \vee y) \geq \mu(x) T \mu(y)$,

(ii) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

We shall denote the set of all $TL$-fuzzy ideals of the lattice $X$ as $TLFI[X]$.

In particular, a $TL$-fuzzy ideal is called an $L$-fuzzy ideal when $T = \wedge$. Moreover, when $L = [0, 1]$, a $TL$-fuzzy ideal and an $L$-fuzzy ideal of $X$ are, respectively, referred to as a $T$-fuzzy ideal and fuzzy ideal of the lattice $X$.

Now, the following result gives an equivalent version of the concept of $TL$-fuzzy ideals in lattices.

**Theorem 3.2** Let $(X, \wedge, \vee, \leq)$ be a lattice and $\mu$ be an $L$-fuzzy subset of $X$. Then $\mu$ is a $TL$-fuzzy ideal of $X$ if and only if it satisfies the following conditions: for all $x, y \in X$,

(i) $\mu(x \vee y) \geq \mu(x) T \mu(y)$,

(ii) if $y \leq x$, then $\mu(y) \geq \mu(x)$.

**Proof** The proof is straightforward.

**Example 3.3** (1) Let $X = \{0, a, b, c, 1\}$ be a lattice, the partial order on $X$ is defined as shown in Fig. 1. Let $L = \{1, 2, 3, 4, 5, 6\}$, the partial order on $L$ is defined as shown in Fig. 2, $T = \wedge$. Define two $L$-fuzzy subsets $\mu$ and $\nu$ of $X$ as follows: $\mu(0) = 6$, $\mu(a) = 2$, $\mu(b) = 5$, $\mu(c) = 2$, $\mu(1) = 2$. By routine calculations, it is easy to check that $\mu$ is a $TL$-fuzzy ideal of $X$. 
(2) Let \( X = \{0, a, b, 1\} \) be a lattice, the partial order on \( X \) is defined as shown in Fig. 3. Let \( L = [0, 1] \) and \( T = T_L \), that is \( x T_L y = \max\{x + y - 1, 0\} \), for any \( x, y \in L = [0, 1] \). Define two \( L \)-fuzzy subsets \( \mu \) and \( \nu \) of \( X \) as follows: \( \mu(0) = \frac{3}{5}, \mu(a) = \frac{3}{5}, \mu(b) = \frac{3}{10}, \mu(1) = \frac{3}{10} \). By routine calculations, it is easy to check that \( \mu \) is a \( T_L \)-fuzzy ideal of \( X \).

In the following, we give some properties of \( T_L \)-fuzzy ideals, which will be used in the sequel.

**Proposition 3.4** Let \( \mu_i (i \in I) \) be \( T_L \)-fuzzy ideals of a lattice \( X \). Then \( \bigcap_{i \in I} \mu_i \) is a \( T_L \)-fuzzy ideal of \( X \).

**Proof** The proof is straightforward.

By the following example we show that the union of two \( T_L \)-fuzzy ideals is not a \( T_L \)-fuzzy ideal.

**Example 3.5** Let \( X = \{0, a, b, 1\} \) be a lattice, the partial order on \( X \) is defined as shown in Fig. 3 in Example 3.3 (2). Let \( L = [0, 1] \) and \( T = T_M \). Define two \( L \)-fuzzy subsets \( \mu \) and \( \nu \) of \( X \) as follows: \( \mu(0) = \frac{2}{5}, \mu(a) = \frac{1}{2}, \mu(b) = \frac{1}{2}, \mu(1) = \frac{1}{5}; \nu(0) = \frac{7}{10}, \nu(a) = \frac{1}{10}, \nu(b) = \frac{3}{10}, \nu(1) = \frac{1}{10} \). Then we can check that both \( \mu \) and \( \nu \) are \( T_L \)-fuzzy ideals of \( X \). Now \( (\mu \cup \nu)(0) = \frac{7}{10}, (\mu \cup \nu)(a) = \frac{1}{2}, (\mu \cup \nu)(b) = \frac{3}{10}, (\mu \cup \nu)(1) = \frac{1}{5} \). Since \( (\mu \cup \nu)(a \lor b) = (\mu \cup \nu)(1) = \frac{1}{5} < (\mu \cup \nu)(a) \land (\mu \cup \nu)(b) = \frac{3}{10} \), \( \mu \cup \nu \) is not a \( T_L \)-fuzzy ideal of \( X \).

4 The Lattice of \( T_L \)-Fuzzy Ideals
Now, we give a procedure to construct the TL-fuzzy ideal generated by an L-fuzzy subset. And we shall discuss the algebraic structure of the set of all TL-fuzzy ideals in lattices.

**Definition 4.1** Let $\mu$ be an L-fuzzy subset in a lattice $X$. A TL-fuzzy ideal $\nu$ of the lattice $X$ is said to be generated by $\mu$, if $\mu \subseteq \nu$ and for any TL-fuzzy ideal $\omega$ of $X$, $\mu \subseteq \omega$ implies $\nu \subseteq \omega$. The TL-fuzzy ideal generated by $\mu$ will be denoted by $(\mu)_{TL}$.

It follows from Definition 4.1 that $(\mu)_{TL}$ is the smallest TL-fuzzy ideal of the lattice $X$ containing $\mu$. And we can easily get that $(\mu)_{TL} = \cap_{i \in I} \{ \mu_i \in TLFI[X] | \mu_i \supseteq \mu, i \in I \}$.

It is easy to verify that for $\mu$ and $\nu$ be L-fuzzy subsets of $X$. Then

1. If $\mu$ is a TL-fuzzy ideal of the lattice $X$, then $(\mu)_{TL} = \mu$,
2. $\mu \subseteq \nu$ implies $(\mu)_{TL} \subseteq (\nu)_{TL}$.

In what follows, we give the formula for calculating the TL-fuzzy ideals generated by L-fuzzy subsets.

**Theorem 4.2** Let $\mu$ be an L-fuzzy subset in the lattice $X$. Then for any $x \in X$,

$$(\mu)_{TL}(x) = \bigvee \{ \mu(a_1)T\mu(a_2)T\cdots T\mu(a_n) | x \leq a_1 \lor a_2 \lor \cdots \lor a_n, a_i \in X, i \in I \}.$$  

**Proof** Let

$$\nu(x) = \bigvee \{ \mu(a_1)T\mu(a_2)T\cdots T\mu(a_n) | x \leq a_1 \lor a_2 \lor \cdots \lor a_n, a_i \in X, i \in I \}.$$  

First, we prove that $\nu$ is a TL-fuzzy ideal of $X$.

For any $x, y \in X$, we have that

$$\nu(x)T\nu(y) = (\bigvee \{ \mu(a_1)T\mu(a_2)T\cdots T\mu(a_n) | x \leq a_1 \lor a_2 \lor \cdots \lor a_n, a_i \in X, i \in I \})T$$

$$= (\bigvee \{ \mu(b_1)T\mu(b_2)T\cdots T\mu(b_n) | y \leq b_1 \lor b_2 \lor \cdots \lor b_n, b_j \in X, j \in I \})$$

$$= \bigvee \{ \mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)T\mu(b_1)T\mu(b_2)T\cdots T\mu(b_n) | x \leq a_1 \lor a_2 \lor \cdots \lor a_n, a_i, b_j \in X, i, j \in I \}$$

$$\leq \bigvee \{ \mu(c_1)T\mu(c_2)T\cdots T\mu(c_l) | x \lor y \leq c_1 \lor c_2 \lor \cdots \lor c_l, c_i \in X, l \in I \}$$

$$= \nu(x \lor y).$$

If $y \leq x$ and $x \leq a_1 \lor a_2 \lor \cdots \lor a_n$ for some $a_1, a_2, \cdots, a_n \in X$, then $y \leq a_1 \lor a_2 \lor \cdots \lor a_n$. It follows that $\nu(y) \geq \nu(x)$. By Theorem 3.2, we can get that $\nu$ is a TL-fuzzy ideal of $X$.

Next, since $x \leq x$, we have that $\nu(x) \geq \mu(x)$. So $\mu \subseteq \nu$.

Finally, suppose that $\omega$ is a TL-fuzzy ideal of $X$ with $\mu \subseteq \omega$. Then for any $x \in X$, $\mu(x) \leq \omega(x)$. Moreover, for any $a_1, a_2, \cdots, a_n \in X$ with $x \leq a_1 \lor a_2 \lor \cdots \lor a_n$, we have $\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n) \leq \omega(a_1)T\omega(a_2)T\cdots T\omega(a_n) \leq \omega(a_1 \lor a_2 \lor \cdots \lor a_n) \leq \omega(x)$, since $\omega$ is a TL-fuzzy ideal of $X$. It follows that $\nu(x) \leq \omega(x)$. Thus, $\nu \subseteq \omega$.

Summarizing the above facts, we obtain that $\nu$ is the smallest TL-fuzzy ideal in the lattice $X$ with $\mu \subseteq \nu$, that is, $\nu$ is the TL-fuzzy ideal generated by $\mu$ in $X$. Therefore, $\nu = (\mu)_{TL}$. 

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Example 4.3 Let $X = \{0, a, b, 1\}$ be a lattice, the partial order on $X$ is defined as shown in Fig. 3 in Example 3.3 (2). Let $L = \{1, 2, 3, 4\}$, the partial order on $L$ is defined as shown in Fig. 4 and $T = \land$. Define an $L$-fuzzy subset $\mu$ of $X$ as follows: $\mu(0) = 3$, $\mu(a) = 2$, $\mu(b) = 4$, $\mu(1) = 1$. One can easily check that the $TL$-fuzzy ideal $(\mu)_{TL}$ generated by $\mu$ as follows: $(\mu)_{TL}(0) = 4$, $(\mu)_{TL}(a) = 2$, $(\mu)_{TL}(b) = 4$, $(\mu)_{TL}(1) = 2$.

Let $X$ be a lattice. For any $\mu_1, \mu_2 \in TLFI[X]$, we define $\mu_1 \oplus \mu_2$ and $\mu_1 \otimes \mu_2$ as follows: $\mu_1 \oplus \mu_2 = \mu_1 \cap \mu_2$, $\mu_1 \otimes \mu_2 = \bigcap \{\mu \in TLFI[X]|\mu \supseteq \mu_1 \cup \mu_2\} = (\mu_1 \cup \mu_2)_{TL}$. In general, for any $\mu_i \in TLFI[X]$, where $i \in I$, we define $\oplus\{\mu_i | i \in I\} = \bigcap \{\mu_i | i \in I\}$, $\otimes\{\mu_i | i \in I\} = \bigcap \{\mu \in TLFI[X]|\mu \supseteq \bigcup_{i \in I} \mu_i\} = (\bigcup_{i \in I} \mu_i)_{TL}$. Therefore, we can get the following result.

Theorem 4.4 $(TLFI[X], \oplus, \otimes)$ is a complete lattice, which is called the lattice of $TL$-fuzzy ideals.

In particular, when $L = [0, 1]$, we can give the simple formulas for calculating the $T$-fuzzy ideals generated by the union of $T$-fuzzy ideals.

Theorem 4.5 Let $\mu_1, \mu_2$ be $T$-fuzzy ideals of a lattice $X$. Then for any $x \in X$,

$$(\mu_1 \cup \mu_2)_T (x) = \sup\{\{\mu_1(a)|x \leq a\} \cup \{\mu_2(b)|x \leq b\} \cup \{\mu_1(a)T\mu_2(b)|x \leq a \lor b\}\}.$$ 

Proof By Theorem 4.2, we have

$$(\mu_1 \cup \mu_2)_T (x) = \sup\{(\mu_1 \cup \mu_2)(a_1)T(\mu_1 \cup \mu_2)(a_2)\cdots T(\mu_1 \cup \mu_2)(a_n)|x \leq a_1 \lor \cdots \lor a_n, a_i \in X, i \in I\}.$$ 

Given an arbitrary small $\epsilon > 0$, we have the following three cases:

Case 1 There exist $a_1, \ldots, a_n \in X$, satisfying

1. $x \leq a_1 \lor a_2 \lor \cdots \lor a_n$,
2. $x \leq a_1 \lor a_2 \lor \cdots \lor a_n$,
3. $x \leq a_1 \lor a_2 \lor \cdots \lor a_n$.

Thus $(\mu_1 \cup \mu_2)_T (x) < \epsilon + \mu_1(a_1)T(\mu_1 \cup \mu_2)(a_2) \cdots T(\mu_1 \cup \mu_2)(a_n)$.

Denote $a = a_1 \lor \cdots \lor a_n$. Since $\mu_1$ is a $T$-fuzzy ideal of a lattice $X$, we have

$$\mu_1(a) \geq \mu_1(a_1)T(\mu_1 \mu_2)(a_2) \cdots T(\mu_1 \mu_2)(a_n),$$

it follows that $x \leq a$ and $(\mu_1 \cup \mu_2)_T (x) < \epsilon + \mu_1(a)$.

Case 2 There exist $b_1, \ldots, b_m \in X$, satisfying

1. $x \leq b_1 \lor b_2 \lor \cdots \lor b_m$,
2. $x \leq b_1 \lor b_2 \lor \cdots \lor b_m$,
3. $x \leq b_1 \lor b_2 \lor \cdots \lor b_m$.

Thus $(\mu_1 \cup \mu_2)_T (x) < \epsilon + \mu_2(b_1)T(\mu_1 \mu_2)(b_2) \cdots T(\mu_1 \mu_2)(b_m)$.

Denote $b = b_1 \lor b_2 \lor \cdots \lor b_m$. Since $\mu_2$ is a $T$-fuzzy ideal of a lattice $X$, we have

$$\mu_2(b) \geq \mu_2(b_1)T(\mu_1 \mu_2)(b_2) \cdots T(\mu_1 \mu_2)(b_m),$$

it follows that $x \leq b$ and $(\mu_1 \cup \mu_2)_T (x) < \epsilon + \mu_2(b)$.

Cases 3 There exist $a_1, \ldots, a_s, b_1, \ldots, b_t \in X$, satisfying
obtain the following main result.

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This completes the proof.

Theorem 4.6 (TLFI(X), ∪, ⊙) is a complete modular lattice if X is a modular lattice.
Proof From Theorem 4.4, we have that $\mathcal{TLFI}[X]$, $\oplus$, $\otimes$ is a complete lattice. To verify that $(\mathcal{TLFI}[X], \oplus, \otimes)$ is a modular lattice, we should show that it satisfies modular law. Now, assume $\mu_1, \mu_2, \mu_3 \in \mathcal{TLFI}[X]$, where $\mu_1 \supseteq \mu_2$ and $x \in X$, the inequality $\mu_1 \oplus (\mu_2 \otimes \mu_3) \supseteq \mu_2 \otimes (\mu_1 \oplus \mu_3)$ is trivial. We only need to prove that $\mu_1 \oplus (\mu_2 \otimes \mu_3) \subseteq \mu_2 \otimes (\mu_1 \oplus \mu_3)$.

Given an arbitrarily small $\epsilon > 0$, by Theorem 4.5, we have the following three cases:

Case 1 There exists $a \in X$ such that $x \leq a$ and $(\mu_2 \otimes \mu_3)(x) < \epsilon + \mu_2(a)$. And so

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_1(x), \mu_2(a)\}.$$ Since $\mu_2(a) \leq \mu_2(x) \leq \mu_1(x)$, it follows that $\min\{\mu_1(x), \mu_2(a)\} = \mu_2(a)$, hence

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \mu_2(a).$$ Combining $x \leq a$, we obtain

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \mu_2(x) \leq \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).$$

Case 2 There exists $b \in X$ such that $x \leq b$ and $(\mu_1 \otimes \mu_3)(x) < \epsilon + \mu_3(b)$. From $x \leq b$ it follows that $\mu_3(b) \leq \mu_3(x)$, hence

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_1(x), \mu_3(x)\}.$$ Combining $x \leq x$ and the definition of $\mu_2 \otimes (\mu_1 \oplus \mu_3)$, we have

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).$$

Case 3 There are $a, b \in X$ such that $x \leq a \lor b$ and $(\mu_2 \otimes \mu_3)(x) < \epsilon + \min\{\mu_2(a), \mu_3(b)\}$. Hence $(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_1(x), \mu_2(a), \mu_3(b)\}$. Denote $b_1 = (x \lor a) \land b$, then $b_1 \leq x \lor a$ and $b_1 \leq b$. Notice that $X$ is a modular lattice, $a \lor b_1 = a \lor ((x \lor a) \land b) = (x \lor a) \land (a \lor b) \geq x$. Since $\mu_1, \mu_3$ are fuzzy ideals, we have $\mu_3(b) \leq \mu_3(b_1)$ and $\mu_1(x \lor a) \leq \mu_1(b_1)$. It follows that

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_2(a), \mu_1(x), \mu_3(b)\}$$

Since $\epsilon$ is arbitrary, we have $(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x)$. Therefore,

$$\mu_1 \oplus (\mu_2 \otimes \mu_3) \subseteq \mu_2 \otimes (\mu_1 \oplus \mu_3).$$
Summarizing the above facts, we get that for any $x \in X$ and given an arbitrarily small $\epsilon > 0$,
\[(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).\]

Therefore, $(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x)$. So $\mu_1 \oplus (\mu_2 \otimes \mu_3) \subseteq \mu_2 \otimes (\mu_1 \oplus \mu_3)$, that is, $(TLF_I[X], \oplus, \otimes)$ is a complete modular lattice.

5 $T$-Product of TL-Fuzzy Ideals

In this section, as a continuation of the work [11], we will explore necessary and sufficient conditions for a fuzzy ideal of a Cartesian product of lattices to be a $T$-product of fuzzy ideals of lattices under a left continuous $t$-norm $T$ on a complete lattice $L$.

First, let us recall the Cartesian product of lattices for the sake of completeness.

Let $(X_1, \wedge_1, \vee_1, \leq_1)$ and $(X_2, \wedge_2, \vee_2, \leq_2)$ be two lattices. Define two binary operations $\wedge$ and $\vee$ on $X_1 \times X_2$ as follows: for any $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$,
\[(x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2), (x_1, x_2) \vee (y_1, y_2) = (x_1 \vee_1 y_1, x_2 \vee_2 y_2).\]

Then $X_1 \times X_2$ is a lattice, which is called the Cartesian product lattice of $X_1$ and $X_2$. The corresponding partial order $\leq$ on $X_1 \times X_2$ as follows:
\[(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq_1 y_1, x_2 \leq_2 y_2.\]

**Definition 5.1** Let $\mu_i \in LF[X_i], i = 1, 2$. Then the $T$-product of $\mu_i$ $(i = 1, 2)$ denoted by $\mu_1 \times_T \mu_2$ is defined as the $L$-fuzzy subset of $X_1 \times X_2$ that satisfies: for any $(x_1, x_2) \in X_1 \times X_2$, $\mu_1 \times_T \mu_2(x_1, x_2) = \mu_1(x_1)T\mu_2(x_2)$.

**Theorem 5.2** Let $\mu_i$ be a TL-fuzzy ideal of a lattice $X_i, i = 1, 2$. Then $\mu_1 \times_T \mu_2$ is a TL-fuzzy ideal of $X_1 \times X_2$.

**Proof** Assume that $\mu_i$ be a TL-fuzzy ideal of a lattice $X_i, i = 1, 2$. For any $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, then we have that
\[
\mu_1 \times_T \mu_2 ((x_1, x_2) \vee (y_1, y_2)) = \mu_1 \times_T \mu_2 ((x_1 \vee_1 y_1, x_2 \vee_2 y_2)) = (\mu_1(x_1)T\mu_2(x_2)) \vee (\mu_1(y_1)T\mu_2(y_2)) = (\mu_1 \times_T \mu_2)(x_1, x_2) \vee (\mu_1 \times_T \mu_2)(y_1, y_2).
\]

On the other hand, if $(x_1, x_2) \leq (y_1, y_2)$, that is $x_1 \leq_1 y_1, x_2 \leq_2 y_2$. Then we have
\[
\mu_1 \times_T \mu_2(x_1, x_2) = \mu_1(x_1)T\mu_2(x_2) \geq \mu_1(y_1)T\mu_2(y_2) = \mu_1 \times_T \mu_2(y_1, y_2).
\]

Therefore, $\mu_1 \times_T \mu_2$ is a TL-fuzzy ideal of $X_1 \times X_2$.

In what follows, we introduce the concepts of the projection and the cut shadow of an $L$-fuzzy set that are instrumental to determine necessary and sufficient conditions under $t$-norm operation.
Definition 5.3 Let \( \mu \in LF[X_1 \times X_2] \). Then the projection of \( \mu \) on \( X_i \) (\( i = 1, 2 \)) denoted by \( \mu_{X_i} \) is defined as the \( L \)-fuzzy subset of \( X_i \) (\( i = 1, 2 \)) that satisfies, respectively, \( \mu_{X_i}(x) = \bigvee_{b \in X_2} \mu(x, b) \) for any \( x \in X_1 \) and \( \mu_{X_i}(y) = \bigvee_{a \in X_1} \mu(a, y) \) for any \( y \in X_2 \).

Theorem 5.4 Let \( X_1 \) and \( X_2 \) be two lattices and \( \mu \) be a \( TL \)-fuzzy ideal of \( X_1 \times X_2 \). Then \( \mu_{X_i} \) is a \( TL \)-fuzzy ideal of \( X_i \), \( i = 1, 2 \).

Proof Assume that \( \mu \) be a \( TL \)-fuzzy ideal of a lattice \( X_1 \times X_2 \). For any \( x, y \in X_1 \), then we have that
\[
\mu_{X_1}(x \lor y) = \bigvee_{b \in X_2} \mu(x \lor y, b) \geq \bigvee_{b_1, b_2 \in X_2} \mu(x \lor y, b_1 \lor b_2)
\]
\[
= \bigvee_{b_1, b_2 \in X_2} \mu((x, b_1) \lor (y, b_2)) \geq \bigvee_{b_1, b_2 \in X_2} [\mu(x, b_1) T \mu(y, b_2)]
\]
\[
= (\bigvee_{b_1 \in X_2} \mu(x, b_1)) T (\bigvee_{b_2 \in X_2} \mu(y, b_2)) = \mu_{X_1}(x) T \mu_{X_1}(y).
\]

On the other hand, we obtain
\[
\mu_{X_1}(x \land y) = \bigvee_{b \in X_2} \mu(x \land y, b) \geq \bigvee_{b_1, b_2 \in X_2} \mu(x \land y, b_1 \land b_2)
\]
\[
= \bigvee_{b_1, b_2 \in X_2} \mu((x, b_1) \land (y, b_2)) \geq \bigvee_{b_1, b_2 \in X_2} [\mu(x, b_1) \lor \mu(y, b_2)]
\]
\[
= (\bigvee_{b_1 \in X_2} \mu(x, b_1)) \lor (\bigvee_{b_2 \in X_2} \mu(y, b_2)) = \mu_{X_1}(x) \lor \mu_{X_1}(y).
\]

Therefore, \( \mu_{X_1} \) is a \( TL \)-fuzzy ideal of \( X_1 \). Dually, we have that \( \mu_{X_2} \) is a \( TL \)-fuzzy ideal of \( X_2 \).

Definition 5.5 Let \( \mu \in LF[X_1 \times X_2] \) and \( a \in X_1 \), \( b \in X_2 \). Then the cut shadow of \( \mu \) with respect to \( b \) denoted by \( \mu|_b \) is defined as the \( L \)-fuzzy subset of \( X_1 \) that satisfies: for any \( x \in X_1 \), \( \mu|_b(x) = \mu(x, b) \). Similarly, the cut shadow of \( \mu \) with respect to \( a \) denoted by \( \mu|_a \) is defined as the \( L \)-fuzzy subset of \( X_2 \) that satisfies: for any \( y \in X_2 \), \( \mu|_a(y) = \mu(a, y) \).

Theorem 5.6 Let \( X_1 \) and \( X_2 \) be two lattices and \( \mu \) be a \( TL \)-fuzzy ideal of \( X_1 \times X_2 \), let \( a \in X_1 \), \( b \in X_2 \). Then \( \mu|_b \) is a \( TL \)-fuzzy ideal of \( X_1 \) and \( \mu|_a \) is a \( TL \)-fuzzy ideal of \( X_2 \).

Proof The proof is straightforward.

In order to obtain necessary and sufficient conditions for a \( TL \)-fuzzy ideal of a Cartesian product lattice to be a \( T \)-product of fuzzy ideals of lattices, we give the following lemma.

Lemma 5.7 Let \( X_1 \) and \( X_2 \) be two lattices and \( \mu \) be a \( TL \)-fuzzy ideal of \( X_1 \times X_2 \) such that \( Im \mu \subseteq D_T \), let \( a \in X_1 \), \( b \in X_2 \). Then \( \mu|_b \times_T \mu|_a \subseteq \mu \subseteq \mu_{X_1} \times_T \mu_{X_2} \).

Proof Assume that \( \mu \) is a \( TL \)-fuzzy ideal of a lattice \( X_1 \times X_2 \).

First, we prove that \( \mu \subseteq \mu_{X_1} \times_T \mu_{X_2} \). For any \( (x, y) \in X_1 \times X_2 \), we have that
\[
\mu(x, y) \leq \bigvee_{b \in X_2} \mu(x, b) = \mu_{X_1}(x)
\]
and \( \mu(x, y) \leq \bigvee_{a \in X_1} \mu(a, y) = \mu_{X_2}(y) \).

Thus, we obtain $\mu(x,y)T\mu(x,y) \leq \mu_{X_1}(x)T\mu_{X_2}(y)$, then $\mu(x,y) \leq \mu_{X_1} \times_T \mu_{X_2}(x,y)$. Hence, $\mu \subseteq \mu_{X_1} \times_T \mu_{X_2}$.

Next, let us check $\mu_{[b \times_T 2]} \mid a \subseteq \mu$. For any $(x,y) \in X_1 \times X_2$, we can have

$$
\mu_{[b \times_T 2]}(x,y) = \mu_{[b \times_T 2]}(y) = \mu(x,b)T\mu(a,y) \leq \mu((x,b) \vee (a,y))
$$

Thus $\mu_{[b \times_T 2]} \subseteq \mu$. Combining the above arguments, we can obtain

$$
\mu_{[b \times_T 2]} \subseteq \mu \subseteq \mu_{X_1} \times_T \mu_{X_2}
$$

for all $a \in X_1, b \in X_2$.

The following theorem gives one of the main results in this paper.

**Theorem 5.8** Let $X_1$ and $X_2$ be two lattices with the bottom element 0 and $\mu$ be a $TL$-fuzzy ideal of $X_1 \times X_2$ such that $Im\mu \subseteq D_T$. Then $\mu$ is the $T$-product of a $TL$-fuzzy ideal of $X_1$ and a $TL$-fuzzy ideal of $X_2$ if and only if $\mu_{[0 \times_T 2]} \subseteq \mu_{X_1} \times_T \mu_{X_2}$.

**Proof** Assume that $\mu = \mu_{1 \times_T 2}$, where $\mu_1$ and $\mu_2$ are $TL$-fuzzy ideals of $X_1$ and $X_2$, respectively. Then $\mu_1(x) \leq \mu_1(0)$ for any $x \in X_1$ and $\mu_2(y) \leq \mu_2(0)$ for any $y \in X_2$. Thus we have $\bigvee_{x \in X_1} \mu_1(x) = \mu_1(0)$ and $\bigvee_{y \in X_2} \mu_2(y) = \mu_2(0)$. Notice this, we can obtain that

$$
\mu_{1 \mid 0}(x) = \mu(x,0) = \mu_1 \times_T \mu_2(2,0) = \mu_1(x)T\mu_2(0) = \mu_1(x)T(\bigvee_{y \in X_2} \mu_2(y))
$$

$$
= \bigvee_{y \in X_2} [\mu_1(x)T\mu_2(y)] = \bigvee_{y \in X_2} \mu(x,y) = \mu_{X_1}(x).
$$

Hence, $\mu_{[0 \mid 2]} = \mu_{X_1}$. Similarly, we can get that $\mu_{2 \mid 0} = \mu_{X_2}$. Therefore,

$$
\mu_{[0 \times_T 2]} = \mu_{X_1} \times_T \mu_{X_2}.
$$

Conversely, assume that $\mu_{[0 \times_T 2]} \subseteq \mu_{X_1} \times_T \mu_{X_2}$. By Lemma 5.7, we can get $\mu = \mu_{X_1} \times_T \mu_{X_2}$. Since $\mu$ is a $TL$-fuzzy ideal of $X_1 \times X_2$, it follows from Theorem 5.4, we have $\mu_{X_i}$ is a $TL$-fuzzy ideal of $X_i$, $i = 1, 2$. That is, $\mu$ is the $T$-product of a $TL$-fuzzy ideal of $X_1$ and a $TL$-fuzzy ideal of $X_2$.

**Open Problem** Whether the lattice of all $TL$-fuzzy ideals of a lattice $X$ forms a distributive lattice or even a modular lattice.

References


