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ON RINGS WITH SYMMETRIC ENDOMORPHISMS AND SYMMETRIC DERIVATIONS

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Abstract: In this paper, we study rings with symmetric endomorphisms and symmetric derivations. By using the property nil($R[x]$) =nil(R)$[x]$, we show that if R is weakly 2-primal, then R is a weak symmetric (σ,δ)-ring if and only if $R[x]$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring, which extend the research on symmetric rings and weak symmetric rings.

Keywords: symmetric ring; symmetric σ-ring; weak symmetric (σ,δ)-ring; weak 2-primal ring

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1 Introduction

Throughout this paper R denotes an associative ring with identity, $\sigma : R \longrightarrow R$ is a nonzero endomorphism. A ring R is called reduced if it has no nonzero nilpotent elements, and a ring R is called an abelian ring if all its idempotents are central. According to Cohn [4], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. Recently, Baser et al. [3] defined a ring R to be right (left) α-shifting if whenever $a\alpha(b) = 0$ ($\alpha(a)b = 0$) for $a, b \in R$, $b\alpha(a) = 0$ ($\alpha(b)a = 0$), which is a generalization of revesible rings. Recall that a ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. Baser et al. [2] extended the notion of semicommutative rings and called a ring R α-semicommutative if $ab = 0$ implies $aR\alpha(b) = 0$ for all $a, b \in R$. Another generalization of semicommutative rings is the semicommutative α-rings. Wang et al. [17] called a ring R right (left) semicommutative α-ring if $a\alpha(b) = 0$ ($\alpha(a)b = 0$) implies $\alpha(a)Rb = 0$ ($aR\alpha(b) = 0$) for all $a, b \in R$, and investigated characterizations of generalized semicommutative rings. According to Lamber

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[13], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. Anderson and Camillo [1] showed that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_\sigma(1) r_\sigma(2) \cdots r_\sigma(n) = 0$ for any permutation σ of the set $\{1, 2, \cdots, n\}$ and $r_i \in R$. There are many papers to study symmetric rings and their generalization (see [6, 8, 11, 14, 16]). In Kwak [12], an endomorphism α of a ring R is called right (left) symmetric if whenever $abc = 0$ for $a, b, c \in R$, $ac\alpha(b) = 0$ ($\alpha(b)ac = 0$). A ring R is called right (left) α-symmetric if there exists a right (left) symmetric endomorphism α of R. The notion of an α-symmetric ring is a generalization of α-rigid rings as well as an extension of symmetric rings. Following [15], a ring R is called a weak symmetric ring if $abc \in \mathrm{nil}(R)$ implies that $acb \in \mathrm{nil}(R)$ for all $a, b, c \in R$, where $\mathrm{nil}(R)$ is the set of all nilpotent elements of R. Let α be an endomorphism, and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. When $\alpha = id_R$, an α-derivation δ is called a derivation of R. A ring R is called a weak α-symmetric provided that $abc \in \mathrm{nil}(R)$ implies $ac\alpha(b) \in \mathrm{nil}(R)$ for $a, b, c \in R$. Moreover, R is called a weak δ-symmetric if for $a, b, c \in R$, $abc \in \mathrm{nil}(R)$ implies that $ac\delta(b) \in \mathrm{nil}(R)$. If R is both weak α-symmetric and weak δ-symmetric, then R is called a weak (α, δ)-symmetric ring. In [15], Ouyang and Chen studed the related properties of weak symmetric rings and weak (σ, δ)-symmetric rings.

Motivated by the above, for an endomorphism σ of a ring R, and a σ-derivation δ of the R, we introduce in this article the notions of symmetric σ-ring and weak symmetric (σ, δ)-rings to extend symmetric rings and weak symmetric rings respectively, and investigate their properties. First, we discuss the relationship between symmetric σ-rings and related rings. Next, we investigate the extension properties of weak symmetric (σ, δ)-rings. Several known results are obtained as corollaries of our results.

2 Symmetric σ-Rings and Related Rings

As a generalization of symmetric rings, we now introduce the notion of a symmetric σ-ring.

Definition 2.1 Let R be a ring, σ a nonzero endomorphism of R. We say that R is a symmetric σ-ring, if $ab\sigma(c) = 0$ implies $ac\sigma(b) = 0$, for any $a, b, c \in R$.

Similarly, a ring R is said to be a left symmetric σ-ring whenever $\sigma(a)bc = 0$ implies $\sigma(b)ac = 0$, for $a, b, c \in R$.

Obviously, if $\sigma = id_R$, the identity endomorphism of R, then a (left) symmetric σ-ring is a symmetric ring.

The next example shows that if $\sigma \neq id_R$, a symmetric σ-ring need not be symmetric and a symmetric σ-ring need not be a left symmetric σ-ring yet. Therefore, the classes of symmetric σ-ring and left symmetric σ-ring are non-trivial extension of symmetric rings, and the symmetric σ-property for a ring is not left-right symmetric, and the concepts of symmetric σ-rings and that of left symmetric σ-rings are independent of each other.

Example 2.2 Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the ring of integers, the endomorphism $\sigma : R \to R$, $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to verify that R is not symmetric. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix} \in R$$

with $\mathbf{AB}\sigma(\mathbf{C}) = 0$, then $a_1a_2a_3 = 0$, so we have $a_1a_3a_2 = 0$ and $\mathbf{AC}\sigma(\mathbf{B}) = 0$, concluding that R is a symmetric σ-ring. For

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R,$$

we have $\sigma(\mathbf{A})\mathbf{BC} = 0$, but $\sigma(\mathbf{B})\mathbf{AC} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$, thus R is not a symmetric σ-ring.

The next example provides that if $\sigma \neq id_R$, then there exists a symmetric ring which is not a symmetric σ-ring.

Example 2.3 Let $\mathbb{Z}_2$ be the ring of integers modulo 2. We consider ring $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a commutative reduced ring, and so R is symmetric. Now let $\sigma : R \longrightarrow R$ given by $\sigma((a, b)) = (b, a)$. Then σ is an endomorphism of R. For $A = (1, 0), B = (0, 1), C = (1, 1) \in R$, we have $AB\sigma(C) = (1, 0)(0, 1)(1, 1) = 0$, but $AC\sigma(B) = (1, 0)(1, 1)(1, 0) = (1, 0) \neq 0$. Thus R is not a symmetric σ-ring.

The next example shows that symmetric σ-rings need not be σ-rigid rings.

Example 2.4 Consider the ring $R = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}\right\}$ and the automorphism $\sigma : R \to R$,

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

R is not reduced and hence not σ-rigid. But R is a symmetric σ-ring. In fact, for any

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \mathbf{B} = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e & f \\ 0 & e \end{pmatrix} \in R$$

with $\mathbf{AB}\sigma(\mathbf{C}) = 0$, we have $ace = 0$, $-acf + ade + bce = 0$, it follows that $a = 0$ or $c = 0$ or $e = 0$. If $a = 0$, then $acf = ade = bce = 0$, and then $aec = -aed + afc + bec = 0$, hence

$$\mathbf{AC}\sigma(\mathbf{B}) = \begin{pmatrix} aec & -aed + afc + bec \\ 0 & aec \end{pmatrix} = 0.$$

Similarly, for $c = 0$ or $e = 0$, we have $\mathbf{AC}\sigma(\mathbf{B}) = 0$.

Proposition 2.5 For a nonzero endomorphism σ of a ring R, the following statements are equivalent:

(1) R is a symmtric σ-ring;

(2) $l_R(b\sigma(c)) \subseteq l_R(c\sigma(b))$ for any $a, b, c \in R$;

(3) $AB\sigma(C) = 0 \iff AC\sigma(B) = 0$ for any $A, B, C \subseteq R$

Proof (1) $\iff$ (3) Suppose $AC\sigma(B) = 0$ for $A, B, C \subseteq R$. Then $ab\sigma(c) = 0$ for any $a \in A, b \in B, c \in C$, and hence $ac\sigma(b) = 0$. Therefore, $AC\sigma(B) = \{\sum a_i c_i \sigma(b_i) | a_i \in A, b_i \in B, c_i \in C\} = 0$. The converse is obvious.

(1) $\iff$ (2) It is clear.

Proposition 2.6 Let σ be a nonzero endomorphism of a ring R. Then we have the following:

(1) If $\sigma^2 = id_R$, then R is a right (left) σ-shifting ring if and only if R is a right (left) semicommutative σ-ring;

(2) If R is a reversible ring, then R is a right (left) σ-shifting ring if and only if R is a right (left) semicommutative σ-ring.

Proof (1) Suppose that R is right σ-shifting and $a\sigma(b) = 0$ for $a, b \in R$. Then we have $b\sigma(a) = 0$, $\sigma(b)\sigma^2(a) = 0$ and $\sigma(b)\sigma^2(a)\alpha(R) = 0$. It implies that $\sigma(a)R\sigma^2(b) = 0$ since R is σ-shifting, and hence $\sigma(a)Rb = 0$ by $\sigma^2 = id_R$.

(2) Suppose that R is left σ-shifting and $\sigma(a)b = 0$ for $a, b \in R$. Then $b\sigma(a) = 0$ since R is reversible, and hence $b\sigma(a)\sigma(r) = b\sigma(ar) = 0$ for all $r \in R$. By the assumption, we have $ar\sigma(b) = 0$, including that R is a left semicommutative σ-ring. Conversely, assume that R is a left semicommutative σ-ring. If $a, b \in R$ with $a\sigma(b) = 0$, then $\sigma(b)a = 0$ since R is reversible. So we obtain that $bR\sigma(a) = 0$ since R is a left semicommutative σ-ring, and hence $b\sigma(a) = 0$. So R is left σ-shifting.

Proposition 2.7 Let σ be a monomorphism of a ring R. If R is a symmetric σ-ring, then R is semicommutative.

Proof Assume that R is a symmetric σ-ring with a monomorphism σ. Since $1 \in R$, R is a right σ-shifting ring. For $a, b \in R$, if $ab = 0$, then $\sigma(a)\sigma(b) = 0$, and hence $b\sigma(\sigma(a)) = 0$. So we have $rb\sigma(\sigma(a)) = 0$ and $\sigma(a)\sigma(rb) = \sigma(arb) = 0$ for any $r \in R$. It shows that $arb = 0$ since σ is a monomorphism of R, entailing that R is semicommutative.

Proposition 2.8 Let σ be an endomorphism of a ring R with $\sigma(e) = e$ for any $e^2 = e \in R$. If R is a symmetric σ-ring, then R, $R[x]$ and $R[x;\sigma]$ are all abelian.

Proof Assume that R is a symmetric σ-ring. Then R is a right σ-shifting ring. For any $r \in R$, we have

$$e\sigma(1-e)\sigma(r) = e\sigma((1-e)r) = 0,$$
$$(1-e)\sigma(e)\sigma(r) = (1-e)\sigma(er) = 0.$$

Hence $(1-e)r\sigma(e) = 0$, $er\sigma(1-e) = 0$ since R is right σ-shifting. Thus we get $re = ere = er$, proving that R is an abelian ring.

Now, suppose that $f^2(x) = f(x) \in R[x;\sigma]$, where $f(x) = \sum\limits_{i=0}^{m} e_i x^i$. Then we have,

$$\sum_{k=0}^{m}(\sum_{i+j=k} e_i\sigma^i(e_j))x^k = \sum_{i=0}^{m} e_i x^i.$$

It follows that the following system of equations:

$$e_0^2 = e_0; \tag{2.1}$$

$$e_0e_1 + e_1\sigma(e_0) = e_1; \tag{2.2}$$

$$e_0e_2 + e_2\sigma^2(e_0) + e_1\sigma(e_1) = e_2; \tag{2.3}$$

$$\vdots$$

$$e_0e_n + e_1\sigma(e_{n-1}) + e_2\sigma^2(e_{n-2}) + \cdots + e_ne_0 = e_n. \tag{2.4}$$

From eq. (2.2), we have $2e_1e_0 = e_1$, $2e_1e_0(1-e_0) = e_1(1-e_0)$ and $e_1 = e_1e_0$, $e_1 = 0$ since $\sigma(e_0) = e_0$ is central. Eq. (2.3) yields $2e_0e_2 = e_2$ and so $e_2 = 0$ by the same method as above. Continuing this procedure implies $e_i = 0$ for $i = 1, 2, \cdots, m$. Consequently, $f(x) = e_0 = e_0^2 \in R$ is central.

Let R_γ be a ring and σ_γ an endomorphism of R_γ for each $\gamma \in \Gamma$. Then $\sigma : \Pi_{\gamma\in\Gamma}R_\gamma \to \Pi_{\gamma\in\Gamma}R_\gamma$, $\sigma((a_\gamma)_{\gamma\in\Gamma}) = (\sigma_\gamma(a_\gamma))_{\gamma\in\Gamma}$ is an endomorphism of the direct product $\Pi_{\gamma\in\Gamma}R_\gamma$ of $R_\gamma, \gamma \in \Gamma$.

The following proposition is a direct verification.

Proposition 2.9 $\Pi_{\gamma\in\Gamma}R_\gamma$ is a symmetric σ-ring if and only if R_γ is a symmetric σ_γ-ring for each $\gamma \in \Gamma$.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

$T(R, M)$ is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R, m \in M$ and the usual matrix operations are used. For an endomorphism σ of a ring R , the map $\bar{\sigma} : T(R, R) \to T(R, R)$ defined by $\bar{\sigma}((a, b)) = (\sigma(a), \sigma(b))$ is an endomorphism of $T(R, R)$, where $(a, b) \in T(R, R),\ a, b \in R$.

Proposition 2.10 Let R be a reduced ring with an endomorphism σ. If R is a symmetric σ-ring, then $T(R, R)$ is a symmetric $\bar{\sigma}$-ring.

Proof Suppose that R is a symmetric σ-ring. Let $\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} a_3 & b_3 \\ 0 & a_3 \end{pmatrix} \in T(R, R)$ with $\mathbf{AB}\bar{\sigma}(\mathbf{C}) = 0$. Then we have

$$a_1a_2\sigma(a_3) = 0; \tag{2.5}$$

$$a_1a_2\sigma(b_3) + a_1b_2\sigma(a_3) + b_1a_2\sigma(a_3) = 0. \tag{2.6}$$

It is known that reduced rings are symmetric rings. Multiplying eq. (2.5) on the right side by b_1 gives $a_1b_1a_2\sigma(a_3) = 0$. If we multiply eq. (2.6) on the left side by a_1, then we have

$$a_1a_1a_2\sigma(b_3) + a_1a_1b_2\sigma(a_3) = 0. \tag{2.7}$$

Multiplying eq. (2.5) on the left side by a_1 and on the right side by b_2 gives $a_1a_1b_2\sigma(a_3)a_2 = 0$. Multiplying eq. (2.7) by a_2 on the right side gives $0 = a_1a_1a_2\sigma(b_3)a_2 = a_1a_2\sigma(b_3)a_1a_2\sigma(b_3) = (a_1a_2\sigma(b_3))^2$, so $a_1a_2\sigma(b_3) = 0$. Thus we have the following equation

$$a_1b_2\sigma(a_3) + b_1a_2\sigma(a_3) = 0. \tag{2.8}$$

If we multiply eq. (2.5) by b_2 on the right side, then we get $a_1b_2\sigma(a_3)a_2 = 0$. Multiplying eq. (2.8) by a_2 on the right side gives $0 = b_1a_2\sigma(a_3)a_2 = b_1a_2\sigma(a_3)a_2b_1\sigma(a_3) = (b_1a_2\sigma(a_3))^2$. Thus we obtain $b_1a_2\sigma(a_3) = 0$, $a_1b_2\sigma(a_3) = 0$, and hence we have $a_1a_3\sigma(a_2) = a_1b_3\sigma(a_2) = a_1a_3\sigma(b_2) = b_1a_3\sigma(a_2) = 0$ since R is a symmetric σ-ring. So $\mathbf{AC}\bar{\sigma}(\mathbf{B}) = 0$, proving that $T(R,R)$ is a symmetric $\bar{\sigma}$-ring.

Corollary 2.11 (see [8], Corollary 2.4) Let R be a reduced ring, then $T(R,R)$ is a symmetric ring.

Proposition 2.12 Let σ be an endomorphism of an abelian ring R with $\sigma(e) = e$ for any $e^2 = e \in R$. Then the following statements are equivalent:

(1) R is a symmetric σ-ring;

(2) eR and $(1-e)R$ are symmetric σ-rings.

Proof (1) $\Rightarrow$ (2) Since $\sigma(eR) \subseteq eR$, $\sigma((1-e)R) \subseteq (1-e)R$, it is obvious by the definition.

(2) $\Rightarrow$ (1) Let $a, b, c \in R$ with $ab\sigma(c) = 0$. Then $eab\sigma(c) = 0$ and $(1-e)ab\sigma(c) = 0$. By the assumption, we get $eab\sigma(c) = e^3ab\sigma(c) = eaebe\sigma(c) = eaeb\sigma(ec) = 0$ and $(1-e)ab\sigma(c) = (1-e)a(1-e)b\sigma((1-e)c) = 0$. Since eR and $(1-e)R$ are symmetric σ-rings, $eaec\sigma(eb) = eac\sigma(b) = 0$ and $(1-e)a(1-e)c\sigma((1-e)b) = (1-e)ac\sigma(b) = 0$, hence $ac\sigma(b) = eac\sigma(b) + (1-e)ac\sigma(b) = 0$, proving that R is a symmetric α-ring.

Corollary 2.13 (see [8], Proposition 3.6(2)) Let R be an abelian ring. Then R is symmetric if and only if eR and $(1-e)R$ are symmetric.

Recall that for a monomorphism σ of a ring R, an over-ring A of R is a Jordan extension of R if σ can be extended to an automorphism of A and $A = \bigcup\limits_{n=0}^{\infty} \sigma^{-n}(R)$ (see [10]).

Proposition 2.14 Let A be the corresponding Jordan extension of a ring R and σ be a monomorphism of R. Then R is a symmetric σ-ring if and only if A is a symmetric σ-ring.

Proof Since $\sigma(R) \subseteq R$, it suffices to obtain the necessity.

Assume that R is a symmetric σ-ring and $ab\sigma(c) = 0$ for $a, b, c \in A$. By the definition of A, there exists $n \geqslant 0$ such that $\sigma^n(a)$, $\sigma^n(b)$, $\sigma^n(c) \in R$. It follows that $\sigma^n(a)\sigma^n(b)\sigma(\sigma^n(c)) = \sigma^n(ab\sigma(c)) = 0$. Since R is a symmetric σ-ring, $\sigma^n(a)\sigma^n(c)\sigma(\sigma^n(b)) = \sigma^n(ac\sigma(b)) = 0$. Then, we have $ac\sigma(b) = 0$ since σ is a monomorphism, and proving that A is a symmetric σ-ring.

Proposition 2.15 Let R be a ring with an endomorphism σ, S a ring and $\tau: R \to S$ a ring isomorphism. Then R is a symmetric σ-ring if and only if S is a symmetric $\tau\sigma\tau^{-1}$-ring.

Proof For $a, b, c \in R$, let $a' = \tau(a)$, $b' = \tau(b)$ and $c' = \tau(c) \in S$. Suppose that R is a symmetric σ-ring and $a'b'\tau\sigma\tau^{-1}(c') = 0$ for $a', b', c' \in S$. Then we have $\tau(a)\tau(b)\tau\sigma\tau^{-1}(\tau(c)) = \tau(ab\sigma(c)) = 0$, hence $ab\sigma(c) = 0$ since τ is a isomorphism. By the assumption, we get $ac\sigma(b) = 0$, so $a'c'\tau\sigma\tau^{-1}(b') = \tau(ac\sigma(b)) = 0$, including that S is a symmetric $\tau\sigma\tau^{-1}$-ring. On the contrary, assume that S is a symmetric $\tau\sigma\tau^{-1}$-ring and $ab\sigma(c) = 0$ for $a, b, c \in R$. Then $a^{'}b^{'}\tau\sigma\tau^{-1}(c^{'}) = \tau(ab\sigma(c)) = 0$. By the assumption, we get $a^{'}c^{'}\tau\sigma\tau^{-1}(b^{'}) = \tau(ac\sigma(b)) = 0$, this implies $ac\sigma(b) = 0$. So R is a symmetric σ-ring.

3 Weak Symmetric (σ, δ)-Rings and their Extensions

As a extended weak symmetric rings, we now introduce the notion of a weak symmetric (σ, δ)-ring.

Definition 3.1 Let σ be an endomorphism and δ a σ-derivation of a ring R. A ring R is called a weak symmetric σ-ring if $ab\sigma(c) \in \mathrm{nil}(R)$ implies $ac\sigma(b) \in \mathrm{nil}(R)$, for $a, b, c \in R$. Moreover, R is called a weak symmetric δ-ring if for $a, b, c \in R, ab\delta(c) \in \mathrm{nil}(R)$ implies $ac\delta(b) \in \mathrm{nil}(R)$. If R is both a weak symmetric σ-ring and a weak symmetric δ-ring, then R is called a weak symmetric (σ, δ)-ring.

Similarly, a ring R is said to be a left weak symmetric (σ, δ)-ring if $\sigma(a)bc \in \mathrm{nil}(R)$ then $\sigma(b)ac \in \mathrm{nil}(R)$, and if $\delta(a)bc \in \mathrm{nil}(R)$ then $\delta(b)ac \in \mathrm{nil}(R)$, for $a, b, c \in R$.

It is easy to see that every subring S with $\sigma(S) \subseteq S$, $\delta(S) \subseteq S$ of a (left) weak symmetric (σ, δ)-ring is also a (left) weak symmetric (σ, δ)-ring.

Consider the R and σ in Example 2.3. Taking $\delta = 0$, then this example shows that the notions of weak symmetric (σ, δ)-rings are not left-right symmetric. Obviously, if $\sigma = id_R$, $\delta = 0$, then a (left) weak symmetric (σ, δ)-ring is a weak symmetric ring. The next example provides that if $\sigma \neq id_R$, $\delta \neq 0$, then there exists a weak symmetric ring which is not a weak symmetric (σ, δ)-ring.

Example 3.2 Let $\mathbb{Z}_2$ be the ring of integers modulo 2, and consider the ring $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a commutative reduced ring, and so R is weak symmetric. Now let $\sigma : R \longrightarrow R$ given by $\sigma((a, b)) = (b, a)$ and $\sigma : R \longrightarrow R$ given by $\delta((a, b)) = (1, 0)(a, b) - \sigma(a, b)(1, 0)$ for each $(a, b) \in \mathbb{Z}_2$. Then σ is an endomorphism of R and δ is a σ-derivation of R. For $A = (1, 0), B = (0, 1), C = (1, 1) \in R$, we have $AB\sigma(C) = (1, 0)(0, 1)(1, 1) = 0 \in \mathrm{nil}(R)$, but $AC\sigma(B) = (1, 0)(1, 1)(1, 0) = (1, 0)$ is not in $\mathrm{nil}(R)$. Thus R is not weak symmetric (σ, δ)-ring.

In the following, we always suppose that σ is an endomorphism and δ a σ-derivation of R.

Now we consider the n-by-n upper triangular matrix ring $T_n(R)$ over R and the ring

$$S_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} | a_i \in R, \ i = 0, 1, \cdots, n-1 \right\}.$$

For an endomorphism σ and a σ-derivation δ of R, the natural extension $\bar{\sigma} : T_n(R) \longrightarrow T_n(R)$ defined by $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ is an endomorphism of $T_n(R)$ and $\bar{\delta} : T_n(R) \longrightarrow T_n(R)$ defined by $\delta((a_{ij})) = (\delta(a_{ij}))$ is a $\bar{\sigma}$-derivation of $T_n(R)$.

Proposition 3.3 The following statements are equivalent:

(1) R is a weak symmetric (σ,δ)-ring;

(2) $T_n(R)$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring;

(3) $S_n(R)$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring.

Proof (1) $\Longrightarrow$ (2) Let $\mathbf{A}=(a_{ij}),\mathbf{B}=(b_{ij}),\mathbf{C}=(c_{ij})\in T_n(R)$, where $a_{ij}=0,b_{ij}=0,c_{ij}=0$, for all $i>j$, with $\mathbf{AB}\sigma(\mathbf{C})\in \mathrm{nil}(T_n(R))$ and $\mathbf{AB}\delta(\mathbf{C})\in \mathrm{nil}(T_n(R))$. Then $a_{ii}b_{ii}\sigma(c_{ii})\in \mathrm{nil}(R)$, $a_{ii}b_{ii}\delta(c_{ii})\in \mathrm{nil}(R)$ for all $0\le i\le n$, and so $a_{ii}c_{ii}\sigma(b_{ii})\in \mathrm{nil}(R)$, $a_{ii}c_{ii}\delta(b_{ii})\in \mathrm{nil}(R)$ since R is a weak symmetric (σ,δ)-ring. It follows that $\mathbf{AC}\bar{\sigma}(\mathbf{B})\in \mathrm{nil}(T_n(R))$ and $\mathbf{AC}\bar{\delta}(\mathbf{B})\in \mathrm{nil}(T_n(R))$. Therefore, $T_n(R)$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring.

(2) $\Longrightarrow$ (1) Suppose that $T_n(R)$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring. For $a,b,c\in R$ with $ab\alpha(c)=0$ and $ab\delta(c)=0$, we have $aEbE\bar{\sigma}(cE)=0$ and $aEbE\bar{\delta}(cE)=0$, and hence $aEcE\bar{\sigma}(bE)=0$ and $aEcE\bar{\delta}(bE)=0$ since $T_n(R)$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring, where E denote the identity matrix. This implies that $ac\sigma(b)=0$ and $ac\delta(b)=0$. So R is a weak symmetric (σ,δ)-ring.

(1) $\Longleftrightarrow$ (3) It is similar to (1) $\Longleftrightarrow$ (2).

Corollary 3.4 The trivial extension $T(R,R)$ of R by R is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring if and only if R is a weak symmetric (σ,δ)-ring.

Proof By the isomorphism $T(R,R)\cong T_2(R)$, we obtain the proof.

Corollary 3.5 $R[x]/\langle x^n\rangle$ is a weak symmetric $(\bar{\sigma},\bar{\delta})$-ring if and only if R is a weak symmetric (σ,δ)-ring, where $\langle x^n\rangle$ is an ideal of R generated by x^n and n is any positive integer.

Proof By the isomorphism $R[x]/\langle x^n\rangle\cong S_n(R)$, we obtain the proof.

An ring R is said to be an NI ring [9] provided that $\mathrm{nil}(R)=\mathrm{Nil}^*(R)$, where $\mathrm{Nil}^*(R)$ denotes the upper nil radical of R.

Proposition 3.6 Let R be an NI ring, $e^2=e\in R$ a central idempotent element of R. If $\sigma(e)=e$, $\sigma(1)=1$, $\delta(e)=\delta(1)=0$, then the following statements are equivalent:

(1) R is a weak symmetric (σ,δ)-ring;

(2) eR and $(1-e)R$ are weak symmetric (σ,δ)-rings.

Proof (1) $\Longrightarrow$ (2) Suppose that $ab\sigma(c)\in \mathrm{nil}(I)$, $ab\delta(c)\in \mathrm{nil}(I)$ for $a,b,c\in I$, where I denotes eR (resp., $(1-e)R$). Then we have $ac\sigma(b)\in \mathrm{nil}(R)$, $ac\delta(b)\in \mathrm{nil}(R)$ since R is a weak symmetric (σ,δ)-ring, and hence $ac\sigma(b)\in(\mathrm{nil}(R)\bigcap I)=\mathrm{nil}(I)$, $ac\delta(b)\in(\mathrm{nil}(R)\bigcap I)=\mathrm{nil}(I)$.

(2) $\Longrightarrow$ (1) Let $a,b,c\in R$ with $ab\sigma(c)\in \mathrm{nil}(R)$, $ab\delta(c)\in \mathrm{nil}(R)$. Then $eaebe\sigma(c)\in \mathrm{nil}(eR)$ and $(1-e)a(1-e)b(1-e)\sigma(c)\in \mathrm{nil}((1-e)R)$ since eR, $(1-e)R$ are ideals of R and $e\in eR$, $1-e\in(1-e)R$. It follows that $eaece\sigma(b)=eac\sigma(b)\in \mathrm{nil}(eR)$, and $(1-e)a(1-e)c\sigma((1-e)b)=(1-e)ac\sigma(b)\in \mathrm{nil}((1-e)R)$ since eR and $(1-e)R$ are weak symmetric (σ,δ)-rings. Hence $ac\sigma(b)\in \mathrm{nil}(R)$ because $\mathrm{nil}(R)$ is an ideal of R. On the other hand, by assumption we have $\delta(ex)=\delta(e)x+\sigma(e)\delta(x)=e\delta(x)$ and $\delta((1-e)x)=(1-e)\delta(x)$ for any $x\in R$. Thus, from $ab\delta(c)\in \mathrm{nil}(R)$ we have $eaeb\delta(ec)\in \mathrm{nil}(R),(1-e)a(1-e)b\delta((1-e)c)\in \mathrm{nil}(R)$. Hence $eaec\sigma(eb)=eac\delta(b)\in \mathrm{nil}(R)$ and $1-e)a(1-e)c\delta((1-e)b)=(1-e)ac\delta(b)\in \mathrm{nil}(R)$ since eR and $(1-e)R$ are weak symmetric (σ,δ)-rings. This implies that $ac\delta(b)\in \mathrm{nil}(R)$ since R

is an NI ring. Therefore, R is a weak symmetric (σ, δ)-ring.

An ideal I of a ring R is said to be (σ, δ)-stable if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$. If I is a (σ, δ)-stable ideal, then $\bar{\sigma} : R/I \longrightarrow R/I$ defined by $\bar{\sigma}(\bar{a}) = \overline{\sigma(a)}$ for $\bar{a} \in R/I$ is an endomorphism of the factor ring R/I, and $\bar{\delta} : R/I \longrightarrow R/I$ defined by $\delta(\bar{a}) = \overline{\delta(a)}$ for $\bar{a} \in R/I$ is an additive map of the ring R/I. We can easily see that $\bar{\delta}$ is a $\bar{\sigma}$-derivation of the ring R/I.

Theorem 3.7 Let I be a (σ, δ)-stable and weak symmetric (σ, δ)-ideal of R. If $I \subseteq \mathrm{nil}(R)$, then R/I is a weak symmetric $(\bar{\sigma}, \bar{\delta})$-ring if and only R is a weak symmetric (σ, δ)-ring.

Proof Suppose $\bar{a}\bar{b}\bar{\sigma}(\bar{c}) \in \mathrm{nil}(R/I)$ and $\bar{a}\bar{b}\bar{\delta}(\bar{c}) \in \mathrm{nil}(R/I)$. Then there exist some positive integer m, n such that $(ab\sigma(c))^n \in I$, $(ab\delta(c))^n \in I$. Thus $ab\sigma(c) \in \mathrm{nil}(R)$ and $ab\delta(c) \in \mathrm{nil}(R)$ since $I \subseteq \mathrm{nil}(R)$. Because R is a weak symmetric (σ, δ)-ring, we get $ac\sigma(b) \in \mathrm{nil}(R)$ and $ac\delta(b) \in \mathrm{nil}(R)$. It follows that $\bar{a}\bar{c}\bar{\sigma}(\bar{b}) \in \mathrm{nil}(R/I)$ and $\bar{a}\bar{c}\bar{\delta}(\bar{b}) \in \mathrm{nil}(R/I)$. Hence R/I is a weak symmetric (σ, δ)-ring.

Conversely, assume that R/I is a weak symmetric $(\bar{\sigma}, \bar{\delta})$-ring. Let $ab\sigma(c) \in \mathrm{nil}(R)$, $ab\delta(c) \in \mathrm{nil}(R)$ for $a, b, c \in R$. Then $\bar{a}\bar{b}\sigma\bar{(c)} \in \mathrm{nil}(R/I)$, $\bar{a}\bar{b}\delta\bar{(c)} \in \mathrm{nil}(R/I)$. Thus we have $\bar{a}\bar{c}\sigma\bar{(b)} = \overline{ac\sigma(b)} \in \mathrm{nil}(R/I)$, and $\bar{a}\bar{c}\delta\bar{(b)} = \overline{ac\delta(b)} \in \mathrm{nil}(R/I)$ since R/I is a weak symmetric (σ, δ)-ring. So there exist some positive integers s and t such that $(ac\sigma(b))^s \in I$ and $(ac\delta(b))^t \in I$. Thus $ac\sigma(b) \in \mathrm{nil}(I)$ and $ac\delta(b) \in \mathrm{nil}(I)$. Therefore, R is a weak symmetric (σ, δ)-ring.

Corollary 3.8 Let σ be an endomorphism and I a weak symmetric σ-ideal of R. If $I \subseteq \mathrm{nil}(R)$, then R/I is a weak symmetric $\bar{\sigma}$-ring if and only R is a weak symmetric σ-ring.

Corollary 3.9 Let δ be a derivation and I a weak symmetric δ-ideal of R. If $I \subseteq \mathrm{nil}(R)$, then R/I is a weak symmetric $\bar{\delta}$-ring if and only R is a weak symmetric δ-ring.

Corollary 3.10 Let I be a weak symmetric ideal of R. If $I \subseteq \mathrm{nil}(R)$, then R/I is a weak symmetric ring if and only R is a weak symmetric ring.

According to Chen et al. [5], a ring R is called weakly 2-primal if the set of nilpotent elements in R coincides with its Levitzki radical, that is, $nil(R) = L\text{-rad}(R)$. Semicommutative rings, 2-primal rings [9] and locally 2-primal rings [6] are weakly 2-primal rings, and weakly 2-primal rings are NI-ring.

Lemma 3.11 If R is a weakly 2-primal ring and $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$. Then $f(x) \in \mathrm{nil}(R[x])$ if and only if $a_i \in \mathrm{nil}(R)$ for each $0 \le i \le n$. that is, we have

$$\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x].$$

Proof Suppose that $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] \in \mathrm{nil}(R[x])$. Then by [7], Proposition 1.3, we obtain $a_i \in \mathrm{nil}(R)$ for each $0 \le i \le n$, and so $\mathrm{nil}(R[x]) \subseteq \mathrm{nil}(R)[x]$. Now assume that

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] \in \mathrm{nil}(R)[x].$$

Consider the finite subset $\{a_0, a_1, \cdots, a_n\}$. Since R is weakly 2-primal and hence $\mathrm{nil}(R) = L$-rad(R). Then the subring $\langle a_0, a_1, \cdots, a_n\rangle$ of R generated by $\{a_0, a_1, \cdots, a_n\}$ is nilpotent, so there exists a positive integer k such that any product of k elements $a_{i1}a_{i2}\cdots a_{ik}$ from

$\{a_0, a_1, \cdots, a_n\}$ is zero. Hence we obtain that $f(x)^{k+1} = 0$ and so $f(x) \in \text{nil}(R[x])$. Thus, we have $\text{nil}(R[x]) = \text{nil}(R)[x]$.

Let σ be an endomorphism and δ a σ-derivation of R. Then the map $\bar{\sigma} : R[x] \longrightarrow R[x]$ defined by $\bar{\sigma}(\sum\limits_{i=0}^{m} a_i x^i) = \sum\limits_{i=0}^{m} \sigma(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends σ, and the σ-derivation δ of R is also extended to $\bar{\delta} : R[x] \longrightarrow R[x]$ defined by $\bar{\delta}(\sum\limits_{i=0}^{m} a_i x^i) = \sum\limits_{i=0}^{m} \delta(a_i)x^i$. We can easily see that $\bar{\delta}$ is a $\bar{\sigma}$-derivation of the ring $R[x]$.

Theorem 3.12 Let R be a weakly 2-primal ring, σ an endomorphism and δ a σ-derivation of R. Then R is a weak symmetric (σ, δ)-ring if and only if $R[x]$ is a weak symmetric $(\bar{\sigma}, \bar{\delta})$-ring.

Proof Since any subring S with $\sigma(S) \subseteq S$, $\delta(S) \subseteq S$ of a (left) weak symmetric (σ, δ)-ring is also a (left) weak symmetric (σ, δ)-ring. Thus it is easy to verify that if $R[x]$ is a weak symmetric $(\bar{\sigma}, \bar{\delta})$-ring, then R is a weak symmetric (σ, δ)-ring.

Conversely, assume that R is a weak symmetric (σ, δ)-ring. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m$, and $h(x) = c_0 + c_1x + \cdots + c_lx^l \in R[x]$ with $fg\bar{\delta}(h) \in \text{nil}(R[x])$. Then we have the following equations by Lemma 3.11:

$$a_0b_0\delta(c_0) = \Delta_0 \in \text{nil}(R); \tag{3.1}$$

$$a_0b_0\delta(c_1) + a_0b_1\delta(c_0) + a_1b_0\delta(c_0) = \Delta_1 \in \text{nil}(R); \tag{3.2}$$

$$\begin{aligned} & a_0b_0\delta(c_2) + a_0b_1\delta(c_1) + a_0b_2\delta(c_0) + a_1b_0\delta(c_1) + a_1b_1\delta(c_0) + a_2b_0\delta(c_0) \\ = {} & \Delta_2 \in \text{nil}(R); \end{aligned} \tag{3.3}$$

$$\vdots$$

$$a_0b_0\delta(c_{n-1}) + a_0b_1\delta(c_{n-2}) + \cdots + a_{n-2}b_1\delta(c_0) + a_{n-1}b_0\delta(c_0) = \Delta_{n-1} \in \text{nil}(R); \tag{3.4}$$

$$a_0b_0\delta(c_n) + a_0b_1\delta(c_{n-1}) + \cdots + a_{n-1}b_1\delta(c_0) + a_nb_0\delta(c_0) = \Delta_n \in \text{nil}(R). \tag{3.5}$$

Since R is NI, $\text{nil}(R)$ is an ideal of R. eq. (3.1) implies $\delta(c_0)a_0b_0 \in \text{nil}(R)$, $b_0\delta(c_0)a_0 \in \text{nil}(R)$. If multiply eq. (3.2) on the left side by $b_0\delta(c_0)$, then we have $b_0\delta(c_0)a_0b_0\delta(c_1) \in \text{nil}(R)$, $b_0\delta(c_0)a_0b_1\delta(c_0) \in \text{nil}(R)$. It implies that $b_0\delta(c_0)a_1b_0\delta(c_0) \in \text{nil}(R)$ and $a_1b_0\delta(c_0) \in \text{nil}(R)$. So we obtain that

$$a_0b_0\delta(c_1) + a_0b_1\delta(c_0) = \Delta_1^{'} \in \text{nil}(R). \tag{3.6}$$

If multiply eq. (3.6) on the right side by a_0b_0, then we have $a_0b_1\delta(c_0)a_0b_0 \in \text{nil}(R)$, $a_0b_0\delta(c_1)a_0b_0 \in \text{nil}(R)$, and hence $a_0b_1\delta(c_0) \in \text{nil}(R)$, $a_0b_0\delta(c_1) \in \text{nil}(R)$.

If multiply eq. (3.3) on the right side by a_0b_0, a_0b_1, a_0b_2, and a_1b_0, respectively, then we obtain $a_0b_0\delta(c_2)$, $a_0b_1\delta(c_1)$, $a_0b_2\delta(c_0)$, $a_2b_0\delta(c_0)$, $a_1b_0\delta(c_1)$, $a_1b_1\delta(c_0) \in \text{nil}(R)$ in turn.

Inductively assume that $a_ib_j\delta(c_k) \in \text{nil}(R)$ for $i + j + k \leq n - 1$. We apply the above method to eq. (3.5). First, If multiply eq. (3.5) on the left side by $b_0\delta(c_0)$, then we have $a_nb_0\delta(c_0) \in \text{nil}(R)$ by the induction hypotheses, and

$$a_0b_1\delta(c_{n-1}) + a_0b_2\delta(c_{n-2}) + \cdots + a_{n-1}b_1\delta(c_0) = \Delta_n^{''} \in \text{nil}(R). \tag{3.7}$$

If we multiply eq. (15) on the right side by a_0b_1, it gives $a_0b_1\delta(c_{n-1}) \in \text{nil}(R)$, and

$$a_0b_2\delta(c_{n-2}) + a_0b_3\delta(c_{n-3}) + \cdots + a_{n-1}b_1\delta(c_0) = \Delta_n^{'''} \in \text{nil}(R). \tag{3.8}$$

If multiply eq. (3.8) on the right side by a_0b_2, $a_0b_3, \cdots, a_{n-1}b_1$, respectively, then we obtain $a_0b_2\delta(c_{n-2}) \in \text{nil}(R)$, $a_0b_3\delta(c_{n-3}) \in \text{nil}(R), \cdots$, $a_{n-1}b_1\delta(c_0) \in \text{nil}(R)$ in turn. By induction, this shows that $a_ib_j\delta(c_k) = 0$ for all i, j and k with $i + j + k = n$, and hence $a_ic_k\delta(b_j) \in \text{nil}(R)$, for all i, j, k with $i + j + k \leq n$ since R is a weak symmetric (σ, δ)-ring. Since the coefficients of $fh\bar{\delta}(g)$ can be written as sums $\sum a_ic_k\delta(b_j)$ and $nil(R)$ is an ideal of R, this yields $fh\bar{\delta}(g) \in \text{nil}(R)$ by Lemma 3.11.

Similarly, if $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m$, and $h(x) = c_0 + c_1x + \cdots + c_lx^l \in R[x]$ with $fg\bar{\sigma}(h) \in \text{nil}(R[x])$, by the same method as above, we can obtain $a_ic_k\sigma(b_j) \in \text{nil}(R)$ for all i, j, k with $i + j + k \leq n$. This yields $fh\bar{\sigma}(g) \in \text{nil}(R)$ by Lemma 3.11. Therefore, $R[x]$ is a weak symmetric $(\bar{\sigma}, \bar{\delta})$-ring.

Corollary 3.13 Let R be a weakly 2-primal ring and σ an endomorphism of R. Then R is a weak symmetric σ-ring if and only if $R[x]$ is a weak symmetric $\bar{\sigma}$-ring.

Corollary 3.14 Let R be a weakly 2-primal ring and δ a derivation of R. Then R is a weak symmetric δ-ring if and only if $R[x]$ is a weak symmetric $\bar{\delta}$-ring.

Corollary 3.15 Let R be a weakly 2-primal ring. Then R is a weak symmetric ring if and only if $R[x]$ is a weak symmetric ring.

Corollary 3.16(see [15], Corollary 3.10) Let R be a semicommutative ring. Then R is weak symmetric if and only if $R[x]$ is weak symmetric.

References

[1] Anderson D D, Camillo V. Semigroups and rings whose zero products commute[J]. Comm. Algebra, 1999, 27(6): 2847–2852.

[2] Baser M, Harmanci A, Kwak T K. Generalized semicommutative rings and their extensions[J]. Bull. Korean Math. Soc., 2008, 45(2): 285–297.

[3] Baser M, Harmanci A, Kwak T K. Ring endomorphisms with the reversible condition[J]. Comm. Korean Math. Soc., 2010, 25(30): 349–364.

[4] Cohn P M. Reversible rings[J]. Bull. London Math. Soc., 1999, 31(6): 641–648.

[5] Chen Weixing, Cui Shuying. On weakly semicommutative rings[J]. Comm. Math. Res., 2011, 27(2): 179–192.

[6] Hong C Y, Kim H K, Kim N K, Kwak T K, Lee Y, Park K S. Rings whose nilpotent elements form a Levitzki radical ring[J]. Comm. Algebra, 2007,35: 1379–1390.

[7] Huh C, Lee C I, Park K S, Ryu S T. On Π-Armendariz rings[J]. Bull. Korean Math. Soc., 2007, 44: 641–649.

[8] Huh C, Kim H K, Kim N K, Lee Y. Basic examples and extensions of symmetric rings[J]. J. Pure Appl. Algebra, 2005, 202(1-3): 154–167.

[9] Hwang S U, Jeon Y C, Lee Y. Structure and topological conditions of NI rings[J]. J. Algebra, 2006,302: 186–199.

[10] Jordan D A. Bijective extensions of injective rings endomorphism[J]. J. Lond. Math. Soc., 1982, 25: 435–448.

[11] Kafkas G, Ungor B, Halicioglu S, Harmanci A. Generalized symmetric rings[J]. Algebra Discrete Math., 2011, 12(2): 72–84.

[12] Kwak T K. Extensions of extended symmetric rings[J]. Bull. Korean Math. Soc., 2007, 44(4): 777–788.

[13] Lambek J. On the representation of modules by sheaves of factor modules[J]. Canad. Math. Bull., 1971, 14: 359–368.

[14] Marks G. Reversible and symmetric rings[J]. J. Pure Appl. Algebra, 2002, 174: 311–318.

[15] Ouyang Lunqun, Chen Huanyin. On weak symmetric rings[J]. Comm. Algebra, 2010, 38: 697–713.

[16] Shin G. Prime ideal and sheaf representation of apseudo symmetric ring[J]. Trans. Amer. Math. Soc., 1973, 184: 43–60.

[17] Wang Yao, Shen Qing, Ren Yanli. Rings with the semicommutative endomorphisms(Chinese)[J]. J. Jilin Univ.(Science Edition), 2013, 51(6): 423–433.

具有对称自同态与对称导子的环

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摘要: 本文研究具有对称自同态和对称导子的环. 利用性质nil($R[x]$) =nil(R)[x], 我们证明了: 如果 R 是弱 2-primal 环, 则 R 是弱对称 (σ,δ) - 环当且仅当 $R[x]$ 是弱对称 $(\bar{\sigma},\bar{\delta})$ -环. 本文结论拓展了关于对称环和弱对称环的研究.

关键词: 对称环; 对称 σ - 环; 弱对称 (σ,δ) - 环; 弱 2-primal 环

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