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PORTFOLIO OPTIMIZATION PROBLEMS WITH LOGARITHMIC UTILITY IN CIR INTEREST RATE MODEL

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Abstract: In this paper, we study the optimal long term investment problem and optimal discounted consumption problem on infinite time horizon with logarithmic utility in CIR interest rate model. By solving the corresponding dynamic programming equations, we obtain the optimal strategies and value functions for the two optimization problems in explicit form.

Keywords: Cox-Ingersoll-Ross interest rate; logarithmic utility; optimal investment; optimal discounted consumption; dynamic programming equation

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1 Introduction

The financial market considered in this paper is complete and consists of one bank account and one risky stock. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space with the augmented Brownian filtration generated by the two dimensional Brownian motion $W := (W_1, W_2)^*$ ($(\cdot)^*$ denotes the transpose of a vector or a matrix). We assume the bank account process $B = (B_t)_{t\geq 0}$ and the price process $S = (S_t)_{t\geq 0}$ of the risky stock evolve respectively according to

$$dB_t = B_t r(Y_t)dt,\ B_0 > 0$$

and

$$dS_t = S_t \left\{\mu(Y_t)dt + e^* dW(t)\right\},\ S_0 > 0,$$

where $r(y),\ \mu(y)$ are two continuous functions and e is defined by

$$e := (\sqrt{1-\rho^2}, \rho)^* \in [0,1] \times [-1,1],$$

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here Y is a stochastic factor process which affects the mean-return-rate $\mu(Y_t)$ of S and the risk-free interest rate $r(Y_t)$ of B, given by

$$dY_t = \left(\frac{b_{-1}}{Y_t} + bY_t\right) dt + \sigma dW_2(t),\ Y_0 = y > 0, \tag{1.1}$$

where $b, b_{-1}, \sigma \in \mathbb{R}$ and $b \neq 0$.

As Hata ect (see [1, 2]), in present paper, we treat the setting

$$r(y) = y^2, \quad \mu(y) = y^2 + \lambda y,$$

where $\lambda \in \mathbb{R}$.

The factor process Y defined in (1.1) is employed in Heston [3] for the modelling of the stochastic volatility of stock price process. According to (1.1), the risk-free interest rate $r_t := r(Y_t) = Y_t^2$ satisfies

$$dr_t = k(\theta - r_t)dt + 2\sigma\sqrt{r_t}dW_2(t),\ r_0 = y^2, \tag{1.2}$$

where $k = -2b,\ \theta = -\frac{2b_{-1}+\sigma^2}{2b}$. The interest rate defined in equation (1.2) is called Cox-Ingersoll-Ross (CIR for short) interest rate which is introduced in Cox, Ingersoll and Ross [4], and the market model described in the above is called CIR interest rate model (see [1, 2] for some details).

Assumption 1.1 $2b_{-1} \geq \sigma^2$.

Remark 1.1 (i) Assumption 1.1 with $Y_0 = y > 0$ ensures that

$$P(Y_t > 0 \text{ for all } t \geq 0) = 1,$$

which means that the interest rate process is always strictly positive [1].

(ii) Applying Itô's formula to $e^{kt}r_t$, where r_t satisfies equation (1.2), we can get

$$E_y(Y_t^2) = y^2 e^{-kt} + \theta\left(1 - e^{-kt}\right), \tag{1.3}$$

where $k = -2b,\ \theta = -\frac{2b_{-1}+\sigma^2}{2b}$ and $E_y(\cdot)$ denotes the conditional expectation about initial value $Y_0 = y$ of the factor process Y.

In the above market model, Hata etc. (see [1, 2]) considered the large deviation control problems which can be transformed to the risk sensitive control problems whose utilities are exponential functionals. In this paper, we consider the optimal long term investment problem and optimal discounted consumption problem on infinite time horizon with logarithmic utility, corresponding to the HARA utility function when risk parameter is 0.

First, we consider the case that there is only investment but not consumption in the CIR interest rate model. Denote π_t the proportion of the investor's wealth invested in the risky stock at time t and X^π the wealth process under strategy π. Then X^π satisfies

$$\frac{dX_t^\pi}{X_t^\pi} = \pi_t \frac{dS_t}{S_t} + (1 - \pi_t)\frac{dB_t}{B_t}, \quad X_0^\pi = x > 0,$$

or equivalently,

$$X_t^\pi = x\exp\left\{\int_0^t \left(Y_s^2 + \lambda Y_s\pi_s - \frac{\pi_s^2}{2}\right)ds + \int_0^t \pi_s e^* dW(s)\right\}. \tag{1.4}$$

One of our purposes in this paper is to solve the following optimal investment problem.

Problem I Maximize the averaging logarithmic cost criterion (the criterion goes back to Kelly [5]) per unit time on infinite time horizon

$$J(x,y;\pi) = \lim_{T\to\infty} \frac{E_{x,y}\left(\log X_T^\pi\right)}{T},$$

where π ranges over the set of all admissible strategies (to be described later), and $E_{x,y}$ denotes the conditional expectation about initial wealth $X_0^\pi = x$ and initial value $Y_0 = y$ of the factor process Y.

Furthermore, we consider the case that there are both investment and consumption in the CIR interest rate model. Denote $X_t^{\pi,c}$ the wealth process at time t, where π_t is the proportion of the investor's wealth invested in the risky stock and c_t is the control variable such that $c_t X_t^{\pi,c}$ is the rate at which wealth is consumed. Then, $X^{\pi,c}$ satisfies the stochastic differential equation

$$\frac{dX_t^{\pi,c}}{X_t^{\pi,c}} = \pi_t\frac{dS_t}{S_t} + (1-\pi_t)\frac{dB_t}{B_t} - c_t dt, \quad X_0^{\pi,c} = x,$$

i.e.,

$$dX_t^{\pi,c} = X_t^{\pi,c}[Y_t^2 + \lambda Y_t\pi_t - c_t]dt + X_t^{\pi,c}\pi_t e^* dW(t), \ X_0^{\pi,c} = x. \tag{1.5}$$

Our another purpose is to consider the optimal consumption problem as follows.

Problem II Maximize the discounted logarithmic cost criterion on infinite time horizon

$$J(x,y;\pi,c) = E_{x,y}\left[\int_0^\infty e^{-\alpha t}\log(c_t X_t^{\pi,c})dt\right], \ \alpha > 0,$$

where (π,c) ranges over the set of all admissible strategies (to be described later).

Problem I and II were studied for some different market models, for example, Matsumoto [6] considered Problem I in finite time horizon for the classical Merton wealth problem in which the risky asset is not completely liquid, Christensen [7] studied Problem I based on impulse control strategies such that number of trades per unit does not exceed a fixed level, Noh and Kim [8] considered Problem II with factor process following geometric Brownian motion, Goll and Kallsen [9] considered Problem II in a general semi-martingale market model and Pang [10] considered Problem II in the market model in which interest rate follows Itô process, etc, but the market models considered in previous works do not include CIR interest rate model.

The contribution of this paper is solving Problems I and II for CIR interest rate model. In previous works, the main approach used to deal with the two problems is dynamic programming method which is a classical method for optimization problems. Similarly, in present paper, we still invoke the approach to solve our problems.

2 Optimal Long Term Investment Problem

In this section, we deal with Problem I. First we give the definition of admissible strategy for this problem.

Definition 2.1 Strategy $(\pi_t)_{t\geq 0}$ is said to be an admissible strategy if π_t is $\sigma(S_s, Y_s, 0 \leq s \leq t)$ - progressively measurable process such that for any $T > 0$,

$$E_{x,y}\left(\int_0^T \pi_t^2 dt\right) < \infty. \tag{2.1}$$

The set of all admissible strategies for Problem I will be denoted by $\mathcal{A}$.

According to (1.4) and (2.1), for any admissible strategy π,

$$\begin{aligned} J(x,y;\pi) &= \lim_{T\to\infty} \frac{E_{x,y}\left(\log X_T^\pi\right)}{T} \\ &= \lim_{T\to\infty} \frac{E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \pi_s - \frac{\pi_s^2}{2}\right) ds\right]}{T} + \lim_{T\to\infty} \frac{E_{x,y}\left(\int_0^T \pi_s e^* dW(s)\right)}{T} \\ &= \lim_{T\to\infty} \frac{1}{T} E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \pi_s - \frac{\pi_s^2}{2}\right) ds\right]. \end{aligned}$$

Hence, Problem I can be rewritten by

$$\Gamma = \sup_{\pi\in\mathcal{A}} J(x,y;\pi) = \sup_{\pi\in\mathcal{A}} \lim_{T\to\infty} \frac{1}{T} E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \pi_s - \frac{\pi_s^2}{2}\right) ds\right]. \tag{2.2}$$

The corresponding dynamic programming equation associated with control problem (2.2) is

$$\Gamma = \frac{\sigma^2}{2} v'' + \left(\frac{b_{-1}}{y} + by\right) v' + \sup_{\pi\in\mathbb{R}} \left\{y^2 + \lambda y \pi - \frac{\pi^2}{2}\right\}, \tag{2.3}$$

where the unknown in equation (2.3) is the pair (Γ, v), and Γ is a constant (see [11, 12]).

In order to solve equation (2.3), we invoke the vanishing discount method discussed by Bensoussan [11] and Morimoto and Okada [12] in ergodic control, that is to construct an infinite time horizon discounted cost problem, and treat (2.3) as a limiting case of the corresponding discounted type Bellman equation as the discount vanishes. First, let us consider the discounted optimal control problem associated with problem (2.2), that is

$$u_\alpha(x) = \sup_{\pi\in\mathcal{A}} I(x,y;\alpha,\pi), \tag{2.4}$$

where

$$I(x,y;\alpha,\pi) = E_{x,y}\left[\int_0^\infty e^{-\alpha s}\left(Y_s^2 + \lambda Y_s \pi_s - \frac{\pi_s^2}{2}\right) ds\right],\ \alpha > 0.$$

The dynamic programming equation associated with control problem (2.4) is

$$\alpha u_\alpha(y) = \frac{\sigma^2}{2} u_\alpha'' + \left(\frac{b_{-1}}{y} + by\right) u_\alpha' + \sup_{\pi\in\mathbb{R}} \left\{y^2 + \lambda y \pi - \frac{\pi^2}{2}\right\} \tag{2.5}$$

(see [11, 12]). By substitution, we can look for the solution of (2.5) as follows

$$u_\alpha(y) = \frac{(\lambda^2+2)(\sigma^2+2b_{-1})}{2\alpha(\alpha-2b)} + \frac{\lambda^2+2}{2(\alpha-2b)}y^2,$$

and the potential optimal control strategy

$$\hat{\pi} = \lambda y.$$

Set $v_\alpha(y) = u_\alpha(y) - u_\alpha(0^+)$. By direct verification, we obtain the following proposition.

Proposition 2.1 If Assumption 1.1 holds, in addition to $b<0$, then

$$\alpha u_\alpha(0^+) \to \tilde{\Gamma} := \frac{(\lambda^2+2)(\sigma^2+2b_{-1})}{-4b}, \quad v_\alpha(y) \to \tilde{v}(y) := \frac{\lambda^2+2}{-4b}y^2, \tag{2.6}$$

as $\alpha \to 0$. Furthermore, the limit $(\tilde{\Gamma}, \tilde{v})$ satisfies (2.3).

Now, we return to problem (2.2).

Theorem 2.1 Under condition of Proposition 2.1, let $\tilde{\Gamma}$ be the constant obtained in (2.6), then for any admissible strategy π,

$$\tilde{\Gamma} \geq J(x,y;\pi) = \lim_{T\to\infty} \frac{E_{x,y}\left(\log X_T^\pi\right)}{T}.$$

Furthermore, the optimal strategy and value function for problem (2.2) are respectively given by $\hat{\pi}_t = \lambda Y_t$ and $\tilde{\Gamma}$, i.e.,

$$\tilde{\Gamma} = \sup_{\pi\in\mathcal{A}} J(x,y;\pi) = J(x,y;\hat{\pi}).$$

Proof For any $\pi \in \mathcal{A}$ and $T>0$, applying Itô's formula to $\tilde{v}(Y_t) = \frac{\lambda^2+2}{-4b}Y_t^2$ between 0 and T, we get

$$\tilde{v}(Y_T) = \tilde{v}(y) + \int_0^T \left[\frac{\sigma^2}{2}\tilde{v}''(Y_s) + \left(\frac{b_{-1}}{Y_s} + bY_s\right)\tilde{v}'(Y_s)\right]ds + \int_0^T \sigma\tilde{v}'(Y_s)dW_2(s).$$

By virtue of (1.3), $\int_0^T \sigma\tilde{v}'(Y_s)dW_2(s)$ is a martingale. Hence

$$E_{x,y}\left[\tilde{v}(Y_T)\right] = \tilde{v}(y) + E_{x,y}\left[\int_0^T \left[\frac{\sigma^2}{2}\tilde{v}'' + \left(\frac{b_{-1}}{Y_s} + bY_s\right)\tilde{v}'\right]ds\right]. \tag{2.7}$$

Since $(\tilde{\Gamma}, \tilde{v})$ satisfies equation (2.3), from (2.7), we get

$$E_{x,y}\left[\tilde{v}(Y_T)\right] \leq \tilde{v}(y) + \tilde{\Gamma}T - E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s\pi_s - \frac{\pi_s^2}{2}\right)ds\right]. \tag{2.8}$$

Thanks to (1.3), we have, as $T\to\infty$,

$$\frac{E_{x,y}\left[\tilde{v}(Y_T)\right]}{T} = \frac{(\lambda^2+2)E_{x,y}(Y_T^2)}{-4bT} = \frac{(\lambda^2+2)\left[y^2e^{2bT} + \theta\left(1-e^{2bT}\right)\right]}{-4bT} \to 0,$$

since $b<0$. Hence, according to (2.8),

$$\tilde{\Gamma} \geq \lim_{T\to\infty} \frac{1}{T} E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \pi_s - \frac{\pi_s^2}{2}\right) ds\right] = \lim_{T\to\infty} \frac{E_{x,y}\left(\log X_T^\pi\right)}{T} = J(x,y;\pi).$$

In particular, if $\hat{\pi}_t = \lambda Y_t$, it is easy from (1.3) to check that $\hat{\pi} \in \mathcal{A}$. Since for any $t \geq 0$,

$$\frac{\sigma^2}{2}\tilde{v}''(Y_t) + \left(\frac{b_{-1}}{Y_t} + bY_t\right)\tilde{v}'(Y_t) = \tilde{\Gamma} - \left(Y_t^2 + \lambda Y_t \hat{\pi}_t - \frac{\hat{\pi}_t^2}{2}\right),$$

when $\pi = \hat{\pi}_t$, inequality (2.8) becomes equality, which implies

$$\tilde{\Gamma} = \lim_{T\to\infty} \frac{1}{T} E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \hat{\pi}_s - \frac{\hat{\pi}_s^2}{2}\right) ds\right] = \sup_{\pi\in\mathcal{A}} J(x,y;\pi).$$

The proof is completed.

3 Optimal Discounted Consumption Problem on Infinite Time Horizon

In this section, we consider Problem II. First we give the definition of admissible strategy for this problem.

Definition 3.1 Strategy $(\pi_t, c_t)_{t\geq 0}$ is said to be an admissible strategy if it is $\sigma(S_s, Y_s, 0 \leq s \leq t)$ -progressively measurable and

(i) $c_t \geq 0$ and there is an upper bounded L which is large enough to guarantee the feasibility of the optimal consumption control,

(ii) for any $T > 0$,

$$E_{x,y}\left(\int_0^T \pi_t^2 dt\right) < \infty, \tag{3.1}$$

and for any $\alpha > 0$,

$$\lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left(\int_0^T \pi_t^2 dt\right) = 0. \tag{3.2}$$

The set of all admissible strategies for Problem II will be denoted by $\mathcal{C}$.

Note that Problem II can be written by

$$V(x,y) = \sup_{(\pi,c)\in\mathcal{C}} J(x,y;\pi,c) = \sup_{(\pi,c)\in\mathcal{C}} E_{x,y}\left[\int_0^\infty e^{-\alpha t}\log(c_t X_t^{\pi,c})dt\right],\ \alpha > 0. \tag{3.3}$$

According to (1.5), the dynamic programming equation associated with problem (3.3) is

$$\begin{aligned}\alpha V =& \sup_{\pi\in\mathbb{R}}\left(\lambda y\pi x V_x + \frac{1}{2}\pi^2 x^2 V_{xx} + \sigma\rho\pi x V_{xy}\right) + xy^2 V_x + \left(\frac{b_{-1}}{y} + by\right) V_y + \frac{\sigma^2}{2} V_{yy} \\ &+ \sup_{c\in\mathbb{R}}\left[-cxV_x + \log(cx)\right].\end{aligned} \tag{3.4}$$

Similar to [10], we can look for the solution of (3.4) of the form

$$\tilde{V}(x,y) = A\log x + W(y),$$

and the candidate optimal control policy

$$\bar{\pi}(x,y) = \lambda y, \quad \bar{c}(x,y) = \alpha.$$

By substitution, we have

$$\tilde{V}(x,y) = \frac{1}{\alpha}\log x + W(y),$$

where $W(y)$ satisfies

$$\alpha W = \frac{\lambda^2+2}{2\alpha}y^2 + \left(\frac{b^{-1}}{y} + by\right)W' + \frac{\sigma^2}{2}W'' + (\log\alpha - 1). \tag{3.5}$$

We can find the solution of (3.5) as follows

$$W(y) = a_0 + a_1 y^2,$$

where

$$a_0 = \frac{(\lambda^2+2)(2b_{-1}+\sigma^2)}{2\alpha^2(\alpha-2b)} + \frac{\log\alpha - 1}{\alpha}, \quad a_1 = \frac{\lambda^2+2}{2\alpha(\alpha-2b)}.$$

Hence,

$$\tilde{V}(x,y) = \frac{1}{\alpha}\log x + a_0 + a_1 y^2 \tag{3.6}$$

is the solution of equation (3.4).

Theorem 3.1 If Assumption 1.1 holds, in addition to $\alpha > 2b$, then for any $(\pi,c) \in \mathcal{C}$, $\tilde{V}(x,y)$ defined in (3.6) satisfies

$$\tilde{V}(x,y) \geq E_{x,y}\left[\int_0^\infty e^{-\alpha t}\log(c_t X_t^{\pi,c})\right]dt. \tag{3.7}$$

Furthermore, the optimal strategy and value function for problem (3.3) are given respectively by

$$\bar{\pi}(x,y) = \lambda y, \quad \bar{c}(x,y) = \alpha,$$

and $\tilde{V}(x,y)$, i.e.,

$$\tilde{V}(x,y) = \sup_{(\pi,c)\in\mathcal{C}} J(x,y;\pi,c) = E_{x,y}\left[\int_0^\infty e^{-\alpha t}\log(\bar{c}_t X_t^{\bar{\pi},\bar{c}})\right]dt. \tag{3.8}$$

Proof By Itô's rule, for any $T > 0$,

$$\begin{aligned} e^{-\alpha T}\tilde{V}(X_T^{\pi,c}, Y_T) =& \tilde{V}(x,y) + \int_0^T e^{-\alpha t}\Bigg[X_t^{\pi,c}(Y_t^2 + \lambda Y_t \pi_t - c_t)\tilde{V}_x + \tilde{V}_y\left(\frac{b_{-1}}{Y_t} + bY_t\right) \\ &+ \frac{\tilde{V}_{xx}}{2}(X_t^{\pi,c})^2\pi_t^2 + \frac{\sigma^2\tilde{V}_{yy}}{2} + \tilde{V}_{xy}X_t^{\pi,c}\rho\sigma\pi_t - \alpha\tilde{V}\Bigg]dt \\ &+ \int_0^T e^{-\alpha t}X_t^{\pi,c}\tilde{V}_x\pi_t e^* dW(t) + \int_0^T e^{-\alpha t}\tilde{V}_y\sigma dW_2(t). \end{aligned}$$

By virtue of (1.3) and (3.1), $\int_0^T e^{-\alpha t} X_t^{\pi,c} \tilde{V}_x \pi_t e^* dW(t)$ and $\int_0^T e^{-\alpha t} \tilde{V}_y \sigma dW_2(t)$ are martingales. Hence,

$$\begin{aligned} e^{-\alpha T} E_{x,y}\left[\tilde{V}(X_T^{\pi,c}, Y_T)\right] =& \tilde{V}(x,y) + E_{x,y}\bigg(\int_0^T e^{-\alpha t}\bigg[X_t^{\pi,c}(Y_t^2 + \lambda Y_t \pi_t - c_t)\tilde{V}_x + \tilde{V}_y\left(\frac{b_{-1}}{Y_t} + bY_t\right) \\ & + \frac{\tilde{V}_{xx}}{2}(X_t^{\pi,c})^2\pi_t^2 + \frac{\sigma^2 \tilde{V}_{yy}}{2} + \tilde{V}_{xy} X_t^{\pi,c}\rho\sigma\pi_t - \alpha\tilde{V}\bigg]dt\bigg). \end{aligned}$$

Since $\tilde{V}(x,y)$ satisfies equation (3.4), from the previous equality, we have

$$\tilde{V}(x,y) \geq E_{x,y}\left[\int_0^T e^{-\alpha t}\log(c_t X_t^{\pi,c})dt\right] + e^{-\alpha T} E_{x,y}\left[\tilde{V}(X_T^{\pi,c}, Y_T)\right]. \tag{3.9}$$

Next, we will show that

$$\begin{aligned} \lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left[\tilde{V}(X_T^{\pi,c}, Y_T)\right] =& \frac{1}{\alpha}\lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left(\log X_T^{\pi,c}\right) \\ & + \lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left(a_0 + a_1 Y_T^2\right) \geq 0. \end{aligned} \tag{3.10}$$

Since $\alpha - 2b > 0$, from (1.3),

$$\lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left[a_0 + a_1 Y_T^2\right] = 0. \tag{3.11}$$

On the other hand, since $X^{\pi,c}$ satisfies (1.5) and there is an sufficiently large upper bounded L for c_t, we have

$$\begin{aligned} E_{x,y}\left(\log X_T^{\pi,c}\right) =& \log x + E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \pi_s - c_s - \frac{\pi_s^2}{2}\right) ds\right] \\ \geq& \log x + E_{x,y}\left[\int_0^T \left(\frac{2-\lambda^2}{2} Y_s^2 - \frac{3}{2}\pi_s^2 - L\right) ds\right] \\ =& \log x + \frac{2-\lambda^2}{2} E_{x,y}\left(\int_0^T Y_s^2 ds\right) - \frac{3}{2} E_{x,y}\left(\int_0^T \pi_s^2 ds\right) - LT. \end{aligned}$$

From the previous inequality, we can get

$$\begin{aligned} & \lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left(\log X_T^{\pi,c}\right) \\ \geq& \frac{2-\lambda^2}{2}\lim_{T\to\infty}\left[e^{-\alpha T} E_{x,y}\left(\int_0^T Y_s^2 ds\right)\right] - \frac{3}{2}\lim_{T\to\infty}\left[e^{-\alpha T} E_{x,y}\left(\int_0^T \pi_s^2 ds\right)\right]. \end{aligned} \tag{3.12}$$

According to (1.3), since $\alpha - 2b > 0$, it is not hard to show that

$$\lim_{T\to\infty}\left[e^{-\alpha T} E_{x,y}\left(\int_0^T Y_t^2 dt\right)\right] = 0. \tag{3.13}$$

Since π is admissible control, according to (3.2),

$$\lim_{T\to\infty}\left[e^{-\alpha T} E_{x,y}\left(\int_0^T \pi_s^2 ds\right)\right] = 0. \tag{3.14}$$

Putting (3.13) and (3.14) into (3.12) , we get

$$\lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left(\log X_T^{\pi,c}\right) \geq 0. \tag{3.15}$$

By virtue of (3.15) and (3.11), we prove (3.10) which implies (3.7), combined with (3.9).

In particular, if

$$\bar{\pi} = \lambda Y_t, \quad \bar{c} = \alpha,$$

then $(\bar{\pi}, \bar{c}) \in \mathcal{C}$. Using the same procedure as the proof of (3.9), we get

$$\tilde{V}(x,y) = E_{x,y}\left[\int_0^T e^{-\alpha t}\log(\bar{c}_t X_t^{\bar{\pi},\bar{c}})dt\right] + e^{-\alpha T} E_{x,y}\left[\tilde{V}(X_T^{\bar{\pi},\bar{c}}, Y_T)\right]. \tag{3.16}$$

Since

$$\begin{aligned} E_{x,y}\left(\log X_T^{\bar{\pi},c}\right) &= \log x + E_{x,y}\left[\int_0^T \left(Y_s^2 + \lambda Y_s \bar{\pi}_s - \bar{c}_s - \frac{\bar{\pi}_s^2}{2}\right) ds\right] \\ &= \log x + E_{x,y}\left[\int_0^T \left(\frac{2+\lambda^2}{2} Y_s^2 - \alpha\right) ds\right], \end{aligned}$$

it is easy from (1.3) to check that

$$\lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left[\tilde{V}(X_T^{\bar{\pi},\bar{c}}, Y_T)\right] = \lim_{T\to\infty} e^{-\alpha T} E_{x,y}\left[\frac{1}{\alpha}\log X_T^{\bar{\pi},\bar{c}} + a_0 + a_1 Y_T^2\right] = 0,$$

which implies (3.8), combined with (3.16). The proof is completed.

In this paper, we consider the case that $r(y)$ and $\mu(y)$ are quadratic functions of y. In this case, we can give the explicit forms for the value functions of Problem I and II. One can try to consider Problem I and II for generalized forms of $r(y)$ and $\mu(y)$. Of course, in the generalized case, the problems become very difficult.

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CIR利率模型中基于对数效用的投资组合最优化问题

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摘要: 本文研究了CIR 利率模型中基于对数效用的最优长期投资问题和无限时间域上的最优折算消费问题. 通过求解相关的动态规划方程, 获得了这两个最优化问题的最优策略及值函数的明确表现形式.

关键词: Cox-Ingersoll-Ross 利率; 对数效用; 最优投资; 最优折算消费; 动态规划方程

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