LOCAL AUTOMORPHISMS AND LOCAL DERIVATIONS OF UPPER TRIANGULAR MATRIX LIE ALGEBRA OVER A COMMUTATIVE RING

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Abstract: The aim of this paper is to characterize the local automorphisms and local derivations of $T_n(R)$. By using the main result about automorphisms and derivations of $T_n(R)$ and the skill of matrix computation, it is proved that every local automorphism of $T_n(R)$ is an automorphism and that each local derivation of $T_n(R)$ is a derivation, which extend the main result about automorphisms and derivations of $T_n(R)$.

Keywords: local automorphisms; local derivations; upper triangular matrix lie algebra; commutative ring

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1 Introduction

Recently, many scholars paid attention to the significant work has been done in studying the local maps. Larson [1] initially considered local maps in his examination of reflexivity and interpolation for subspaces of $B(H)$, where $H$ is a Hilbert space. The notion of local derivations (resp., local automorphisms) was introduced independently by Larson and Sourour [2] and Kadison [3] (resp., Larson and Sourour [2]). Recall that a linear map $\delta$ from an algebra $\mathcal{A}$ into itself is called a local derivation (resp., local automorphism) if for every $a \in \mathcal{A}$, there exists a derivation (resp., an automorphism) $\delta_a$ of $\mathcal{A}$, depending on $a$, such that $\delta(a) = \delta_a(a)$. If every local derivation (resp., local automorphism) of an algebra is a derivation (resp., an automorphism), then we can say that the derivations (resp., automorphisms) of those structures are, in a certain sense, completely determined by their local actions.

Local derivations, local automorphisms and other local maps have been studied in a variety of contexts. Larson and Sourour [2] showed that every local derivation (resp., every

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surjective linear local automorphism) on $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a Banach space $\mathcal{H}$, is a derivation (an automorphism). Zhao, Yao and Wang [4] proved that every local Jordan derivation (resp., local Jordan automorphism) of upper triangular matrix algebra is an inner derivation (resp., a Jordan automorphism). Other work on the description of the local derivations or local automorphisms on operator algebras can be found in [5–7]. In those articles all local derivations or local automorphisms are actually global derivations or automorphisms. A nontrivial local derivation on an operator algebra was found by Crist in [8]. Crist [9] showed that any linear local automorphism of a finite dimensional CSL algebra $\mathcal{A}$ is either an automorphism or can be factored as an automorphism and the transpose of a self-adjoint summand of $\mathcal{A}$.

The algebra $T_n(R)$ of all upper triangular matrices over a commutative ring $R$ is an interesting topic for many researchers. Significant research has been done in studying various linear maps of $T_n(R)$. In 1990, Kezlan [10] showed that every $R$-algebra automorphism of $T_n(R)$ is inner. Cao [11] and Wang and You [12] gave a description of the Lie automorphisms of $T_n(R)$. Tang, Cao and Zhang [13] determined all Jordan isomorphisms of $T_n(R)$. Wang and Yu [14] determined the derivations of any Lie subalgebra of the general linear Lie algebra containing $T_n(R)$. In this paper, we regard $T_n(R)$ as a Lie algebra and we shall study the local automorphisms and local derivations of $T_n(R)$.

Let $R$ be a commutative ring with identity, $R^*$ the group of invertible elements of $R$. In the following of this paper, we use $T_n(R)$ (resp., $D_n(R)$) denote the Lie algebra of all upper triangular (resp., diagonal) $n$ by $n$ matrices over $R$, $T_n^*(R)$ the set of all invertible elements in $T_n(R)$. We denote by $\mathfrak{n}$ the subalgebra of $T_n(R)$ consisting of all strictly upper triangular matrices. Let $e$ be the identity matrix of $T_n(R)$, $e_{i,j}$ the matrix with 1 at the position $(i,j)$ and zero elsewhere for $1 \leq i, j \leq n$. For $x \in T_n(R)$, denote by $x^t$ the transpose of $x$. Let $S_k = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \in T_n(R) \mid x \in T_{n-k}(R) \right\}, k = 1, 2, \cdots, n - 1$. Obviously, each $S_k$ is a subalgebra of $T_n(R)$.

2 Local Automorphisms

Cao [11] and Wang and You [12] gave an explicit description of the automorphisms of $T_n(R)$, respectively. For convenience of the proof of the main result in this section, we give another description of the automorphisms of $T_n(R)$ by the following lemma. Before giving the lemma, let us introduce some standard automorphisms of $T_n(R)$ as follows. In this section, 2 is an unit in $R$.

(A) Inner automorphisms

Let $a \in T_n^*(R)$, the map $\theta_a : x \mapsto axa^{-1}$ for all $x \in T_n(R)$ is an automorphism of $T_n(R)$, which is called an inner automorphism.

(B) Central automorphisms
Regarding $R$ as an abelian Lie algebra. Let
\[ F = \{ f \in \text{Hom}_R(T_n(R), R) \mid 1 + f(e) \in R^* \}. \]
For $f \in F$, we define a map $\eta_f : x \mapsto x + f(x)e$ for all $x \in T_n(R)$. It can be checked that $\eta_f$ is an automorphism of $T_n(R)$, which is called a central automorphism. Since $\eta_f(n) = n$, $f(y) = 0$ for any $y \in n$.

(C) Graph automorphisms

Set $r = e_{1n} + e_{2,n-1} + \cdots + e_{n-1,2} + e_{n1}$. It is clear that $r^2 = e$ and $r^t = r$. Let $\Upsilon$ be the set of all idempotents in $R$. For $\varepsilon \in \Upsilon$, it is easy to check that the map $w_\varepsilon : x \mapsto \varepsilon x - (1 - \varepsilon)rx^tr$ is an automorphism of $T_n(R)$. We call $w_\varepsilon$ a graph automorphism.

**Lemma 2.1** (the main theorem of [11] and [12]) Let $\psi$ be an automorphism of $T_n(R)$. Then there exist an inner automorphism $\theta_r$, a graph automorphism $w_\varepsilon$ and a central automorphism $\eta_f$ of $T_n(R)$ such that $\psi = \theta_r w_\varepsilon \eta_f$ for $n \geq 3$; $\psi = \theta_r \eta_f$ when $n = 2$; $\psi = \eta_f$ for $n = 1$.

The following lemma is obvious.

**Lemma 2.2** Let $\theta$ be an inner automorphism of $T_n(R)$. Then $\theta(E) = \theta(E)^2$ for every idempotent $E$ in $T_n(R)$.

We will prove our main result in this section via the following lemmas.

**Lemma 2.3** Let $\varphi$ be a local automorphism of $T_n(R)$ ($n \geq 3$). If $\varphi(e_{1i}) = e_{11}$, then we may find an inner automorphism $\theta = \prod_{j=2}^n \theta_{e_j}$ and a central automorphism $\eta_f$ such that $\eta_f^{-1} \theta^{-1} \varphi(e_{1i}) = e_{1i}$ for $i = 1, 2, \cdots, n$.

**Proof** For $e_{1i} \in T_n(R)$, since $\varphi$ is a local automorphism, there exists an automorphism $\varphi_{e_{1i}}$, depending on $e_{1i}$, such that $\varphi(e_{1i}) = \varphi_{e_{1i}}(e_{1i})$. By Lemma 2.1, we know there exist $\varepsilon_i \in \Upsilon$, $f_i \in F$ and $a_i \in T_n^*(R)$ such that
\[ \varphi(e_{1i}) = \varphi_{e_{1i}}(e_{1i}) = \theta_{\varepsilon_i} w_{\varepsilon_i} \eta_f(e_{1i}). \]  
(2.1)

In the following, we first prove that $\varepsilon_i = 1$ for $i = 2, 3, \cdots, n$ in (2.1).

From (2.1) we get
\[ \varphi(e_{11} + e_{1i}) = e_{11} + \varphi(e_{1i}) \]
\[ \equiv e_{11} + \varepsilon_i e_{1i} + (2\varepsilon_i - 1)f_i(e_{1i})e - (1 - \varepsilon_i)e_{n+1-i,n+1-i} \text{ mod } n. \]  
(2.2)

On the other hand, we have $\varphi(e_{11} + e_{1i}) = \varphi_{e_{11} + e_{1i}}(e_{11} + e_{1i})$, where $\varphi_{e_{11} + e_{1i}}$ is an automorphism depending on $e_{11} + e_{1i}$. By Lemma 2.1, we have $\varphi_{e_{11} + e_{1i}} = \theta_{\varepsilon'_i} w_{\varepsilon'_i} \eta_{f'_i}$ for some $a'_i \in T_n(R)$, $\varepsilon'_i \in \Upsilon$ and $f'_i \in F$, so
\[ \varphi(e_{11} + e_{1i}) = \theta_{\varepsilon'_i} w_{\varepsilon'_i} \eta_{f'_i}(e_{11} + e_{1i}) \]
\[ \equiv \varepsilon'_i (e_{11} + e_{1i}) + (2\varepsilon'_i - 1)f'_i(e_{11} + e_{1i})e - (1 - \varepsilon'_i)(e_{nn} + e_{n+1-i,n+1-i}) \text{ mod } n. \]  
(2.3)

From (2.2) and (2.3), we have $\varepsilon_i = \varepsilon'_i = 1$ for $i = 2, 3, \cdots, n$. That is to say
\[ \varphi(e_{1i}) = \theta_{\varepsilon_i} \eta_f(e_{1i}), i = 2, 3, \cdots, n. \]  
(2.4)
Next we use induction to prove that there exists an inner automorphism $\theta$ such that $\theta^{-1} \varphi(e_{ij}) = e_{ii} + f_i(e_{ii})e$ for $i = 2, \ldots, n$, and $\theta^{-1} \varphi(e_{11}) = e_{11}$. Let $i = 2$ in (2.2) and (2.3), we have

$$f_2(e_{12}) = f'_2(e_{11} + e_{22}).$$

So

$$\varphi(e_{11} + e_{22}) = \theta_{a_2} \varphi (e_{11} + e_{22}) = \theta_{a_2} (e_{11} + e_{22}) + f_2(e_{22})e. \quad (2.5)$$

On the other hand, by (2.4) we have

$$\varphi (e_{11} + e_{22}) = e_{11} + \varphi (e_{22}) = e_{11} + \theta_{a_2} (e_{22}) + f_2(e_{22})e. \quad (2.6)$$

(2.5) and (2.6) imply that $\theta_{a_2} (e_{11} + e_{22}) = e_{11} + \theta_{a_2} (e_{22})$. The idempotence of $e_{11} + e_{22}$ shows that the image of it under $\varphi$ is also idempotent. So $\theta_{a_2} (e_{22}) \in S_1$. Suppose $a_2 = (a^{(2)}_{ij})_{n \times n}$. Let $b_2 = (b^{(2)}_{ij})_{n \times n}$, where $b^{(2)}_{11} = a^{(2)}_{11}, b^{(2)}_{ij} = a^{(2)}_{ij}$ for $2 \leq i \leq j \leq n$, and $b^{(2)}_{ij} = 0$ for $2 \leq j \leq n$. Then $\theta_{b_2} (e_{22}) = \theta_{a_2} (e_{22})$. So

$$\theta_{b_2}^{-1} \varphi (e_{22}) = e_{22} + f_2(e_{22})e \quad \text{and} \quad \theta_{b_2}^{-1} \varphi (e_{11}) = e_{11}.$$ 

Denote $\theta_{b_2}^{-1} \varphi$ by $\varphi_1$.

By induction we assume that there are $\theta_{b_j}, j = 3, 4, \ldots, k - 1$ such that

$$(\prod_{j=3}^{k-1} \theta_{b_j})^{-1} \varphi_1 (e_{11}) = e_{11}$$

and $(\prod_{j=3}^{k-1} \theta_{b_j})^{-1} \varphi_1 (e_{ii}) = e_{ii} + f_i(e_{ii})e$, where $f_i \in F, i = 2, 3, \ldots, k - 1$. Denote $(\prod_{j=3}^{k-1} \theta_{b_j})^{-1} \varphi_1$ by $\varphi_{k-2}$. By (2.4), we know there exist some $u \in T^*_n(R)$ and $f_k \in F$ such that

$$\varphi_{k-2} (e_{kk}) = \theta_u (e_{kk} + f_k(e_{kk})e) = \theta_u (e_{kk}) + f_k(e_{kk})e.$$  

So

$$\varphi_{k-2} (e_{11} + \cdots + e_{k-1,k-1} + e_{kk}) = e_{11} + \cdots + e_{k-1,k-1} + (f_2(e_{22}) + \cdots + f_k(e_{kk}))e + \theta_u (e_{kk}) \quad (2.7)$$

$$\equiv e_{11} + \cdots + e_{kk} + (f_2(e_{22}) + \cdots + f_k(e_{kk}))e \pmod{n}. \quad (2.8)$$

On the other hand, since $\varphi_{k-2}$ is a local automorphism, there exists an automorphism $\psi = \theta_t w_{\alpha} \eta_\sigma$, where $t \in T^*_n(R)$, $\alpha \in \mathcal{Y}$ and $\sigma \in F$, depending on $e_{11} + \cdots + e_{kk}$, such that

$$\varphi_{k-2} (e_{11} + \cdots + e_{kk}) = \psi (e_{11} + \cdots + e_{kk}) = \alpha (e_{11} + \cdots + e_{kk}) + (2\alpha - 1) \sigma (e_{11} + \cdots + e_{kk})e$$

$$- (1 - \alpha) (e_{n+1-k,n+1-k} + \cdots + e_{n+1-k,n+1-k}) \pmod{n}. \quad (2.9)$$
Similarly, there exists some \( \beta \in \mathcal{S}_{k-1} \). Suppose \( u = (u_{ij})_{n \times n} \). Let \( b_k = (b_{ij}^{(k)})_{n \times n}, \) where \( b_{ii}^{(k)} = u_{ii}, \ i = 1, 2, \cdots, k-1, b_{ij}^{(k)} = u_{ij} \) for \( k \leq i \leq j \leq n \), and \( b_{ts}^{(k)} = 0 \) for \( t = 1, 2, \cdots, k-1, t < s \leq n \). By calculating, we have \( \theta_{b_k}(e_{kk}) = \theta_u(e_{kk}) \).

When \( k = n \), let \( \theta = \prod_{j=2}^n \theta_h_j \). Then \( \theta^{-1} \varphi(e_{11}) = e_{11}, \) and \( \theta^{-1} \varphi(e_{ii}) = e_{ii} + f_i(e_{ii})e \) for \( i = 2, \cdots, n \).

Let \( f \) be an \( R \)-linear map satisfying \( f(e_{11}) = 0, f(e_{ii}) = f_i(e_{ii}) \) for \( i = 2, \cdots, n \) and \( f(e_{ij}) = 0 \) for \( 1 \leq i < j \leq n \). Then \( f \in \text{Hom}_R(T_n(R), R) \). It is easy to check that \( 1 + f(e) \in R^* \). Thus \( f \in F \) and \( \eta_j^{-1} \varphi(e_{ii}) = e_{ii} \) for \( i = 2, \cdots, n \).

**Lemma 2.4** Let \( \varphi \) be a local automorphism of \( T_n(R) \) satisfying \( \varphi(e_{ii}) = e_{ii} \) for \( i = 1, 2, \cdots, n \). Then \( \varphi(e_{ij}) = a_{ij}e_{ij} \), where \( 1 \leq i < j \leq n \) and \( a_{ij} \in R^* \).

**Proof** For \( 1 \leq i < j \leq n \), there exists an automorphism \( \varphi_{e_{ii} + e_{ij}} \) which agree with \( \varphi \) at \( e_{ii} + e_{ij} \). By Lemma 2.1, we know there exist \( \beta_{ij} \in T, \tau_{ij} \in F \) and \( u_{ij} \in T_n^*(R) \) such that \( \varphi_{e_{ii} + e_{ij}} = \theta_{u_{ij}}w_{\beta_{ij}}\eta_{\tau_{ij}}(e_{ii} + e_{ij}) \).

So \( \beta_{ij} = 1 \) and \( \tau_{ij}(e_{ii} + e_{ij}) = 0 \) follow from (2.11) and (2.12). Thus

\[
\varphi(e_{ii} + e_{ij}) = \varphi(e_{ii} + e_{ij}) = \theta_{a_{ij}}(e_{ii} + e_{ij}).
\]

Similarly, there exists some \( h_{ij} \in T_n^*(R) \) such that

\[
e_{jj} + \varphi(e_{ij}) = \varphi(e_{jj} + e_{ij}) = \theta_{h_{ij}}(e_{jj} + e_{ij}).
\]

Since \( e_{ii} + e_{ij} \) and \( e_{jj} + e_{ij} \) are idempotents, by Lemma 2.2, we know that the image of them under \( \varphi \) are also idempotent, which imply that \( \varphi(e_{ij}) = a_{ij}e_{ij} \) for some \( a_{ij} \in R \). Clearly, \( a_{ij} \in R^* \).
Lemma 2.5 Let \( \varphi \) be a local automorphism of \( T_n(R) \) satisfying \( \varphi(e_{ii}) = e_{ii} \) for \( i = 1, 2, \cdots, n \). Then there exists an inner automorphism \( \theta_d \) such that \( \theta_d \varphi(e_{i,i+1}) = e_{i,i+1} \) for \( i = 1, 2, \cdots, n-1 \), and \( \theta_d \varphi(e_{ii}) = e_{ii} \) for \( i = 1, 2, \cdots, n \).

Proof By Lemma 2.4, we have \( \varphi(e_{i,i+1}) = a_{i,i+1}e_{i,i+1} + a_{i,i+1} \in R^* \). Let

\[
d = \text{diag}(1, a_{12}^{-1}, (a_{12}a_{23})^{-1}, \cdots, (a_{12}a_{23} \cdots a_{n-1,n})^{-1}).
\]

Then \( \theta_d^{-1} \varphi(e_{i,i+1}) = e_{i,i+1} \) for \( i = 1, 2, \cdots, n-1 \), and \( \theta_d^{-1} \varphi(e_{ii}) = e_{ii} \) for \( i = 1, 2, \cdots, n \).

Lemma 2.6 Let \( \varphi \) be a local automorphism of \( T_n(R) \). If \( \varphi(e_{ii}) = e_{ii} \) for \( i = 1, 2, \cdots, n \), and \( \varphi(e_{i,i+1}) = e_{i,i+1} \) for \( i = 1, 2, \cdots, n-1 \), then for any \( e_{i,i+k} \in T_n(R) \), we have

\[
\varphi(e_{i,i+k}) = e_{i,i+k}.
\]

Proof We will prove this lemma by induction on \( k \) (\( k \geq 2 \)). When \( k = 2 \), since \( \varphi \) is a local automorphism, we have

\[
\varphi(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) = \phi^{(2)}_{i}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}),
\]

where \( \phi^{(2)}_{i} \) is an automorphism corresponding to \( e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1} \). By Lemma 2.1, we know there exist \( \gamma_i^{(2)} \in \mathcal{Y} \), \( \sigma_i^{(2)} \in F \) and \( x_i^{(2)} \in T_n^*(R) \) such that \( \phi^{(2)}_{i} = \theta_{x_i^{(2)}}w_{\gamma_i^{(2)}}\eta_{\sigma_i^{(2)}} \). So

\[
\varphi(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1})
\equiv \gamma_i^{(2)}e_{i+1,i+1} - (1 - \gamma_i^{(2)})e_{n-i,n-i} + (2\gamma_i^{(2)} - 1)\sigma_i^{(2)}(e_{i,i+1} + e_{i,i+1}) \pmod n. \tag{2.13}
\]

On the other hand, by Lemma 2.4, we have

\[
\varphi(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1})
\equiv e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + a_{i,i+2}e_{i,i+2} \equiv e_{i,i+1} + e_{i,i+2} \pmod n. \tag{2.14}
\]

From (2.13) and (2.14), we have \( \gamma_i^{(2)} = 1 \) and \( \sigma_i^{(2)}(e_{i+1,i+1}) = 0 \). So

\[
\theta_{x_i^{(2)}}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) = e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1} \pmod n.
\]

The idempotency of \( e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1} \) and Lemma 2.2 imply that \( a_{i,i+2} = 1 \). So \( \varphi(e_{i,i+2}) = e_{i,i+2} \).

By induction we assume that \( \varphi(e_{i,i+m}) = e_{i,i+m} \) for \( m = 2, \cdots, k - 1 \). For \( e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+1} \), similar to the case \( k = 2 \), we can get that there exists some \( x_i^{(k)} \in T_n^*(R) \) such that

\[
\theta_{x_i^{(k)}}(e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+1}) = e_{i,i+1} + e_{i+1,i+k} + e_{i+1,i+1} + a_{i,i+k}e_{i,i+k}.
\]

Also by the idempotence of \( e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+1} \) and Lemma 2.2, we can prove that \( a_{i,i+k} = 1 \). That is to say \( \varphi(e_{i,i+k}) = e_{i,i+k} \) for any \( e_{i,i+k} \in T_n(R) \).
**Theorem 2.1** Let $R$ be a commutative ring with identity 1 and unit 2, $T_n(R)$ the Lie algebra consisting of all upper triangular $n \times n$ matrices over $R$. Then every local automorphism $\varphi$ of $T_n(R)$ is an automorphism.

**Proof** Let $\varphi$ be a local automorphism of $T_n(R)$. When $n \geq 3$, for $e_{11} \in T_n(R)$, by the definition of $\varphi$, there exists an automorphism $\varphi_{e_{11}}$, depending on $e_{11}$, such that $\varphi(e_{11}) = \varphi_{e_{11}}(e_{11})$. So $\varphi_{e_{11}}^{-1}\varphi(e_{11}) = e_{11}$. Obviously, $\varphi_{e_{11}}^{-1}$ is also a local automorphism of $T_n(R)$. By Lemmas 2.3, 2.5 and 2.6, there are $\eta_f^{-1}, \theta^{-1}$ and $\theta_d^{-1}$ such that

$$
\theta_d^{-1}\eta_f^{-1}\theta^{-1}\varphi_{e_{11}}^{-1}\varphi(i_{ij}) = e_{ij} \text{ for } 1 \leq i \leq j \leq n,
$$

which mean that $\varphi = \varphi_{e_{11}}, \eta_f \theta_d$. So $\varphi$ is an automorphism.

When $n = 1$, suppose that $\varphi(1) = a$, then for any $x \in T_1(R) = R$ we have $\varphi(x) = x\varphi(1) = xa = \eta_f(x)$, where $f : R \to R, x \mapsto (a - 1)x$ is an $R$–linear map from $R$ to $R$. So $\varphi$ is an automorphism.

When $n = 2$, similar to the case $n \geq 3$, there is an automorphism $\varphi_{e_{11}}$, depending on $e_{11}$, such that $\varphi_{e_{11}}^{-1}\varphi(e_{11}) = e_{11}$. Denote $\varphi_{e_{11}}^{-1}$ by $\varphi_1$. Clearly, $\varphi_1$ is also a local automorphism of $T_2(R)$, by Lemma 2.1, there exist an inner automorphism $\theta_{e_{22}}$ and a central automorphism $\eta_{e_{22}}$, corresponding to $e_{22}$, such that $\varphi_1(e_{22}) = \theta_{e_{22}}\eta_{e_{22}}(e_{22}) = \theta_{e_{22}}(e_{22}) + f_2(e_{22})e$. So

$$
\varphi_1(e_{11} + e_{22}) = e_{11} + \theta_{e_{22}}(e_{22}) + f_2(e_{22})e \equiv e + f_2(e_{22})e \mod n. \tag{2.15}
$$

On the other hand, there exist an inner automorphism $\theta_{e_{22}}$ and a central automorphism $\eta_{e_{22}}$, depending on $e_{11} + e_{22}$, such that

$$
\varphi_1(e_{11} + e_{22}) = \theta_{e_{22}}\eta_{e_{22}}(e_{11} + e_{22}) = e + g_2(e)e. \tag{2.16}
$$

From (2.15) and (2.16), we get $\theta_{e_{22}}(e_{22}) = e_{22}$. Now we have $\varphi_1(e_{11}) = e_{11}$ and $\varphi_1(e_{22}) = e_{22} + f_2(e_{22})$. Let $f$ be an $R$–linear map satisfying $f(e_{11}) = 0, f(e_{22}) = f_2(e_{22})$, it is easy to check that $f \in F$ and $\eta_f^{-1}\varphi_1(e_{11}) = e_{11}$ for $i = 1, 2$.

Denote $\eta_f^{-1}\varphi_1$ by $\varphi_2$. Since $\varphi_2$ is a local automorphism, by Lemma 2.1, we have $\varphi_2(e_{12}) = \theta_a(e_{12}) = ae_{12}$, where $\theta_a$ is an inner automorphism depending on $e_{12}$ and $a \in R^*$. Let $z = \text{diag}(1, a^{-1})$, then $\theta_a^{-1}\varphi_2(i_{ij}) = e_{ij}, 1 \leq i \leq j \leq 2$, which mean $\theta_a^{-1}\eta_f^{-1}\varphi_1 = 1$, that is $\varphi = \varphi_{e_{11}}, \eta_f \theta_a$. So $\varphi$ is an automorphism.

3 Local Derivation

In [14], Wang and Yu characterized the derivations of $T_n(R)$ by the following lemma. Before giving this lemma, we first introduce two standard derivations of $T_n(R)$.

(A) Inner derivations

Let $t \in T_n(R)$, then $ad t : x \mapsto [t, x], x \in T_n(R)$ is a derivation of $T_n(R)$, which is called an inner derivation of $T_n(R)$ induced by $t$.

(B) Central derivations

We denote by $\text{Hom}(D_n(R), R)$ the set of all $R$-module homomorphisms from $D_n(R)$ to $R$. For any $\sigma \in \text{Hom}(D_n(R), R)$, $\sigma$ may be extended to a derivation $\eta_\sigma$ of $T_n(R)$ by:
Lemma 3.1 (the theorem of [14]) Let $R$ be a commutative ring with identity. Then

(1) every derivation of $T_n(R)$ can be uniquely written as the sum of an inner derivation and a central derivation when $n \geq 2$.

(2) every derivation of $T_n(R)$ is a central derivation when $n = 1$.

Proof By the definition of $\delta$ and Lemma 3.1, there exists a derivation $\delta_{e_{22}} = ad t_2 + \eta_{s_2}$, corresponding to $e_{22}$, such that

$$\delta(e_{11} + e_{22}) = \delta(e_{11}) + \delta(e_{22}) = 0 + \delta_{e_{22}}(e_{22}) = ad t_2(e_{22}) + \sigma_2(e_{22})e.$$

(3.1)

On the other hand, there is a derivation $\delta_{e_{11} + e_{22}} = ad s_2 + \eta_{s_2}$, depending on $e_{11} + e_{22}$, such that

$$\delta(e_{11} + e_{22}) = \delta_{e_{11} + e_{22}}(e_{11} + e_{22}) = ad s_2(e_{11} + e_{22}) + \alpha_2(e_{11} + e_{22})e.$$

(3.2)

Suppose $t_2 = (t_{ij}^{(2)})_{n \times n}$, from (3.1) and (3.2), we have $t_{12}^{(2)} = 0$. Let $m_2 = (m_{ij}^{(2)})_{n \times n}$, where $m_{ij}^{(2)} = t_{ij}^{(2)}$ for $2 \leq i \leq j \leq n$, and $m_{ij}^{(2)} = 0$ for $1 \leq j \leq n$. Then $(\delta - ad m_2)(e_{11}) = 0$ and $(\delta - ad m_2)(e_{22}) = \sigma_2(e_{22})e$. Denote $\delta - ad m_2$ by $\delta_1$.

By induction we assume that there are ad $m_j$, $j = 3, 4, \cdots, k - 1$ such that

$$(\delta_i - \sum_{j=3}^{k-1} ad m_j)(e_{11}) = 0 \text{ and } (\delta_i - \sum_{j=3}^{k-1} ad m_j)(e_{ii}) = \sigma_i(e_{ii})e,$$

where $\sigma_i \in \text{Hom}(D_n(R), R), i = 2, \cdots, k - 1$. Denote $\delta_1 = \sum_{j=3}^{k-1} ad m_j$ by $\delta_{k-2}$. It is obvious that $\delta_{k-2}$ is also a local derivation. By Lemma 3.1, there exist an inner derivation $ad t_k$ and a central derivation $\eta_{s_k}$, depending on $e_{kk}$, such that

$$\delta_{k-2}(e_{11} + e_{22} + \cdots + e_{kk}) = \sigma_2(e_{22})e + \cdots + \sigma_{k-1}(e_{k-1,k-1})e + ad t_k(e_{kk}) + \sigma_k(e_{kk})e.$$

(3.3)

On the other hand, since $\delta_{k-2}$ is a local derivation, we have

$$\delta_{k-2}(e_{11} + e_{22} + \cdots + e_{kk}) = ad s_k(e_{11} + e_{22} + \cdots + e_{kk}) + \alpha_k(e_{11} + e_{22} + \cdots + e_{kk})e,$$

(3.4)

where $s_k \in T_n(R)$ and $\alpha_k \in \text{Hom}(D_n(R), R)$, depending on $e_{11} + e_{22} + \cdots + e_{kk}$. Suppose $t_k = (t_{ij}^{(k)})_{n \times n}$. By (3.3) and (3.4), we have $t_{ij}^{(k)} = 0$ for $1 \leq j \leq k - 1$. Let $m_k = (m_{ij}^{(k)})_{n \times n}$,
where \( m_{ij}^{(k)} = t_{ij}^{(k)} \) for \( k \leq i \leq j \leq n \), and \( m_{st}^{(k)} = 0 \) for \( 1 \leq s \leq k-1, s \leq t \leq n \). Then 
\[
(\delta_{k-2} - \text{ad } m_k)(e_{11}) = 0 \text{ and } (\delta_{k-2} - \text{ad } m_k)(e_{ii}) = \sigma_i(e_{ii})e \text{ for } 2 \leq i \leq k.
\]

When \( k = n \), let \( m = \sum_{j=2}^n m_j \). Then 
\[
(\delta - \text{ad } m)(e_{11}) = 0, \text{ and } (\delta - \text{ad } m)(e_{ii}) = \sigma_i(e_{ii})e \text{ for } i = 2, 3, \cdots, n.
\]

Let \( \sigma \) be an \( R \)-linear map from \( D_n(R) \) to \( R \), and define \( \sigma(e_{11}) = 0, \sigma(e_{ii}) = \sigma_i(e_{ii}) \) for \( i = 2, 3, \cdots, n \). Then \( \sigma \in \text{Hom} (D_n(R), R) \) and \((\delta - \text{ad } m - \eta_n)(e_{ii}) = 0 \) for \( i = 1, 2, \cdots, n \).

**Lemma 3.3** Let \( \delta \) be a local derivation of \( T_n(R) \) satisfying \( \delta(e_{ii}) = 0 \) for \( i = 1, 2, \cdots, n \). Then \( \delta(e_{ij}) = a_{ij}e_{ij} \) for some \( a_{ij} \in R \) and \( 1 \leq i < j \leq n \).

**Proof** For \( e_{ii} + e_{ij}, j \neq i \), since \( \delta \) is a local derivation, from Lemma 3.1 we know there exist an inner derivation \( \text{ad } x_{ij} \) and a central derivation \( \eta_{ij} \), depending on \( e_{ii} + e_{ij} \), such that 
\[
\delta(e_{ii} + e_{ij}) = (\text{ad } x_{ij} + \eta_{ij})(e_{ii} + e_{ij}) = \text{ad } x_{ij}(e_{ii} + e_{ij}) + \gamma_{ij}(e_{ii} + e_{ij})e.
\]

On the other hand, by the definition of \( \delta \) and Lemma 3.1, we have 
\[
\delta(e_{ii} + e_{ij}) = \delta(e_{ii}) + \delta(e_{ij}) = \delta(e_{ij}) = \text{ad } p_{ij}(e_{ij}) \in n,
\]

where \( p_{ij} \in T_n(R) \) depending on \( e_{ij} \). By (3.5) and (3.6), we have 
\[
\text{ad } x_{ij}(e_{ii} + e_{ij}) = \text{ad } p_{ij}(e_{ij}).
\]

Similarly, there exists some \( y_{ij} \in T_n(R) \) such that 
\[
\text{ad } y_{ij}(e_{jj} + e_{ij}) = \text{ad } p_{ij}(e_{ij}).
\]

(3.7) and (3.8) imply that \( \delta(e_{ij}) = a_{ij}e_{ij} \) for some \( a_{ij} \in R \) and \( 1 \leq i < j \leq n \).

**Lemma 3.4** Let \( \delta \) be a local derivation of \( T_n(R) \). If \( \delta(e_{ii}) = 0 \) for \( i = 1, 2, \cdots, n \), then there exists some \( h \in T_n(R) \) such that \((\delta - \text{ad } h)(e_{ii+1}) = 0 \) for \( i = 1, 2, \cdots, n-1 \), and \((\delta - \text{ad } h)(e_{ii}) = 0 \) for \( i = 1, 2, \cdots, n \).

**Proof** By Lemma 3.3, we have \( \delta(e_{i,i+1}) = a_{i,i+1}e_{i,i+1} \) for some \( a_{i,i+1} \in R \). Let 
\[
h = \text{diag}(0, -a_{12}, -(a_{12} + a_{23}), \cdots, -(a_{12} + a_{23} + \cdots + a_{n-1,n})).
\]

Then \((\delta - \text{ad } h)(e_{i,i+1}) = 0 \) for \( i = 1, 2, \cdots, n-1 \), and \((\delta - \text{ad } h)(e_{ii}) = 0 \) for \( i = 1, 2, \cdots, n \).

**Lemma 3.5** Let \( \delta \) be a local derivation of \( T_n(R) \) satisfying \( \delta(e_{ii}) = 0 \) for \( i = 1, 2, \cdots, n \), and \( \delta(e_{i,i+1}) = 0 \) for \( i = 1, 2, \cdots, n-1 \). Then we have \( \delta(e_{i,i+k}) = 0 \) for any \( e_{i,i+k} \in T_n(R) \).

**Proof** We will prove this lemma by induction on \( k, k \geq 2 \).

When \( k = 2 \), for \( e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1} \), since \( \delta \) is a local derivation, by Lemma 3.1, there exist an inner derivation \( \text{ad } q_i^{(2)} \) and a central derivation \( \eta_{i,(2)}^{(2)} \), depending on \( e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1} \), such that 
\[
\delta(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}) = \text{ad } q_i^{(2)}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1})
\]
\[
\eta_{i,(2)}^{(2)}(e_{i,i+1} + e_{i+1,i+2} + e_{i,i+2} + e_{i+1,i+1}).
\]

(3.9)
On the other hand, By Lemma 3.3, we have
\[
\delta(e_{i,i+1} + e_{i+1,i} + e_{i,i+2} + e_{i+1,i+2}) = a_{i,i+2}e_{i,i+2}.
\] (3.10)
From (3.9) and (3.10), we have
\[
\text{ad} q_i^{(2)}(e_{i,i+1} + e_{i+1,i} + e_{i,i+2} + e_{i+1,i+2}) = a_{i,i+2}e_{i,i+2},
\]
this forces that \(a_{i,i+2} = 0\), that is to say \(\delta(e_{i,i+2}) = 0\).
By induction we assume that \(\delta(e_{i,i+m}) = 0\) for \(m = 2, 3, \cdots, k - 1\). For
\[
e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+k},
\]
similar to the case \(k = 2\), we can get there exists some \(q_i^{(k)} \in T_n(R)\) such that
\[
\text{ad} q_i^{(k)}(e_{i,i+1} + e_{i+1,i+k} + e_{i,i+k} + e_{i+1,i+k}) = a_{i,i+k}e_{i,i+k},
\]
which means that \(a_{i,i+k} = 0\). So \(\delta(e_{i,i+k}) = 0\) for any \(e_{i,i+k} \in T_n(R)\).
By those lemmas, we can prove the following theorem.

**Theorem 3.1** Let \(R\) be a commutative ring with identity, \(T_n(R)\) the Lie algebra consisting of all upper triangular \(n \times n\) matrices over \(R\). Then every local derivation \(\delta\) of \(T_n(R)\) is a derivation.

**Proof** Let \(\delta\) be a local derivation of \(T_n(R)\). When \(n \geq 2\), for \(e_{11} \in T_n(R)\), there exists a derivation \(\delta_{e_{11}}\), depending on \(e_{11}\), such that \(\delta(e_{11}) = \delta_{e_{11}}(e_{11})\). So \((\delta - \delta_{e_{11}})(e_{11}) = 0\). Clearly, \(\delta - \delta_{e_{11}}\) is also a local derivation of \(T_n(R)\). By Lemmas 3.2–3.5, we know there exist \(\eta_\sigma, \text{ad} m\) and \(\text{ad} h\) such that
\[
(\delta - \text{ad} m - \eta_\sigma - \text{ad} h)(e_{ij}) = 0 \text{ for } 1 \leq i \leq j \leq n,
\]
which imply that \(\delta = \text{ad} m + \eta_\sigma + \text{ad} h\), so \(\delta\) is a derivation.

When \(n = 1\), suppose that \(\delta(1) = b\), then for any \(x \in T_1(R) = R\), we have
\[
\delta(x) = x\delta(1) = xb = \eta_\sigma(x),
\]
where \(\sigma : R \to R, x \mapsto bx\) is an \(R\)-linear from \(R\) to \(R\). So \(\delta\) is a derivation.

**References**


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