U-ABUNDANT SEMIGROUPS WITH LEFT CENTRAL IDEMPOTENTS

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Abstract: In this paper, we study the semilattice decomposition of U-abundant semigroups with left central idempotents. By using this semilattice decomposition, it is proved that a semigroup $S$ is a U-abundant semigroup with left central idempotents if and only if it is a strong semilattice of a direct product $M_\alpha \times \Lambda_\alpha$, where $M_\alpha$ is a unipotent monoid and $\Lambda_\alpha$ is a right zero band. This result is the basis of the establishing of the structure theorem of U-abundant semigroups with left central idempotents.

Keywords: U-abundant semigroup; left central idempotent; unipotent monoid; $\sim$-Green’s relation

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1 Introduction

On a semigroup $S$ the relation $\tilde{L}$ is defined by the rule that for any elements $a, b$ of $S$, $a \tilde{L} b$ if and only if for all $e \in E$, $ae = a \Leftrightarrow be = b$. The relation $\tilde{R}$ is dually defined. The relations $\tilde{L}$ and $\tilde{R}$ on a semigroup $S$ are generalizations of the familiar $*$-Green’s relations $\mathcal{L}^*$ and $\mathcal{R}^*$. As usual, the join of $\tilde{L}$ and $\tilde{R}$ is denoted by $\tilde{D}$ and the intersection of them is denoted by $\tilde{H}$. Clearly, $\tilde{L}$ and $\tilde{R}$ are equivalences, but $\tilde{L}$ is generally not right compatible and $\tilde{R}$ is generally not left compatible. The $\tilde{L}$-class containing the element $a$ of the semigroup $S$ is denoted by $\tilde{L}_a$ or by $\tilde{L}_a(S)$ in case of ambiguity. One can see that there is at most one idempotent contained in each $\tilde{H}$-class. Furthermore, if $a$ and $b$ are both regular elements of a semigroup $S$, then $(a, b) \in \tilde{L}$ if and only if $(a, b) \in \mathcal{L}$. In particular, if $S$ itself is a regular semigroup, then $\tilde{L} = \mathcal{L}$ [1]. Dually, we also have $\tilde{R} = \mathcal{R}$ on a regular semigroup $S$.

A monoid is a semigroup with identity. A monoid is called unipotent if it does not contain any idempotents except identity. A semigroup in which each $\mathcal{L}$*-class and each $\mathcal{R}$*-class contains at least one idempotent is called abundant [2]. A semigroup $S$ is called U-semiabundant [3] if each $\tilde{L}$-class and each $\tilde{R}$-class of $S$ contains at least one idempotent. All

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abundant semigroups and \( U \)-semiabundant semigroups form two important classes of generalized regular semigroups. Moreover, we easily see that \( L \subseteq L^* \subseteq \tilde{L} \) and \( R \subseteq R^* \subseteq \tilde{R} \). Thus, abundant semigroups are obviously \( U \)-semiabundant semigroups, but \( U \)-semiabundant semigroups may not be abundant semigroups. This means that \( U \)-semiabundant semigroups are the generalizations of abundant semigroups in the range of generalized regular semigroups.

It is well known that a Clifford semigroup is a strong semilattice of groups \([4]\). And Fountain proved that an adequate semigroup with idempotents lying in the center is a strong semilattice of cancellative monoids \([5]\). Later on, the semilattice decomposition on abundant semigroups with left central idempotents has been investigated by Shum and Ren \([6]\).

In this paper, we will extend the above results to the class of \( U \)-abundant semigroups in which all idempotents are left central. Thus, the results of Clifford, Fountain and Shum are all amplified. The main techniques that we use in the study are the \( \sim \)-Green’s relations.

For terminologies and notations not given in this paper, the reader is referred to Lawson \([3]\) and Howie \([7]\).

2 Preliminaries

In this section, we first give some basic definitions and results concerning \( U \)-abundant semigroups with left central idempotents.

Definition 2.1 An idempotent \( e \) of a semigroup \( S \) is called a left central idempotent if \( xey = exy \) for all \( x, y \in S^1 \) and \( y \neq 1 \).

Definition 2.2 A \( U \)-semiabundant semigroup \( S \) is called a \( U \)-abundant semigroup if \( S \) satisfies the congruence condition, that is, \( \tilde{L} \) is a right congruence and \( \tilde{R} \) is a left congruence on \( S \) respectively.

Next, \( S \) is always a \( U \)-abundant semigroup with left central idempotents, that is, \( S \) is a \( U \)-abundant semigroup and all idempotents of \( S \) are left central.

Lemma 2.3 Each \( \tilde{L} \)-class of \( S \) contains a unique idempotent.

Proof Suppose \((e, f) \in \tilde{L} \) for \( e, f \in E \). Then, we have \( ef \sim f \). This leads to \( ef = e \) and \( fe = f \). Since \( f \) is a left central idempotent, we immediately have \( ef = fe \). Thereby \( e = ef = f \).

Remark We now denote a unique idempotent in the \( \tilde{L} \)-class containing the element \( a \) of \( S \) by \( a^* \). Then we have the following lemmas.

Lemma 2.4 The relation \( \tilde{L} \) is a congruence on \( S \).

Proof Let \((a, b) \in \tilde{L} \) for \( a, b \in S \). In order to show that \((ca, cb) \in \tilde{L} \) for any \( c \in S \), we suppose that \( caa = ca \) for any \( c \in E \). By Lemma 2.3, then there exists a unique idempotent \( c^* \) in \( \tilde{L}_c \) such that \((c, c^*) \in \tilde{L} \). Since \( \tilde{L} \) is a right congruence on \( U \)-abundant semigroups, we have \((ca, c^*) \in \tilde{L} \) for \( a \in S \). Thus, by the definition of \( \tilde{L} \), we get \( c^*ac = c^*a \). Thereby \( ac^*a^*e = ac^*a^*e \), because \( c^* \) is a left central idempotent and \( a = aa^* \). Furthermore, since \((a, b) \in \tilde{L} \) and \( \tilde{L} \) is a right congruence on a \( U \)-abundant semigroup, we also have \((ac^*a^*, bc^*a^*) \in \tilde{L} \). Thus, by the definition of \( \tilde{L} \), we immediately have \( bc^*a^*e = bc^*a^* \).
Hence, by the left centrality of $c^*$, we deduce that $c^*ba^*e = c^*ba^*$. Finally, and again, since $(e, c^*) \in \tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ is a right congruence on a $U$-abundant semigroup, we have $(cba^*, c^*ba^*) \in \tilde{\mathcal{L}}$. Thus, by the definition of $\tilde{\mathcal{L}}$, we can obtain that $cba^*e = cba^*$. As a result, we have $cbe = cb$, since $a^* = b^*$ and $b = bb^*$. Similarly, if $cbe = cb$ for any $e \in E$, then $cae = ca$. This leads to $(ca, cb) \in \tilde{\mathcal{L}}$. Hence, $\tilde{\mathcal{L}}$ is a right congruence on $S$. It is well known that $\tilde{\mathcal{L}}$ is a right congruence on a $U$-abundant semigroup $S$. This shows that $\tilde{\mathcal{L}}$ is a congruence on $S$.

**Lemma 2.5** $(ab)^* = a^*b^*$ for any $a, b \in S$.

**Proof** It is trivial that $(b, b^*) \in \tilde{\mathcal{L}}$ for any $b \in S$. Since $\tilde{\mathcal{L}}$ is a congruence on a $U$-abundant semigroup with left central idempotents, we have $(ab, ab^*) \in \tilde{\mathcal{L}}$ for any $a \in S$. Thus, it is obvious that $(ab)^*(ab^*) = (ab^*)^*$ by Lemma 2.3. Similarly, we know that $(ab^*, a^*b^*) \in \tilde{\mathcal{L}}$. By using Lemma 2.3 again, we get $(ab^*)^* = (a^*b^*)^* = a^*b^*$. Consequently, $(ab)^* = (ab^*)^* = a^*b^*$.

**Lemma 2.6** Define a relation $\sigma$ on $S$ by $a\sigma b$ if and only if $a^*b^* = b^*a^*$ for any $a, b \in S$. Then the relation $\sigma$ is a semilattice congruence on $S$.

**Proof** It is easy to see that $\sigma$ is an equivalent relation on $S$. Suppose that $a\sigma b$ for $a, b \in S$. Then $a^*b^* = b^*a^*$. By applying Lemma 2.5, we get $(ac)^*(bc) = a^*c^*b^*c^* = a^*b^*c^* = b^*c^* = (bc)^*$. By using similar arguments, we can also prove that $(bc)^*(ac)^* = (ac)^*$. This shows that $(ac, bc) \in \sigma$. Similarly, $(ca, cb) \in \sigma$. Hence, $\sigma$ is a congruence on $S$. Now, by Lemma 2.5 and the left centrality of $a^*$, we have $(ab)^*(ba)^* = a^*b^*a^* = b^*a^* = (ba)^*$ and $(ba)^*(ab)^* = (ab)^*$. Thus, we know immediately that $(ab, ba) \in \sigma$ by the definition of $\sigma$. In addition, it is obvious that $a^*a^*$ and $a^*a^2$ for any $a \in S$. We prove that $\sigma$ is indeed a semilattice congruence on $S$.

**Lemma 2.7** On a $U$-abundant semigroup $S$ with left central idempotents, we have

(i) $\tilde{\mathcal{L}} = \tilde{\mathcal{H}}$ and $\sigma = \tilde{\mathcal{R}} = \tilde{\mathcal{D}}$;

(ii) $\tilde{\mathcal{H}}, \tilde{\mathcal{R}}$ and $\tilde{\mathcal{D}}$ are all congruences on $S$.

**Proof** (i) Since $S$ is a $U$-abundant semigroup, there exists $e \in E$ such that $(a, e) \in \tilde{\mathcal{R}}$ for any $a \in S$ and so $ea = a$. Thus, by Lemma 2.5, we have $ea^* = a^*$. On the other hand, since $(a, a^*) \in \tilde{\mathcal{L}}$ and $a^*$ is a left central idempotent, we deduce that $a = aa^* = aa^*a^* = a^*aa^* = a^*a = a$. Thus, by using $(a, e) \in \tilde{\mathcal{R}}$ and the definition of $\tilde{\mathcal{R}}$, we have $ae = e$. From this, together with $ea^* = a^*$, we can deduce that $(a^*, e) \in \mathcal{R}$. This leads to $(a^*, e) \in \tilde{\mathcal{R}}$. Hence, $(a, a^*) \in \tilde{\mathcal{R}}$. Again since $(a, a^*) \in \tilde{\mathcal{L}}$, we have $(a, a^*) \in \tilde{\mathcal{H}}$. Now let $(a, b) \in \tilde{\mathcal{L}}$ for $a, b \in S$. Then, we have known that $(a, a^*) \in \tilde{\mathcal{H}}$ and $(b, b^*) \in \tilde{\mathcal{H}}$. By Lemma 2.3, we have $a^* = b^*$ and so $(a, b) \in \tilde{\mathcal{H}}$. This shows that $\tilde{\mathcal{L}} \subseteq \tilde{\mathcal{H}}$. Clearly, $\tilde{\mathcal{L}} = \tilde{\mathcal{H}} \subseteq \tilde{\mathcal{R}}$.

Let $(a, b) \in \tilde{\mathcal{R}}$ for $a, b \in S$. Then $(a^*, b^*) \in \tilde{\mathcal{R}}$, since $\tilde{\mathcal{L}} \subseteq \tilde{\mathcal{R}}$. Thus, we have $a^*b^* = b^*a^*$ and $b^*a^* = a^*$. Hence, we have $(a, b) \in \sigma$ by the definition of $\sigma$. This leads to $\tilde{\mathcal{R}} \subseteq \sigma$.

On the other hand, let $(a, b) \in \sigma$ for $a, b \in S$. Then $a^*b^* = b^*a^*$ and $b^*a^* = a^*$. In order to show that $(a, b) \in \tilde{\mathcal{R}}$, we suppose $ea = a$ for any $e \in E$. By Lemma 2.5, we have $ea^* = a^*$. Thus $ea^*b^* = a^*b^*$. This implies that $eb^* = b^*$. And again, since $(b, b^*) \in \tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}} \subseteq \tilde{\mathcal{R}}$, we immediately have $(b, b^*) \in \tilde{\mathcal{R}}$. Thus, by using the definition of $\tilde{\mathcal{R}}$, we know that $eb = b$. Similarly, if $eb = b$ for any $e \in E$, then $ea = a$. As a result, we obtain $(a, b) \in \tilde{\mathcal{R}}$. This leads
to \( \sigma \subseteq \widehat{R} \) and so \( \sigma = \widehat{R} \). Now, it is easy to see that

\[
\widehat{D} = \widehat{L} \lor \widehat{R} = \widehat{H} \lor \widehat{R} = \widehat{R}.
\]

This shows that \( \sigma = \widehat{R} = \widehat{D} \).

(ii) By using Lemma 2.4 and Lemma 2.6, we immediately know that \( \widehat{H}, \widehat{R} \) and \( \widehat{D} \) are all congruences on \( S \).

3 Main Results

We are now going to give the characteristic theorem for \( U \)-abundant semigroups with left central idempotents.

**Theorem 3.1** Let \( S \) be a semigroup. Then the following statements are equivalent:

(i) \( S \) is a \( U \)-abundant semigroup with left central idempotents;

(ii) \( S \) is a semilattice of direct products \( M_\alpha \times \Lambda_\alpha \), where \( M_\alpha \) is a unipotent monoid and \( \Lambda_\alpha \) is a right zero band for every \( \alpha \in \gamma \). Moreover, \( E \) is a right normal band;

(iii) \( S \) is a strong semilattice of direct products \( M_\alpha \times \Lambda_\alpha \), where \( M_\alpha \) is a unipotent monoid and \( \Lambda_\alpha \) is a right zero band for every \( \alpha \in \gamma \).

**Proof** (i)\(\Rightarrow\) (ii) Clearly, \( S = \bigcup_{\alpha \in \gamma} S_\alpha \), where \( S_\alpha \) is a \( \sigma \)-class of \( S \) on a semilattice \( Y \). We know that \( S_\alpha \cap E \neq \emptyset \), since \( \sigma = \widehat{R} \) and \( S \) is a \( U \)-abundant semigroup. Let \( M_\alpha = S_\alpha e_\alpha \) for some \( e_\alpha \in S_\alpha \cap E \). To show that \( M_\alpha \) is a unipotent monoid with an identity element \( e_\alpha \). Let \( y = xe_\alpha \in M_\alpha \) for any \( x \in S_\alpha \). Then \( ye_\alpha = xe_\alpha = y \). On the other hand, by using \( \sigma = \widehat{R} \), we get \((y, e_\alpha) \in \widehat{R}\) and so \( e_\alpha y = y \). Hence, \( M_\alpha \) is indeed a monoid with an identity element \( e_\alpha \). In particular, \( \Lambda_\alpha = \widehat{L} = \widehat{H} \). This means that \( M_\alpha \) contains a unique idempotent. Thus, we know that \( M_\alpha \) also is a unipotent monoid. Now, let \( \Lambda_\alpha \) be the set of all idempotents of \( S_\alpha \), that is, \( \Lambda_\alpha = S_\alpha \cap E \). Then it is clear that \( e\widehat{R}f \) for all \( e, f \in \Lambda_\alpha \). This implies that \( \Lambda_\alpha \) is a right zero band. We define a mapping \( \varphi : M_\alpha \times \Lambda_\alpha \rightarrow S_\alpha \) by \( \varphi(x, f) = xf \) for any \( (x, f) \in M_\alpha \times \Lambda_\alpha \). Then, for any \( (x, f), (y, k) \in M_\alpha \times \Lambda_\alpha \), we have

\[
\varphi(x, f)\varphi(y, k) = xfyk = xyfk = xyk = \varphi[(x, f)(y, k)].
\]

Thus, \( \varphi \) is a morphism. Furthermore, if \( \varphi(x, f) = \varphi(y, k) \), that is, \( xf = yk \), then \( xfe_\alpha = yke_\alpha \). Since \( \Lambda_\alpha \) is a right zero band and \( e_\alpha \) is an identity element in \( M_\alpha \), we have \( x = y \). In the meantime, we also have \( xf = xk \). By using Lemma 2.5, we obtain that \( x^*f = x^*k \). Again since \( \Lambda_\alpha \) is a right zero band and \( x^* \in \Lambda_\alpha \), we immediately have \( f = k \). This shows that \( \varphi \) is a monomorphism as well. In order to show that the mapping \( \varphi \) is onto. Let any \( a \in S_\alpha \). Then there exists a unique idempotent \( a^* \) such that \((a, a^*) \in \widehat{L} \). By Lemma 2.7, we know that \( \widehat{L} \subseteq \widehat{R} \) and \( \sigma = \widehat{R} \). This means that \((a, a^*) \in \sigma \) such that \( a^* \in S_\alpha \). Hence, \( a^* \in \Lambda_\alpha = S_\alpha \cap E \). Moreover, by using \( S_\alpha \cap E \neq \emptyset \), it is natural for us to know \( ae_\alpha \in M_\alpha \) for some \( e_\alpha \in S_\alpha \cap E \). In this case, we always have

\[
(ae_\alpha, a^*)\varphi = ae_\alpha a^* = aa^* = a,
\]
since $\Lambda_\alpha$ is a right zero band and $(a, a^*) \in \hat{L}$. This shows that the mapping $\varphi$ is onto. In conclusion, we prove that $S_\alpha \cong M_\alpha \times \Lambda_\alpha$.

Finally, since any $e \in E$ is a left central idempotent, we have $ehg = heg$ for all $e, h, g \in E$. This show that $E$ is a right normal band.

(ii)$\Rightarrow$(iii) To show that $S = \bigcup_{\alpha \in Y} S_\alpha$ is a strong semilattice of direct products $M_\alpha \times \Lambda_\alpha$, we pick any $\alpha, \beta \in Y$ with $\alpha \geq \beta$. First, let $a \in S_\alpha$ and $e_\beta \in S_\beta \cap E$. Then $e_\beta a \in S_\beta$.

According to this fact, we define a mapping $\theta_{\alpha, \beta} : S_\alpha \rightarrow S_\beta$ by $a\theta_{\alpha, \beta} = e_\beta a$ for any $a \in S_\alpha$ and some $e_\beta \in S_\beta \cap E$. Write $e_\beta a = (u, i) \in S_\beta$. Let $g = (1_\beta, i) \in S_\beta \cap E$ and $e_\beta = (1_\beta, j) \in S_\beta \cap E$, where $1_\beta$ is the identity element of $M_\beta$. Then

$$e_\beta ag = (u, i)(1_\beta, i) = (u, i) = e_\beta a$$

and

$$g = (1_\beta, i) = (1_\beta, j)(1_\beta, i) = e_\beta g.$$  \hfill (2.1)

Similarly, we let $b = (v, l) \in S_\alpha$ and $h = (1_\alpha, l) \in S_\alpha \cap E$. Then

$$hb = b.$$  \hfill (2.2)

Furthermore, since $E$ is a right normal band, we also have

$$ge_\beta h = e_\beta gh.$$  \hfill (2.4)

By using (2.1), (2.2), (2.3) and (2.4), we can proof that

$$e_\beta ae_\beta b = e_\beta age_\beta bb = e_\beta ae_\beta gbb = e_\beta agb = e_\beta ab.$$  

Thus, $a\theta_{\alpha, \beta}b\theta_{\alpha, \beta} = (ab)\theta_{\alpha, \beta}$. This shows that $\theta_{\alpha, \beta}$ is a morphism.

On the other hand, it is easy to prove that $\theta_{\alpha, \alpha} = 1_{S_\alpha}$ for any $\alpha \in Y$.

Now, let $a = (w, k) \in S_\alpha$ and $p = (1_\alpha, k) \in S_\alpha \cap E$ for $\alpha \in Y$. Then we have $pa = a$.

By the right normality of $E$, for some $e_\beta \in S_\beta \cap E$ and $e_\gamma \in S_\gamma \cap E$ with $\alpha \geq \beta \geq \gamma$, we can deduce that

$$e_\gamma e_\beta p = e_\beta e_\gamma \cdot e_\gamma p = e_\gamma p$$

so that

$$a\theta_{\alpha, \beta}e_{\beta, \gamma} = e_\gamma (e_\beta a) = e_\gamma e_\beta pa = e_\gamma pa = e_\gamma a = a\theta_{\alpha, \gamma}.$$  

Thus, $\theta_{\alpha, \gamma} = \theta_{\alpha, \beta}\theta_{\beta, \gamma}$.

Finally, let $a \in S_\alpha, b \in S_\beta$ for any $\alpha, \beta \in Y$. Then $ab = e_\alpha e_\beta(ab) \in S_\alpha \cap E$ for some $e_\alpha e_\beta \in S_\alpha \cap E$. Since $e_\alpha e_\beta a \in S_\alpha \beta$, by using (2.1), we known that there exists $f^2 = f \in S_\alpha \beta$ such that $e_\alpha e_\beta af = e_\alpha e_\beta a$ for $e_\alpha e_\beta \in S_\alpha \beta$. Moreover, by using (2.3), we also know that there exists $e^2 = e \in S_\beta$ such that $eb = b$ for $b \in S_\beta$. Thus, by the right normality of $E$ again, we have

$$e_\alpha e_\beta ac_{\alpha, \beta} b = e_\alpha e_\beta af c_{\alpha, \beta} eb = e_\alpha e_\beta ac_{\alpha, \beta} fe b = e_\alpha e_\beta af eb = e_\alpha e_\beta ab = ab.$$
This shows that \( ab = aθ_{α,β}bθ_{β,α} \). Hence, \( S = \bigcup_{α ∈ Y} S_α \) is indeed a strong semilattice of direct products \( S_α = M_α × Λ_α \), and denote it by \( S = [Y; S_α, θ_{α,β}] \).

(iii)⇒(i) Let \( S = [Y; S_α, θ_{α,β}] \) be a strong semilattice of direct products \( M_α × Λ_α \) for \( α ∈ Y \), where \( M_α \) is a unipotent monoid and \( Λ_α \) is a right zero band. And let \( x, y ∈ S^1 \) with \( y ≠ 1 \) and \( e ∈ E \). Then \( x ∈ S^1_α \), \( y ∈ S^1_β \) and \( e ∈ S_γ ∩ E \) for some \( α, β, γ ∈ Y \). Write \( αβγ = δ \), where \( δ ∈ Y \). Now, we suppose that

\[
\begin{align*}
xθ_{α,δ} &= (m, t) ∈ S_δ, \\
yθ_{β,δ} &= (v, l) ∈ S_δ, \\
eθ_{γ,δ} &= (1, q) ∈ S_δ.
\end{align*}
\]

Then

\[
xy = xθ_{α,δ} eθ_{γ,δ} yθ_{β,δ} = (m, t)(1, q)(v, l) = (mv, l).
\]

Similarly, we have \( ey = (mv, l) \). Thus, \( xey = exy \). This shows that the element \( e \) is a left central idempotent for any \( e ∈ E \). In other words, \( S \) is a semigroup with left central idempotents.

It remains to show that each \( \tilde{R} \)-class of \( S \) contains at least one idempotent. Let \( a = (u, i) ∈ S_α \) and \( f = (1, i) ∈ S_α ∩ E \) for any \( α ∈ Y \). In order to proof that \( (a, f) ∈ \tilde{R} \), we assume that \( ea = a \) for any \( e ∈ E \). Then \( e = (1, γ, k) ∈ S_γ ∩ E \) for some \( γ ∈ Y \). Because \( S \) is the semilattice of \( S_α \), we can immediately see that \( α ≤ γ \) for \( α, γ ∈ Y \). Thus,

\[
ef = (1, γ, k)θ_{γ,α} · (1, i)θ_{α,α} = (1, i) = f.
\]

Similarly, if \( ef = f \) for any \( e ∈ E \), then \( ea = a \). Hence, we know that \( (a, f) ∈ \tilde{R} \). This implies that each \( \tilde{R} \)-class of \( S \) contains at least one idempotent.

On the other hand, we need to proof that each \( \tilde{L} \)-class of \( S \) contains at least one idempotent. Let \( b = (w, h) ∈ S_α \) and \( g = (1, h) ∈ S_α ∩ E \) for any \( α ∈ Y \). In order to proof that \( (b, g) ∈ \tilde{L} \), we assume that \( be = b \) for any \( e ∈ E \). Then \( e = (1, γ, k) ∈ S_γ ∩ E \) for some \( γ ∈ Y \) and \( γ ≥ α \). Again, since \( be = b \), we have

\[
(w, h)(1, γ, k) = (w, h)θ_{α,α} · (1, γ, k)θ_{γ,α} = (w, kθ_{γ,α}) = (w, h).
\]

This implies that \( kθ_{γ,α} = h \). Hence, we have

\[
ge = (1, a, h)(1, γ, k) = (1, a, h)θ_{α,α} · (1, γ, k)θ_{γ,α} = (1, a, kθ_{γ,α}) = (1, a, h) = g.
\]

Similarly, if \( ge = g \) for any \( e ∈ E \), then \( be = b \). Thereby, we obtain that \( (b, g) ∈ \tilde{L} \). This implies that each \( \tilde{L} \)-class of \( S \) contains at least one idempotent. Thus, \( S \) is a \( U \)-semiabundant semigroup.

Finally, we shall show that \( \tilde{L} \) is a right congruence. We first let \( (a, b) ∈ \tilde{L} \) for \( a, b ∈ S \). Then \( a \) and \( b \) are the element of the same \( S_α \) for \( α ∈ Y \). Clearly, if \( (a, b) ∈ \tilde{L} \) but
$a \in S_\alpha, b \in S_\beta$, since $S$ is a $U$-semiabundant semigroup, there exist $f^2 = f \in S_\alpha \cap E$ and $g^2 = g \in S_\beta \cap E$ such that $(a, f) \in \tilde{\mathcal{L}}$ and $(b, g) \in \tilde{\mathcal{L}}$. This implies that $(f, g) \in \tilde{\mathcal{L}}$. Thus, $fg = f$ and $gf = g$, where the elements $fg, gf \in S_\alpha \cap E$, $f \in S_\alpha$ and $g \in S_\beta$. Hence, $\alpha = \beta$. According to this fact, we let $(a, b) \in \tilde{\mathcal{L}}$ for $a = (u, i) \in S_\alpha, b = (w, h) \in S_\alpha$. To show that $(ac, bc) \in \tilde{\mathcal{L}}$ for any $c \in S$, we assume that $ace = ac$ for any $e \in E$. Then $c = (v, j) \in S_\beta$ and $e = (1, k) \in S_\gamma$ for some $\beta, \gamma \in Y$. In the meantime, we have $\alpha, \beta \leq \gamma$ for $\alpha, \beta, \gamma \in Y$, since $S$ is a strong semilattice of $S_\alpha$. Furthermore, we obtain that

$$(u, i)(v, j)(1, k) = (u, i)\theta_{\alpha, \beta} \cdot (v, j)\theta_{\beta, \alpha} (1, k)\theta_{\gamma, \alpha}$$

$$= (u\theta_{\alpha, \beta} \cdot v\theta_{\beta, \alpha}, k\theta_{\gamma, \alpha})$$

$$= (u\theta_{\alpha, \beta} \cdot v\theta_{\beta, \alpha}, j\theta_{\beta, \alpha}).$$

This implies that $k\theta_{\gamma, \alpha} = j\theta_{\beta, \alpha}$. Hence, we immediately have that

$$bce = (w, h)(v, j)(1, k) = (w, h)\theta_{\alpha, \beta} \cdot (v, j)\theta_{\beta, \alpha} (1, k)\theta_{\gamma, \alpha}$$

$$= (w\theta_{\alpha, \beta} \cdot v\theta_{\beta, \alpha}, k\theta_{\gamma, \alpha}) = (w\theta_{\alpha, \beta} \cdot v\theta_{\beta, \alpha}, j\theta_{\beta, \alpha}) = bc.$$  

By using similar arguments, if $bce = bc$ for any $e \in E$, then $ace = ac$. Thereby, $(ac, bc) \in \tilde{\mathcal{L}}$. This show that $\tilde{\mathcal{L}}$ is a right congruence. Similarly, $\tilde{\mathcal{R}}$ is also a left congruence. In other words, $S$ is a $U$-abundant semigroup. Now summing up the above facts, we show that the semigroup $S$ is indeed a $U$-abundant semigroup with left central idempotents. The proof is completed.

**Remark** We notice here that the above theorem extends a known result of Shum and Ren (see [6], Theorem 3.1).

The following theorem can be proved by using the similar method as Theorem 3.1.

**Theorem 3.2** Let $S$ be a semigroup. Then the following statements are equivalent:

(i) $S$ is a regular semigroup with left central idempotents;

(ii) $S$ is a strong semilattice of right groups;

(iii) $S$ is a right Clifford semigroup and $E$ is a right normal band.

**Proof** We first note that $\tilde{\mathcal{L}} = \mathcal{L}$ and $\tilde{\mathcal{R}} = \mathcal{R}$ on regular semigroups. If $S$ is a regular semigroup with left central idempotents, then by the results of Lemma 2.7, we know immediately that $\mathcal{L} = \mathcal{H} \subseteq \mathcal{R} = \mathcal{D}$ and $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}$ are all congruences on $S$. Moreover, $S$ is a completely regular semigroup. In particular, for any $a \in S$, $a^*$ is a unique idempotent of the $\mathcal{H}$-class containing $a$. It following that each $\sigma$-class of $S$ is a direct product of a group and a right zero band, which is called a right group. Hence, by using the same arguments as Theorem 3.1, we can prove that (i)$\Rightarrow$(ii). The details are omitted. For (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i), it is the dual of Theorem 4.1 in [8].

**Remark** We notice here that the above theorem extends a known result of Clifford (see [7], Theorem 2.1).
References