A NOTE ON FOUR-PARAMETRIZED QUARTIC THUE EQUATIONS

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Abstract: In this paper, we study four-parametric quartic Thue equations. An effective upper bound of the solutions $(x, y)$ is obtained for the four-parametric quartic Thue equations by using simpler method to approximate certain algebraic numbers, which extends the number of parameters from 2 to 4.

Keywords: Thue equation; parametric Diophantine equation; rational approximation

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1 Introduction

Let $F(x, y) \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree $n \geq 3$. Thue [1] proved that the number of solutions of Thue equations $F(x, y) = k$ is finite. Furthermore, an explicit Thue equations can be solved by the method of A. Baker’s [2] linear form in logarithms of algebraic numbers. Therefore, researchers focused on the parametrized Thue equations and inequalities, and abundant results are obtained. For cubic Thue equations and inequalities, see Thomas [3], Xia-Chen-Zhang [4] and Hoshi [5]. For quartic case, see Chen-Voutier [6] and Xia-Chen-Zhang [7]. While for the sextic case, we refer to Wakabayashi [8].

However, all of the above Thue equations and inequalities have at most two parameters. An interesting question is that whether or not there exist a solvable strategy for Thue equations with more than two parameters. In this research, we use the method proposed in [7] to study the Thue equations which consists of four integral parameters. Our main result is the following:

Theorem Define

$$f(x, y) = sx^4 + 4tbdx^3y + 6b^2dx^2y^2 + 4b^3d^2txy^3 + sb^4d^2y^4 = N,$$

and let $\theta$ be the real root of $f(x, 1) = 0$, then we have

$$|y|^{3-\lambda} \leq \frac{cN\varepsilon^2}{\rho^3b^3\sqrt{d}},$$

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where $\lambda = \frac{\log(8\sqrt{s})}{\log((8\sqrt{s}a)^s)}$, $c = 32.3144d\sqrt{d}\left(4\sqrt{d^2+1}\right)^\lambda$, $\varepsilon = t\sqrt{d + \sqrt{t^2 + s^2}}$, and $\rho = \sqrt{\varepsilon^2 - s^2}$. If $\varepsilon > 64ds^3$, this is an effective upper bound for $|y|$.

### 2 Some Lemmas

Let $n$ be a positive integer, suppose $\alpha, w \in (0, 1)$ are real numbers,

\[ p_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{n - \alpha}{\alpha}\right)^{k} x^k, \quad q_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{n - \alpha}{\alpha}\right)^{n-k} x^k, \]

and $R_n(x) = x^\alpha q_n(x) - p_n(x)$.

**Lemma 1** Let $R_n(w)$ be defined above, we have $|R_n(w)| \leq (1 - w^\alpha)(1 - \sqrt{w})^{2n}$.

**Proof** Put $f(t) = \frac{(1-t)(w-t)}{t}$. By Lemma 1 in [7], we get

\[ |R_n(w)| \leq \left|\left(n + \alpha\right)\left(\frac{\alpha}{n + 1}\right)\right| \int_{1}^{w} f(t)^n t^{\alpha-1} dt. \]

Since $|f'(t)| = \frac{w-t^2}{t^2}$. It is easy to obtain $|f(t)| < |f(\sqrt{w})| = (1 - \sqrt{w})^2$.

While

\[ \left|\left(n + \alpha\right)\left(\frac{\alpha}{n + 1}\right)\right| = |\alpha| \cdot \frac{(n^2 - \alpha^2)(n+1)^2}{(n)^2} \leq \alpha, \]

we obtain

\[ |R_n(w)| \leq \alpha \int_{1}^{w} |1 - \sqrt{w}|^{2n} t^{\alpha-1} dt = (1 - w^\alpha)(1 - \sqrt{w})^{2n}. \]

Thus Lemma 1 is proved.

**Lemma 2** Let $q_n(w)$ be defined above, then we have

\[ |q_n(w)| < \frac{(1 + \sqrt{w})^{2n}}{\sqrt{w}^\alpha} \left(1 + \frac{1 - \sqrt{w}}{\pi}\right). \]

**Proof** By Lemma 2 in [7], we get

\[ q_n(w) = \frac{(-1)^n}{2\pi i} \oint_c (1 - wt)^{n-\alpha} (1 - t)^{n+\alpha} t^{-n-1} dt, \]

where $c$ denotes the integration path encircles the origin once in the positive sense.

In order to get the estimation of $q_n(w)$, we cut the complex plain from 1 to $1/\sqrt{w}$, and consider the integration path $c = c_1 + c_2 + c_3$, where $c_1$ denotes the path that starts from 1 and proceeds along the positive real axis to $1/\sqrt{w}$, $c_2$ denotes the path that circles the origin in positive sense by a circle of radius $1/\sqrt{w}$, and $c_3$ denotes the path that starts from $1/\sqrt{w}$ back to 1 along the lower part of real axis.

While putting $f_1(t) = \frac{(1-w)(1-t)}{t}$ and $g_1(t) = \frac{(1-w)^{1-\alpha}(1-t)^{1+\alpha}}{t^2}$, we have

\[ |q_n(w)| = \frac{1}{2\pi} \left|\oint_c f_1(t)^{n-1} g_1(t) dt\right| \tag{2.1} \]

\[ \leq \frac{1}{2\pi} \left(\int_{c_1} f_1(t)^{n-1} g_1(t) dt + \int_{c_2} f_1(t)^{n-1} g_1(t) dt + \int_{c_3} f_1(t)^{n-1} g_1(t) dt\right). \]
On the upper part of real axis, we have $\arg t = 0$, so it leads to
\[
| \int_{c_1} f_1(t)^{n-1} g_1(t) dt | = \int_1^{1/\sqrt{w}} |f_1(t)|^{n-1} |g_1(t)| dt.
\]
It’s easy to get
\[
|f_1(t)| < |f_1(\frac{1}{\sqrt{w}})| = (1 - \sqrt{w})^2;
\]
and
\[
|g_1(t)| = (\frac{1}{t} - w)^{-\alpha}(1 - \frac{1}{t})^{1+\alpha} < g(\frac{1}{\sqrt{w}}) = \sqrt{w}^{-\alpha}(1 - \sqrt{w})^2.
\]
Straight forward computation shows that
\[
| \int_{c_1} f_1(t)^{n-1} g_1(t) dt | \leq \sqrt{w}^{-\alpha}(1 - \sqrt{w})^{2n+1}.
\]
While on the lower part of real axis, we have $\arg t = 2\pi$. Similarly, it leads to
\[
| \int_{c_3} f_1(t)^{n-1} g_1(t) dt | = | \int_{1/\sqrt{w}}^{1} |f_1(te^{2\pi i})|^{n-1} |g_1(te^{2\pi i})| dt|,
\]
and
\[
| \int_{c_3} f_1(t)^{n-1} g_1(t) dt | \leq \sqrt{w}^{-\alpha}(1 - \sqrt{w})^{2n+1}.
\]
On the circle, after putting $t = e^{i\theta}/\sqrt{w}$, we have
\[
| \int_{c_2} f_1(t)^{n-1} g_1(t) dt | = | \int_{0}^{2\pi} f_1(e^{i\theta})^{n-1} g_1(e^{i\theta}) \frac{1}{\sqrt{w}} d\theta | = | \int_{0}^{2\pi} f_1(e^{i\theta})^{n-1} g_1(e^{i\theta}) \frac{1}{\sqrt{w}} d\theta |.
\]
It is easy to get
\[
|f_1(e^{i\theta})| \leq |f_1(\frac{1}{\sqrt{w}})| = (1 + \sqrt{w})^2
\]
and
\[
|g_1(e^{i\theta})| = |(\sqrt{w}e^{-i\theta} - w)^{1-\alpha}(\sqrt{w}e^{-i\theta} - 1)^{1+\alpha}|
\]
\[
= \sqrt{(w - 2\sqrt{w} \cos \theta + w^2)^{1-\alpha}(w - 2\sqrt{w} \cos \theta + 1)^{1+\alpha}}
\]
\[
= \sqrt{w}^{1-\alpha}(w + 1 - 2\sqrt{w} \cos \theta)
\]
\[
\leq \sqrt{w}^{1-\alpha}(1 + \sqrt{w})^2.
\]
So we have
\[
| \int_{c_3} f_1(t)^{n-1} g_1(t) dt | \leq \frac{2\pi}{\sqrt{w}}(1 + \sqrt{w})^{2n}.
\]
By (2.1), we complete the proof of Lemma 2.
Lemma 3 Let $p_n(w)$ be denoted above, then we have

$$|p_n(w)| < |(1 + \sqrt{w})^{2n} 2^{\alpha} (1 + \frac{1 - \sqrt{w}}{\pi})|.$$ 

Proof Note that

$$p_n(w) = w^n q_n\left(\frac{1}{w}\right),$$

and we know that the estimation of $|p_n(w)|$ is decided by that of $|q_n\left(\frac{1}{w}\right)|$.

By Lemma 2 in [7], we have

$$|p_n(w)| = w^n \left|\frac{(-1)^n}{2\pi i c} \oint_c (1 - \frac{t}{w})^{n-\alpha} (1 - t)^{n+\alpha} t^{-n-1} dt\right|,$$

where $c = c_1 + c_2 + c_3$. In detail, $c_1$ denotes the path that starts from $w$ and proceeds along the positive real axis to $\sqrt{w}$, $c_2$ denotes the path that circles the origin in positive sense by a circle of radius $\sqrt{w}$, and $c_3$ denotes the path that starts from $\sqrt{w}$ and return to $w$ along the lower part of real axis.

Now we put

$$f_2(t) = \frac{(1 - \frac{t}{w})(1-t)}{t} \text{ and } g_2(t) = \frac{(1 - \frac{t}{w})^{1-\alpha}(1-t)^{1+\alpha}}{t^2},$$

then we have

$$|p_n(w)| = w^n \frac{1}{2\pi} \left|\oint_c f_2(t)^n g_2(t) dt\right|$$

$$\leq w^n \frac{1}{2\pi} \left|\int_{c_1} f_2(t)^n g_2(t) dt\right| + \left|\int_{c_2} f_2(t)^n g_2(t) dt\right| + \left|\int_{c_3} f_2(t)^n g_2(t) dt\right|.$$ 

In a similar way as the proof in Lemma 2, we have

$$\left|\int_{c_2} f_2(t)^n g_2(t) dt\right| = \left|\int_{\sqrt{w}} f_2(t)^n g_2(t) dt\right|$$

$$\leq \left|\int_{\sqrt{w}} \left(1 - \frac{\sqrt{w}}{w}\right)^{n-1} \left(1 - \frac{1}{\sqrt{w}}\right)^{2} \left(\frac{1}{\sqrt{w}}\right)^{1-\alpha} dt\right|$$

$$\leq \frac{(1 - \sqrt{w})^{2n}}{w^n} (1 - \sqrt{w}) \sqrt{w^n},$$

and also

$$\left|\int_{c_3} f_2(t)^n g_2(t) dt\right| \leq \frac{(1 - \sqrt{w})^{2n}}{w^n} (1 - \sqrt{w}) \sqrt{w^n}.$$ 

On the circle, putting $t = \sqrt{w} e^{i\theta}$, we have

$$\left|\int_{c_2} f_2(t)^n g_2(t) dt\right| = \left|i \int_0^{2\pi} f_2(\sqrt{w} e^{i\theta})^{n-1} g_2(\sqrt{w} e^{i\theta}) \sqrt{w} e^{i\theta} d\theta\right|$$

$$= \left|\int_0^{2\pi} f_2(\sqrt{w} e^{i\theta})^{n-1} g_2(\sqrt{w} e^{i\theta}) \sqrt{w} d\theta\right|$$

$$\leq \left|\int_0^{2\pi} f_2(-\sqrt{w})^{n-1} g_2(-\sqrt{w}) \sqrt{w} d\theta\right|$$

$$= 2\pi \frac{(1 + \sqrt{w})^{2n}}{w^n} \sqrt{w^n}.$$
From (2.2), it is direct to prove Lemma 3.

**Lemma 4** (see [6]). Let \( \theta \) be an algebraic number. Suppose that there exists \( k_0 > 0, l_0, Q > 1, E > 1 \) such that for all \( n \) there are rational integers \( P_n, y_n \) with \( |Q_n| < k_0Q^n \) and \( |Q_n\theta - P_n| \leq l_0E^{-n} \) and suppose further that \( P_nQ_{n+1} \neq Q_nP_{n+1} \). Then, for any rational integers \( x \) and \( y, y \geq c/(2l_0) \), we have

\[
|x - y\theta| > \frac{c}{cy^n}, \quad \text{where } c = 2k_0Q(2l_0E)^{\lambda}, \quad \lambda = \frac{\log Q}{\log E}.
\]

**3 Proof of Theorem**

Let \( f(x, y), \varepsilon \) be as in Theorem and assume \( \varepsilon > 64dt^3, d > 1 \), where \( d, s, t, b, d \in \mathbb{Z} \) and \( t > 0 \) without loss of generality. Then we have

\[
f(x, 1) = sx^4 + 4tbdx^3 + 6b^2d^2x^2 + 4b^3d^2tx + sb^4d^2
\]

denoted by \( f(x) \). If denoting the root of \( f(x) = 0 \) as \( \theta \), it is easy to show that \( \theta \) satisfies

\[
\left( \frac{\theta + b\sqrt{d}}{\theta - b\sqrt{d}} \right)^4 = \frac{\sqrt{d} - s}{\sqrt{d} + s}.
\]

For simplicity, we denote \( z = t\sqrt{d} - s, u = t\sqrt{d} + s, \) and \( w = \frac{z}{u} \).

It is straightforward to get

\[
\sqrt{w} = \frac{t\sqrt{d} + \sqrt{t^2d - s^2} - s}{t\sqrt{d} + \sqrt{t^2d - s^2} + s} = \frac{\varepsilon - s}{\varepsilon + s}.
\]

Putting \( \rho = \sqrt{\varepsilon^2 - s^2} \), we have \( \sqrt{w} = \pm \frac{\varepsilon - s}{\rho}, \pm \frac{\varepsilon + s}{\rho} i \). Hence, the roots of \( f(x) = 0 \) are

\[
\begin{align*}
\theta_0 &= b\sqrt{d} - \frac{e - s + 1}{\rho} = b\sqrt{d} \frac{\varepsilon - s + \rho}{\varepsilon - s - \rho}, \\
\theta_1 &= b\sqrt{d} - \frac{e - s + 1}{\rho} = b\sqrt{d} \frac{\varepsilon - s - \rho}{\varepsilon - s + \rho}, \\
\theta_2 &= b\sqrt{d} - \frac{e - s + 1}{\rho} = b\sqrt{d} \frac{\varepsilon - s + \rho}{\varepsilon - s - \rho}, \\
\theta_3 &= b\sqrt{d} - \frac{e - s + 1}{\rho} = b\sqrt{d} \frac{\varepsilon - s - \rho}{\varepsilon - s + \rho}.
\end{align*}
\]

Let \( \delta_i = |x - y\theta_i| \) (\( i = 0, 1, 2, 3 \)), then we have

\[
\delta_2 = |x - \theta_2y| = |x - b\sqrt{d}\frac{\varepsilon - s - \rho}{\varepsilon} y| > \frac{\rho}{\varepsilon} b\sqrt{d}|y|.
\]

Similarly, we have \( \delta_3 > \frac{\rho}{\varepsilon} b\sqrt{d}|y| \). If \( \delta_0 < \delta_1 \), then we know that

\[
\delta_1 > \frac{\delta_0 + \delta_1}{2} > \frac{|x - y\theta_0 - (x - y\theta_1)|}{2} = |y||\theta_0 - \delta_1| = \frac{\rho}{s}\sqrt{\frac{d}{s}}|y|.
\]
If $\delta_1 < \delta_0$, we similarly have $\delta_0 > \frac{\sqrt{\sqrt{d}}}{s}|y|$. Since $|f(x, y)| = s\delta_0\delta_2\delta_3$, we obtain that

$$\min\{\delta_0, \delta_1\} < \frac{N}{s(\frac{\sqrt{\sqrt{d}}}{s}|y|)^2} = \frac{Ne^2}{\rho^3b^3\sqrt{d}^3|y|^3}. \quad (3.1)$$

Thus we obtain an upper bound for $\delta_0$ or $\delta_1$. Thereafter, we will get a lower bound for them by proving Theorem.

**Proof of Theorem**

Since

$$\left(\frac{\theta + b\sqrt{d}}{\theta - b\sqrt{d}}\right)^4 = \frac{t\sqrt{d} - s}{t\sqrt{d} + s} = w,$$

it is easy to write $\theta_0, \theta_1$ as $\theta_i = b\sqrt{d}\frac{aw^i + a'}{aw^i - a'}$ where $a = 1$, if $i = 0$; $a = \sqrt{d}$, if $i = 1$, and $'$ denote conjugate in $\mathbb{Q}[\sqrt{d}]$ that maps $Z + Z\sqrt{d}$ into $Z - Z\sqrt{d}$.

Since $0 < w < 1$, the lemmas can be applied into Theorem 1 after putting $\alpha = 1/4 \in (0, 1)$. Now we use the Padè approximation method to formulate rational integers approximation so as to give an effective measure of the irrationality of $\theta$.

Let $p_n(w), q_n(w), R_n(w)$ be defined above and $\theta$ is one of $\theta_0, \theta_1$. We know that

$$\theta = b\sqrt{d}\frac{aw^i + a'}{aw^i - a'} = b\sqrt{d}\frac{aw^i q_n(w) + a'q_n(w)}{aw^i q_n(w) - a'q_n(w)} = b\sqrt{d}a R_n(w) + b\sqrt{d}ap_n(w) + b\sqrt{d}a'q_n(w) \quad \frac{a R_n(w) + ap_n(w) - a'q_n(w)}{a R_n(w) + ap_n(w) - a'q_n(w)} = \frac{4^n(\sqrt{d})^{n+1}u^nb\sqrt{d}a R_n(w) + 4^n(\sqrt{d})^{n+1}u^nb\sqrt{d}ap_n(w) + b\sqrt{d}a'q_n(w)}{4^n(\sqrt{d})^{n+1}u^n a R_n(w) + 4^n(\sqrt{d})^{n+1}u^n (ap_n(w) - a'q_n(w))}.$$ 

After putting

$$P_n = 4^n(\sqrt{d})^{n+1}u^n(b\sqrt{d}a p_n(w) + b\sqrt{d}a'q_n(w))$$

and

$$Q_n = 4^n(\sqrt{d})^{n+1}u^n(ap_n(w) - a'q_n(w)),$$

we have

$$|P_n - Q_n\theta| = |4^n(\sqrt{d})^{n+1}u^n a R_n(w)(b\sqrt{d} - \theta)|,$$

and denoted it as $R_n$.

Since $P_n, Q_n \in Z$, from the estimation of $p_n(w)$ and $q_n(w)$ in Lemma 2 and Lemma 3,
we obtain

\[ |Q_n| = |4^n(\sqrt{d})^{n+1}u^n(a p_n(w) - a' q_n(w))| \]
\[ \leq 4^n(\sqrt{d})^{n+1}u^n|a|(|p_n(w)| + |q_n(w)|) \]
\[ \leq 4^n\sqrt{d}^{n+1}u^n(1 + \sqrt{w})^{2n} |a|(|\sqrt{w}^\alpha (1 + \frac{1 - \sqrt{w}}{\pi}) + \sqrt{w}^{-\alpha} (1 + \frac{1 - \sqrt{w}}{\pi}))| \]
\[ \leq 4^n\sqrt{d}^{n+1}u^n|a||(1 + \sqrt{w})^{2n}(\sqrt{w}^\alpha + \sqrt{w}^{-\alpha})(1 + \frac{1 - \sqrt{w}}{\pi})| \]
\[ = C_Q(\sqrt{d}u(1 + \sqrt{w})^2)^n = C_Q(8\sqrt{d}e)^n, \]

where

\[ C_Q = \sqrt{d}|a|(\sqrt{w}^\alpha + \sqrt{w}^{-\alpha})(1 + \frac{1 - \sqrt{w}}{\pi}). \]

Since \( |a| \leq \sqrt{d} \) and \( \varepsilon > 64ds^3 \), we have \( \sqrt{w} = \frac{e\varepsilon}{\sqrt{ds}} > 63/65 \), so we can estimate that \( \sqrt{w}^\alpha + \sqrt{w}^{-\alpha} < 2.00006 \), and \( 1 + \frac{1 - \sqrt{w}}{\pi} < 1.00979 \). Hence \( C_Q < 2.01965d \).

On the other hand, from Lemma 1, we have

\[ |R_n| = |4^n(\sqrt{d})^{n+1}u^n a R_n(w)(b\sqrt{d} - \theta)| \]
\[ < 4^n(\sqrt{d})^{n+1}u^n a|(1 - w^\alpha)(1 - \sqrt{w})^{2n}(b\sqrt{d} - \theta)| \]
\[ = C_R(\sqrt{d}u(1 - \sqrt{w})^2)^n = C_R(8\sqrt{ds^2/\varepsilon})^n, \]

where

\[ C_R = \sqrt{d}|a|(1 - w^\alpha)(b\sqrt{d} - \theta) \leq \sqrt{d}|a|(b\sqrt{d} + 1) \leq d(b\sqrt{d} + 1). \]

Let \( Q = 8\sqrt{d}e, k_0 = 2.01965b, E = \frac{\varepsilon}{s\sqrt{d}^2}, l_0 = d(b\sqrt{d} + 1) \), result in Lemma 4 yields to

\[ \delta_0, \delta_1 > \frac{1}{c|y|^\lambda}, \quad \text{(3.2)} \]

where \( \lambda = \frac{\log(8\sqrt{d}e)}{\log(\varepsilon/(8\sqrt{d}^2))} \) and \( c = 32.3144d\sqrt{\varepsilon(\sqrt{d}(b\sqrt{d}+1)\varepsilon)^\lambda}. \)

By (3.1) and (3.2), we obtain \( |y|^{\lambda-\lambda} \leq \frac{cN^2}{\rho^b\sqrt{d}}. \)

Thus Theorem follows.

Actually, when \( \varepsilon > 64ds^3 \), it directly leads to \( \lambda < 3 \), so we are able to derived an effective upper bound for \( y \).

In this research, a four-parametrized quartic Thue equations is solved by approximation certain crucial algebraic numbers in an elementary way. A computable upper bound for solutions is obtained as well, which is quite effective when \( \varepsilon \) is much greater than \( 64ds^3 \). In the mean time, the value of \( \frac{1}{\pi - \lambda} \) decreases dramatically when \( w \) approximate to 1.
References


含四个参数的四次Thue方程

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摘要：本文研究了含四个参数的四次Thue方程. 利用简单的代数数有理逼近方法给出了该方程解的有效上界, 从而将参数个数由两个推导到四个.

关键词：Thue方程; 含参丢番图方程; 有理逼近