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SZEGÖ KERNELS ON CERTAIN UNBOUNDED DOMAINS

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Abstract: The Szegö kernel of the unbounded domain D_ψ which was built on an arbitrary irreducible bounded circled homogeneous domain is considered. Using an explicit expression of certain integral over Cartan domain, we obtain the Szegö kernel on domain D_ψ in explicit formula.

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1 Introduction

Let $\phi(z)$ be a non-negative function on $\mathbb{C}^d$, and

$$D_\phi := \left\{(z_0, z) \in \mathbb{C} \times \mathbb{C}^d : \mathrm{Im} z_0 > \phi(z)\right\}. \tag{1.1}$$

The Hardy space $H^2(D_\phi)$ is defined by

$$H^2(D_\phi) := \left\{ f \in H(D_\phi) : \sup_{s>0} \int_{\mathbb{R}} \int_{\mathbb{C}^d} |f(z, t + \sqrt{-1}\phi(z) + \sqrt{-1}s)|^2 dt dm(z) < +\infty \right\},$$

where $H(D_\phi)$ denotes all holomorphic functions on the domain D_ϕ, and $dm(z)$ is the Lebesgue measure on $\mathbb{C}^d$. The closed subspace $H^2(\partial D_\phi)$ of $L^2(\partial D_\phi)$ consisting of boundary values of holomorphic functions $f \in H^2(D_\phi)$. The Szegö projection is the orthogonal projection

$$S : L^2(\partial D_\phi) \to H^2(\partial D_\phi),$$

and the Szegö kernel $S(z, t; u, s)$ is the distribution kernel on $\partial D_\phi \times \partial D_\phi$ give by

$$Sf(z, t) := \int_{\partial D_\phi} S(z, t; u, s) f(u, s) dm(u) ds, \tag{1.2}$$

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where the boundary ∂D_ϕ of D_ϕ be identified with $\mathbb{C}^d \times \mathbb{R}$, the coordinates are (z,t).

Let $d_1, d_2, \cdots, d_n$ be positive integers, $s_1, s_2, \cdots, s_n$ be positive real numbers, $\phi(z) = \sum\limits_{i=1}^{n} ||z_i||^{\frac{2}{s_i}}$ with $z_i = (z_{i1}, z_{i2}, \cdots, z_{id_i}) \in \mathbb{C}^{d_i}$, where $||z_i||^2 := \sum\limits_{j=1}^{d_i} |z_{ij}|^2$. In [1], Francsics and Hanges obtained the Szegö kernel of the unbounded domain D_ϕ. For special $\phi(z)$, the Szegö kernel of D_ϕ, see also the references of [1].

Let Ω_i be irreducible bounded symmetric domains (Cartan domains) in $\mathbb{C}^{d_i}$ in its Harish-Chandra realization, s_i be positive real numbers, $1 \le i \le n$. The following we assume that

$$D_\psi := \left\{(z_0, z) \in \mathbb{C} \times \mathbb{C}^d : \mathrm{Im} z_0 > \psi(z)\right\}, \tag{1.3}$$

where $\psi(z) = \sum\limits_{i=1}^{n} \|z_i\|^{\frac{2}{s_i}}$, $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_n}$, $d = \sum\limits_{j=1}^{n} d_j$ and $\|z_i\|$ are the spectral norms of z_i, see (3.5). If the ranks of Ω_i equal to 1 for all $1 \le i \le n$, then here $\psi(z)$ same as the above $\phi(z)$.

For convenience, we list classical domains and corresponding the generic norms $N(z, \overline{z})$ and the spectral norms $\|z\|$ as following (see [2, 3])

$$\mathfrak{R}_I(m,n) = \{z \in \mathbb{C}^{m\times n} : \|z\| < 1\}\ (m \le n), N(z,\overline{z}) = \det(I - zz^\dagger), \|z\| = \sqrt{\sup_{uu^\dagger=1} uzz^\dagger u^\dagger},$$

where $z^\dagger$ denotes the conjugation transposition of z. If the rank m of $\mathfrak{R}_I(m,n)$ equal to 1, then $\|z\| = \sqrt{zz^\dagger}$.

$$\mathfrak{R}_{II}(n) = \{z \in \mathbb{C}^{n\times n} : z = z^t, \|z\| < 1\}, N(z,\overline{z}) = \det(I - zz^\dagger), \|z\| = \sqrt{\sup_{uu^\dagger=1} uzz^\dagger u^\dagger},$$

where z^t denotes the transposition of z.

$$\mathfrak{R}_{III}(n) = \{z \in \mathbb{C}^{n\times n} : z = -z^t, \|z\| < 1\}, N(z,\overline{z}) = \sqrt{\det(I - zz^\dagger)}, \|z\| = \sqrt{\sup_{uu^\dagger=1} uzz^\dagger u^\dagger},$$

$$\mathfrak{R}_{IV}(n) = \{z \in \mathbb{C}^{n} : \|z\| < 1\}, N(z,\overline{z}) = 1 - 2zz^\dagger + zz^t\overline{zz^t}, \|z\| = \sqrt{zz^\dagger + \sqrt{(zz^\dagger)^2 - zz^t\overline{zz^t}}}.$$

In this note, by using the method of [1], we will calculate the Szegö kernel of D_ψ. In the following Section 2 and Section 3, we collect basic material about the generalized Selberg formula, integrals of Jack polynomials time the certain weight, and integrals of K_λ over the Cartan domain. In Section 4, we compute the Szegö kernel of D_ψ.

2 Integrals of Jack Polynomials

Lemma 2.1 (Generalized Selberg formula [4–6]) For give $\mathrm{Re}(x) > -1, \mathrm{Re}(y) > -1, \alpha > 0$, let λ be any partition of length $\ell(\lambda) \le n$, $P_\lambda^{(\alpha)}$ be the symmetric Jack polynomial, then we have

$$\begin{aligned}&\int_{[0,1]^n} P_\lambda^{(\alpha)}(x_1, x_2, \cdots, x_n) \prod_{i=1}^{n} x_i^x (1-x_i)^y \prod_{1\le j<k\le n} |x_j - x_k|^{\frac{2}{\alpha}} \prod_{j=1}^{n} dx_j \\ = \;& P_\lambda^{(\alpha)}(1_n) \frac{[x+1+\frac{n-1}{\alpha}]_\lambda^{(\alpha)}}{[x+y+2+2\frac{n-1}{\alpha}]_\lambda^{(\alpha)}} S_n(x,y;\alpha),\end{aligned} \tag{2.1}$$

where

$$S_n(x,y;\alpha)=\prod_{j=0}^{n-1}\frac{\Gamma(x+1+\frac{j}{\alpha})\Gamma(y+1+\frac{j}{\alpha})\Gamma(1+\frac{j+1}{\alpha})}{\Gamma(x+y+2+\frac{n+j-1}{\alpha})\Gamma(1+\frac{1}{\alpha})} \tag{2.2}$$

and

$$[s]_\lambda^{(\alpha)}:=\prod_{j=1}^{\ell(\lambda)}\left(s-\frac{j-1}{\alpha}\right)_{\lambda_j},$$

$P_\lambda^{(\alpha)}(1_n)$ denotes the value of the function $P_\lambda^{(\alpha)}(x_1,x_2,\cdots,x_n)$ at $(x_1,x_2,\cdots,x_n)=(1,1,\cdots,1)$.

The following we compute the integral of Jack polynomial times the certain weight using the generalized Selberg formula, to this end, we first give the following results.

Lemma 2.2 Let $\varphi(x_1,x_2,\cdots,x_n)$ and $f(x_1,x_2,\cdots,x_n)$ be continuous real functions such that

(1) $\forall x_i\geq 0, 1\leq i\leq n$, $\varphi(x)\geq 0$. $\varphi(x)=0$ if and only if $x=0$.

(2) $\forall\alpha\in\mathbb{R},\varphi(\alpha x)=|\alpha|\varphi(x)$.

(3) $\forall t\in\mathbb{R}$, $f(tx)=t^d f(x)$. $\forall x_i\geq 0,1\leq i\leq n$, $f(x)\geq 0$.

Then for all positive real numbers s,s_1,s_2, we have

(1)

$$\int_{\substack{x\geq 0\\ \varphi(x)<1}} f(x)\left(1-\varphi(x)^{\frac{1}{s_2}}\right)^{s_1}dx=\frac{\Gamma(s_1+1)\Gamma(s_2(d+n)+1)}{\Gamma(s_1+s_2(d+n)+1)}\int_{\substack{x\geq 0\\ \varphi(x)<1}} f(x)dx, \tag{2.3}$$

where $x=(x_1,x_2,\cdots,x_n)$, $dx:=dx_1dx_2\cdots dx_n$, and $x\geq 0$ mean $\forall i,1\leq i\leq n,x_i\geq 0$.

(2)

$$\int_{x\geq 0} f(x)\exp\{-\varphi(x)^{\frac{1}{s}}\}dx=\Gamma(s(d+n)+1)\int_{\substack{x\geq 0\\ \varphi(x)<1}} f(x)dx. \tag{2.4}$$

Proof (1) Let

$$I=\int_{\substack{u+\varphi(x)^{\frac{1}{s_2}}<1\\ u\geq 0,x\geq 0}} u^{s_1-1}f(x)dudx. \tag{2.5}$$

On the one hand,

$$\begin{aligned} I &= \int_0^1 u^{s_1-1}du\int_{\substack{\varphi(x)<(1-u)^{s_2}\\ x\geq 0}} f(x)dx=\int_0^1 u^{s_1-1}(1-u)^{s_2(d+n)}du\int_{\substack{\varphi(x)<1\\ x\geq 0}} f(x)dx\\ &= \frac{\Gamma(s_1)\Gamma(s_2(d+n)+1)}{\Gamma(s_1+s_2(d+n)+1)}\int_{\substack{\varphi(x)<1\\ x\geq 0}} f(x)dx. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} I &= \int_{\substack{\varphi(x)<1\\ x\geq 0}} f(x)dx\int_{0\leq u<1-\varphi(x)^{\frac{1}{s_2}}} u^{s_1-1}du=\int_{\substack{\varphi(x)<1\\ x\geq 0}} f(x)(1-\varphi(x)^{\frac{1}{s_2}})^{s_1}dx\int_0^1 u^{s_1-1}du\\ &= \frac{1}{s_1}\int_{\substack{\varphi(x)<1\\ x\geq 0}} f(x)(1-\varphi(x)^{\frac{1}{s_2}})^{s_1}dx. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we obtain (2.3).

(2) In (2.3), we set $s_1 = m^{\frac{1}{s}}, s_2 = s$, by the change of variables $x = \frac{y}{m}, m > 0$, we have

$$\int_{\substack{y\geq 0\\ \varphi(y)<m}} f(y)\left(1-\frac{1}{m^{\frac{1}{s}}}\varphi(y)^{\frac{1}{s}}\right)^{m^{\frac{1}{s}}} dy = m^{d+n}\frac{\Gamma(m^{\frac{1}{s}}+1)\Gamma(s(d+n)+1)}{\Gamma(m^{\frac{1}{s}}+s(d+n)+1)}\int_{\substack{x\geq 0\\ \varphi(x)<1}} f(x)dx. \tag{2.8}$$

When $m \to +\infty$, limit of L.H.S. of (2.8) is $\int_{y\geq 0} f(y)\exp\{-\varphi(y)^{\frac{1}{s}}\}dy$, and limit of R.H.S. of (2.8) is

$$\Gamma(s(d+n)+1)\int_{\substack{x\geq 0\\ \varphi(x)<1}} f(x)dx.$$

Here we use $\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}(x \to +\infty)$. So we obtain (2.4). The proof is completed.

For $s > 0, x = (x_1, x_2, \cdots, x_n)$, setting $\|x\|_s := (\sum\limits_{i=1}^{n} |x_i|^s)^{\frac{1}{s}}$. It is easy to see $\|x\|_1 := \sum\limits_{i=1}^{n} |x_i|$ and $\|x\|_\infty = \max\limits_{1\leq i\leq n}\{|x_i|\}$. Let

$$P(\lambda,\alpha,b,n,x) := P_\lambda^{(\alpha)}(x_1,x_2,\cdots,x_n)\prod_{i=1}^{n} x_i^b \prod_{1\leq j<k\leq n} |x_j - x_k|^{\frac{2}{\alpha}}.$$

Since homogeneous the function $P(\lambda,\alpha,b,n,x)$ of degree $d = |\lambda| + nb + \frac{n(n-1)}{\alpha}$ satisfies Lemma 2.2. By Lemma 2.2, we have the following Corollary 2.3.

Corollary 2.3 For all positive real numbers s, t, we have

$$\begin{aligned}&\int_{x\geq 0} P(\lambda,\alpha,b,n,x)\exp\{-\|x\|_s^{\frac{1}{t}}\}dx\\ =\ &\Gamma(t(|\lambda|+nb+\frac{n(n-1)}{\alpha}+n)+1)\int_{\substack{x\geq 0\\ \|x\|_s<1}} P(\lambda,\alpha,b,n,x)dx.\end{aligned} \tag{2.9}$$

As a consequence of Lemma 2.1, we have

$$\begin{aligned}&\int_{\substack{x\geq 0\\ \|x\|_\infty\leq 1}} P(\lambda,\alpha,b,n,x)\\ =\ &P_\lambda^{(\alpha)}(1_n)\frac{[b+1+\frac{n-1}{\alpha}]_\lambda^{(\alpha)}}{[b+2+2\frac{n-1}{\alpha}]_\lambda^{(\alpha)}}\prod_{j=0}^{n-1}\frac{\Gamma(b+1+\frac{j}{\alpha})\Gamma(1+\frac{j}{\alpha})\Gamma(1+\frac{j+1}{\alpha})}{\Gamma(b+2+\frac{n+j-1}{\alpha})\Gamma(1+\frac{1}{\alpha})}.\end{aligned} \tag{2.10}$$

From Corollary 2.3 and (2.10) we get

Corollary 2.4 For all positive real numbers s, we have

$$\begin{aligned}&\int_{x\geq 0} P(\lambda,\alpha,b,n,x)\exp\{-\|x\|_\infty^{\frac{1}{s}}\}dx\\ =\ &\Gamma(s(|\lambda|+nb+\frac{n(n-1)}{\alpha}+n)+1)P_\lambda^{(\alpha)}(1_n)\frac{[b+1+\frac{n-1}{\alpha}]_\lambda^{(\alpha)}}{[b+2+2\frac{n-1}{\alpha}]_\lambda^{(\alpha)}}S_n(b,0;\alpha).\end{aligned} \tag{2.11}$$

3 Integrals of K_λ over the Cartan Domain

Let $\Omega \subset \mathbb{C}^d$ be Cartan domain, we denote by r, a, b, d, p and $N(z, \overline{w})$ the rank, the characteristic multiplicities, the dimension, the genus, and the generic norm of Ω, respectively. Let $\mathcal{G}$ stand for the identity connected component of the group of biholomorphic self-maps of Ω, and $\mathcal{K}$ for the stabilizer of the origin in $\mathcal{G}$.

Under the action $f \mapsto f \circ k (k \in \mathcal{K})$ of $\mathcal{K}$, the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^d$ admits the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\lambda} \mathcal{P}_\lambda,$$

where the spaces $\mathcal{P}_\lambda$ are $\mathcal{K}$-invariant and irreducible.

Let

$$\langle f, g\rangle_{\mathcal{F}} := \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z)\overline{g(z)} e^{-|z|^2} dm(z), \tag{3.1}$$

where $dm(z)$ denotes the Lebesgue measure on $\mathbb{C}^d$.

For every partition λ, let $K_\lambda(z_1, \overline{z_2})$ be the reproducing kernel of $\mathcal{P}_\lambda$ with respect to (3.1). The kernels $K_\lambda(z_1, \overline{z_2})$ are related to the generic norm $N(z_1, \overline{z_2})$ by the Faraut- Korányi formula

$$N(z_1, \overline{z_2})^{-s} = \sum_{\lambda} (s)_\lambda K_\lambda(z_1, \overline{z_2}), \tag{3.2}$$

where $(s)_\lambda$ denotes the generalized Pochhammer symbol

$$(s)_\lambda := \prod_{j=1}^{r} \left(s - \frac{j-1}{2}a\right)_{\lambda_j}. \tag{3.3}$$

Here $(s)_m$ denotes the raising factorial

$$(s)_m := \frac{\Gamma(s+m)}{\Gamma(s)} = s(s+1)\cdots(s+m-1).$$

Let $e_1, e_2, \cdots, e_r \in \mathbb{C}^d$ be a Jordan frame. Then each $z \in \mathbb{C}^d$ has the polar decomposition

$$z = k \cdot (t_1 e_1 + t_2 e_2 + \cdots + t_r e_r), \quad k \in \mathcal{K}, t_1 \geq t_2 \geq \cdots \geq t_r \geq 0. \tag{3.4}$$

The numbers $t_1, t_2, \cdots, t_r$ are called the singular values of z. the spectral norm of z is defined by

$$\|z\| := \max\{t_1, t_2, \cdots, t_r\}. \tag{3.5}$$

It is known that

$$K_\lambda(z, \overline{z}) = K_\lambda(\sum_{j=1}^{r} t_j^2 e_j, \overline{e}), \tag{3.6}$$

For the proofs of above facts and additional details, we refer e.g. to [7].

Lemma 3.1 For give a positive real number s, we have

$$\int_{\mathbb{C}^d} K_\lambda(z, \overline{z}) \exp\{-\|z\|^{\frac{2}{s}}\} dm(z) = \Gamma(s(|\lambda| + d) + 1) \frac{\dim \mathcal{P}_\lambda}{(p)_\lambda} V(\Omega), \tag{3.7}$$

where $V(\Omega)$ is the volume with respect to the Euclidean measure of Ω.

Proof We recall the formula for integration in polar coordinates (see [7])

$$\int_{\mathbb{C}^d} f(z)dm(z) = c\int_{[0,+\infty)^r} 2^r \prod_{j=1}^r t_j^{2b+1} \prod_{1\le j<k\le r} |t_j^2 - t_k^2|^a \prod_{j=1}^r dt_j \int_{\mathcal{K}} f(k\cdot \sum_{j=1}^r t_j e_j)dk, \quad (3.8)$$

where c is a constant. By using (3.8), (3.5) and (3.6), we have

$$\text{L.H.S. of (3.7)} = c\int_{[0,+\infty)^r} K_\lambda(\sum_{j=1}^r t_j e_j, \overline{e})\exp\{-\|t\|_\infty^{\frac{1}{s}}\} \prod_{j=1}^r t_j^b \prod_{1\le j<k\le r} |t_j - t_k|^a \prod_{j=1}^r dt_j. \quad (3.9)$$

For each partition λ, since the polynomial $K_\lambda(\sum\limits_{j=1}^r t_j e_j, \overline{e})$ in $t_1, t_2, \cdots, t_r$ is proportional to the Jack polynomial $P_\lambda^{(\frac{2}{a})}(t_1, t_2, \cdots, t_r)$, by (2.11) for (3.9), we get

$$\text{L.H.S. of (3.7)} = \Gamma(s(|\lambda| + d) + 1)cS_r(b, 0; 2/a)\frac{K_\lambda(e, \overline{e})(\frac{d}{r})_\lambda}{(p)_\lambda}, \quad (3.10)$$

where we have used $d = rb + \frac{a}{2}r(r-1) + r, p = (r-1)a + b + 2, (s)_\lambda = [s]_\lambda^{(\frac{2}{a})}$.

It is well known that (see [8])

$$\dim \mathcal{P}_\lambda = K_\lambda(e, \overline{e})(\frac{d}{r})_\lambda. \quad (3.11)$$

Applying (3.8) we obtain

$$V(\Omega) := \int_\Omega dm(z) = cS_r(b, 0; 2/a). \quad (3.12)$$

Substituting (3.11), (3.12) into (3.10), we have (3.7). The proof is completed.

4 The Szegö Kernel of D_ψ

Let $\|z_i\|$ be the spectral norms of Carta domains $\Omega_i (1 \le i \le n)$, in this section, we will calculate the Szegö kernel of a domain D_ψ

$$D_\psi := \left\{(z_0, z) \in \mathbb{C}\times\mathbb{C}^d : \mathrm{Im} z_0 > \psi(z)\right\},$$

where

$$\psi(z) := \sum_{i=1}^n \|z_i\|^{\frac{2}{s_i}},\ z = (z_1, z_2, \cdots, z_n),\ d = \sum_{j=1}^n \dim\Omega_j.$$

Theorem 4.1 Let $r_i, a_i, b_i, d_i, p_i, (s)_\lambda^{(i)}$, $V(\Omega_i)$ and N_i be ranks, characteristic multiplicities, dimensions , genuses, generalized Pochhammer symbols, volumes and generic norms of Cartan domains Ω_i, $1 \le i \le n$, respectively. The Szegö kernel of D_ψ is

$$\begin{aligned} & S(z_0, t_1 z_1, \cdots, t_n z_n; \overline{u_0}, \overline{u_1}, \cdots, \overline{u_n}) \\ = & \frac{1}{4\pi}\prod_{i=1}^n \frac{1}{V(\Omega_i)} \frac{1}{A^{1+\sum\limits_{i=1}^n s_i d_i}} \varphi(t_1\frac{d}{dt_1}, \cdots, t_n\frac{d}{dt_n}) \prod_{i=1}^n \left\{\frac{1}{N_i(\frac{t_i z_i}{A^{s_i}}, \overline{u_i})}\right\}^{p_i}, \end{aligned} \quad (4.1)$$

where $\forall 1 \leq i \leq n, t_i \in [0,1]$, the function $\varphi(t_1, \cdots, t_n)$ is given

$$\varphi(t_1, \cdots, t_n) = \frac{\Gamma(\sum\limits_{i=1}^{n} s_i(d_i + t_i) + 1)}{\prod\limits_{i=1}^{n} \Gamma(s_i(d_i + t_i) + 1)}, \tag{4.2}$$

$\varphi(t_1 \frac{d}{dt_1}, \cdots, t_n \frac{d}{dt_n})$ is defined by

$$\varphi(t_1 \frac{d}{dt_1}, \cdots, t_n \frac{d}{dt_n}) t_1^{i_1} \cdots t_n^{i_n} := \varphi(i_1, \cdots, i_n) t_1^{i_1} \cdots t_n^{i_n}, \tag{4.3}$$

here $i_1, \cdots, i_n \in \mathbb{N}$, and

$$A = -\frac{\sqrt{-1}}{2}(z_0 - \overline{u_0}) = \frac{1}{2}\left(\sum_{i=1}^{n}\left(\|z_i\|^{\frac{2}{s_i}} + \|u_i\|^{\frac{2}{s_i}}\right) - \sqrt{-1}\mathrm{Re}(z_0 - u_0)\right). \tag{4.4}$$

In particular for $n = 1$, the Szegö kernel of D_ψ is

$$S(z_0, z_1; \overline{u_0}, \overline{u_1}) = \frac{1}{4\pi V(\Omega_1) A^{1+s_1 d_1}} \left\{\frac{1}{N_1(\frac{z_1}{A^{s_1}}, \overline{u_1})}\right\}^{p_1}. \tag{4.5}$$

Proof By [1], the Szegö kernel of D_ψ is written as

$$S(z_0, z; \overline{u_o}, \overline{u}) = \int_0^{+\infty} \exp\{-4\pi t A\} K_t(z, \overline{u}) dt, \tag{4.6}$$

where $K_t(z, \overline{u})$ is the reproducing kernels of the Hilbert spaces $L_a^2(\mathbb{C}^d, \rho_t)$, here

$$L_a^2(\mathbb{C}^d, \rho_t) := \{f \in H(\mathbb{C}^d) | (f, f) < +\infty\}, \quad \rho_t(z) := \exp\{-4\pi t \phi(z)\},$$

where $H(\mathbb{C}^d)$ denotes the space of holomorphic functions on $\mathbb{C}^d$, and the inner product $(\cdot, \cdot)$ is defined by

$$(f, g) := \int_{\mathbb{C}^d} f(z)\overline{g(z)} \rho_t(z) dm(z).$$

Let $\mathcal{G}_i$ stand for the identity connected components of groups of biholomorphic self-maps of $\Omega_i \subset \mathbb{C}^{d_i}$, and $\mathcal{K}_i$ for stabilizers of the origin in $\mathcal{G}_i$, $1 \leq i \leq n$, respectively. For any $k = (k_1, \cdots, k_n) \in \mathcal{K} := \mathcal{K}_1 \times \cdots \times \mathcal{K}_n$, we define the action

$$\pi(k) f(z_1, \cdots, z_n) \equiv f \circ k(z_1, \cdots, z_n) := f(k_1 \circ z_1, \cdots, k_n \circ z_n)$$

of $\mathcal{K}$, then the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}$ admits the decomposition

$$\mathcal{P} = \bigoplus_{\substack{\ell(\lambda^i) \leq r_i \\ 1 \leq i \leq n}} \mathcal{P}_{\lambda^1}^{(1)} \otimes \cdots \otimes \mathcal{P}_{\lambda^n}^{(n)},$$

where spaces $\mathcal{P}_{\lambda^i}^{(i)}$ are $\mathcal{K}_i$-invariant and irreducible subspaces of spaces of holomorphic polynomials on $\mathbb{C}^{d_i} (1 \leq i \leq n)$.

Since $\mathbb{C}^d=\mathbb{C}^{d_1}\times\cdots\times\mathbb{C}^{d_n}$ invariant under the action of $\mathcal{K}_1\times\cdots\times\mathcal{K}_n$, $L_a^2(\mathbb{C}^d,\rho_t)$ admits an irreducible decomposition (see ref. [9])

$$L_a^2(\mathbb{C}^d,\rho_t)=\widehat{\bigoplus_{\substack{\ell(\lambda^i)\le r_i\\ 1\le i\le n}}}\mathcal{P}_{\lambda^1}^{(1)}\otimes\cdots\otimes\mathcal{P}_{\lambda^n}^{(n)},$$

where $\widehat{\bigoplus}$ denotes the orthogonal direct sum.

For every partition λ^i of length less than or equal to r_i, let $K_{\lambda^i}^{(i)}(z_i,\overline{u_i})$ be the reproducing kernels of $\mathcal{P}_{\lambda^i}^{(i)}$ with respect to (3.1). By Schur's lemma, there exist positive constants $c_{\lambda^1\cdots\lambda^n}$ such that $c_{\lambda^1\cdots\lambda^n}\prod\limits_{i=1}^n K_{\lambda^i}^{(i)}(z_i,\overline{u_i})$ is reproducing kernels of $\mathcal{P}_{\lambda^1}^{(1)}\otimes\cdots\otimes\mathcal{P}_{\lambda^n}^{(n)}$ with respect to the above inner product $(\cdot,\cdot)$. According to the definition of reproducing kernel, we have

$$\int_{\mathbb{C}^d}c_{\lambda^1\cdots\lambda^n}\prod_{i=1}^n K_{\lambda^i}^{(i)}(z_i,\overline{z_i})\rho_t(z_1,\cdots,z_n)\prod_{i=1}^n dm(z_i)=\prod_{i=1}^n\dim\mathcal{P}_{\lambda^i}^{(i)}.$$

Therefore, the reproducing kernels of $L_a^2(\mathbb{C}^d,\rho_t)$ can be written as

$$K_t(z_1,\cdots,z_n;\overline{u_1},\cdots,\overline{u_n})=\sum_{\substack{\ell(\lambda^i)\le r_i\\ 1\le i\le n}}\frac{\prod\limits_{i=1}^n\dim\mathcal{P}_{\lambda^i}^{(i)}}{<\prod\limits_{i=1}^n K_{\lambda^i}^{(i)}(z_i,\overline{z_i})>}\prod_{i=1}^n K_{\lambda}^{(i)}(z_i,\overline{u_i}),\tag{4.7}$$

where $<f>$ denotes integral

$$\int_{\mathbb{C}^d}f(z_1,\cdots,z_n)\rho_t(z_1,\cdots,z_n)\prod_{i=1}^n dm(z_i).$$

From (3.7), we have

$$\begin{aligned}&\int_{\mathbb{C}^d}\prod_{i=1}^n K_{\lambda^i}^{(i)}(z_i,\overline{z_i})\rho_t(z_1,\cdots,z_n)\prod_{i=1}^n dm(z_i)\\ =&\prod_{i=1}^n(4\pi t)^{-s_i(|\lambda^i|+d_i)}\Gamma(s_i(|\lambda^i|+d_i)+1)\frac{\dim\mathcal{P}_{\lambda^i}^{(i)}}{(p_i)_{\lambda^i}^{(i)}}V(\Omega_i).\end{aligned}\tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\begin{aligned}&K_t(z_1,\cdots,z_n;\overline{u_1},\cdots,\overline{u_n})\\ =&\sum_{\substack{\ell(\lambda^i)\le r_i\\ 1\le i\le n}}(4\pi t)^{\sum_{i=1}^n s_i(|\lambda^i|+d_i)}\prod_{i=1}^n\frac{1}{\Gamma(s_i(|\lambda^i|+d_i)+1)V(\Omega_i)}\prod_{i=1}^n(p_i)_{\lambda^i}^{(i)}K_{\lambda}^{(i)}(z_i,\overline{u_i}).\end{aligned}\tag{4.9}$$

Now substituting (4.9) into (4.6), by

$$\int_0^{+\infty}\exp\{-4\pi tA\}(4\pi t)^{\sum\limits_{i=1}^n s_i(|\lambda^i|+d_i)}dt=\frac{1}{4\pi}A^{-\sum\limits_{i=1}^n s_i(|\lambda^i|+d_i)-1}\Gamma(\sum_{i=1}^n s_i(|\lambda^i|+d_i)+1),\tag{4.10}$$

and (3.2), we get

$$\begin{aligned}
&S(z_0,t_1z_1,\cdots,t_nz_n;\overline{u_o},\overline{u_1},\cdots,\overline{u_n})\\
=\;&\frac{1}{4\pi}\prod_{i=1}^{n}\frac{1}{V(\Omega_i)}\frac{1}{A^{1+\sum\limits_{i=1}^{n}s_id_i}}\sum_{\substack{\ell(\lambda^i)\le r_i\\1\le i\le n}}\varphi(|\lambda^1|,\cdots,|\lambda^n|)\prod_{i=1}^{n}(p_i)^{(i)}_{\lambda^i}t_i^{|\lambda^i|}K^{(i)}_{\lambda}(\frac{z_i}{A^{s_i}},\overline{u_i})\\
=\;&\frac{1}{4\pi}\prod_{i=1}^{n}\frac{1}{V(\Omega_i)}\frac{1}{A^{1+\sum\limits_{i=1}^{n}s_id_i}}\sum_{\substack{\ell(\lambda^i)\le r_i\\1\le i\le n}}\varphi(t_1\frac{d}{dt_1},\cdots,t_n\frac{d}{dt_n})\prod_{i=1}^{n}(p_i)^{(i)}_{\lambda^i}t_i^{|\lambda^i|}K^{(i)}_{\lambda}(\frac{z_i}{A^{s_i}},\overline{u_i})\\
=\;&\frac{1}{4\pi}\prod_{i=1}^{n}\frac{1}{V(\Omega_i)}\frac{1}{A^{1+\sum\limits_{i=1}^{n}s_id_i}}\varphi(t_1\frac{d}{dt_1},\cdots,t_n\frac{d}{dt_n})\prod_{i=1}^{n}\left\{\frac{1}{N_i(\frac{t_iz_i}{A^{s_i}},\overline{u_i})}\right\}^{p_i},
\end{aligned}$$

where $0\le t_i\le 1$, this proves Theorem 4.1.

To provide a concrete expression of (4.1), we need Lemma 4.2 below.

Lemma 4.2 If $n>1$, $s_i\in\mathbb{N}_+$ and $N_i(z_i,\overline{u_i})=1-z_iu_i^\dagger$ for $1\le i\le n-1$, φ same as (4.2), let $\partial_x=\frac{1}{\prod\limits_{i=1}^{n-1}s_id_i!}\frac{\partial^{d_1+\cdots+d_{n-1}}}{\partial x_1^{d_1}\cdots\partial x_n^{d_{n-1}}}$, then we have

$$\begin{aligned}
&\varphi(t_1\frac{d}{dt_1},\cdots,t_n\frac{d}{dt_n})\left\{\prod_{i=1}^{n}\frac{1}{N_i(t_iz_i,\overline{u_i})^{p_i}}\right\}\Bigg|_{t_1=\cdots=t_n=1}\\
=\;&\partial_x\sum_{j_1=0}^{s_1-1}\cdots\sum_{j_{n-1}=0}^{s_{n-1}-1}\frac{1}{\left(1-\sum\limits_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i}\right)^{s_nd_n+1}}\left\{\frac{1}{N_n\left(\frac{z_n}{\left(1-\sum\limits_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i}\right)^{sn}},\overline{u_n}\right)}\right\}^{p_n},
\end{aligned}$$

where $x_i=z_iu_i^\dagger$, $\omega_i=\exp\{\frac{2\pi\sqrt{-1}}{s_i}\}$ and symbols $u_i^\dagger$ denote the conjugation transposition of the row vectors u_i.

Proof For Cartan domains $\Omega_i(1\le i\le n-1)$, its ranks $r_i=1$, genuses $p_i=d_1+1$, and the reproducing kernels $K^{(i)}_k=\frac{(z_iu_i^\dagger)^k}{k!}$ of $\mathcal{P}^{(i)}_k$ with respect to (3.1), we have

$$\begin{aligned}
&\varphi(t_1\frac{d}{dt_1},\cdots,t_n\frac{d}{dt_n})\left\{\prod_{i=1}^{n}\frac{1}{N_i(t_iz_i,\overline{u_i})^{p_i}}\right\}\Bigg|_{t_1=\cdots=t_n=1}\\
=\;&\sum_{\ell(\lambda)\le r_n}\sum_{\substack{k_i=0\\1\le i\le n-1}}^{+\infty}\varphi(k_1,\cdots,k_{n-1},|\lambda|)\prod_{i=1}^{n-1}\frac{(k_i+1)_{d_i}(z_iu_i^\dagger)^{k_i}}{d_i!}(p_n)^{(n)}_\lambda K^{(n)}_\lambda(z_n,\overline{u_n})\\
=\;&\prod_{i=1}^{n-1}s_i\sum_{\ell(\lambda)\le r_n}\partial_x\sum_{\substack{k_i=0\\1\le i\le n-1}}^{+\infty}\varphi(k_1,\cdots,k_{n-1},|\lambda|)\prod_{i=1}^{n-1}x_i^{k_i+d_i}(p_n)^{(n)}_\lambda K^{(n)}_\lambda(z_n,\overline{u_n})\\
=\;&\prod_{i=1}^{n-1}s_i\partial_x\sum_{\ell(\lambda)\le r_n}\sum_{\substack{k_i=0\\1\le i\le n-1}}^{+\infty}\frac{\Gamma(\sum\limits_{i=1}^{n-1}s_ik_i+s_n(d_n+|\lambda|)+1)}{\prod\limits_{i=1}^{n-1}\Gamma(s_ik_i+1)\Gamma(s_n(d_n+|\lambda|)+1)}\prod_{i=1}^{n-1}x_i^{k_i}(p_n)^{(n)}_\lambda K^{(n)}_\lambda(z_n,\overline{u_n}),
\end{aligned}$$

where $x_i = z_i u_i^\dagger$.

Let

$$\omega_i := \exp\{\frac{2\pi\sqrt{-1}}{s_i}\}(1 \le i \le n-1),$$

using (3.2), we obtain

$$\begin{aligned}
&\varphi(t_1\frac{d}{dt_1},\cdots,t_n\frac{d}{dt_n})\{\prod_{i=1}^{n}\frac{1}{N_i(t_iz_i,\overline{u_i})^{p_i}}\}\Big|_{t_1=\cdots=t_n=1}\\
=\ &\partial_x\sum_{j_1=0}^{s_1-1}\cdots\sum_{j_{n-1}=0}^{s_{n-1}-1}\sum_{\ell(\lambda)\le r_n}\sum_{\substack{k_i=0\\1\le i\le n-1}}^{+\infty}\frac{\Gamma(\sum\limits_{i=1}^{n-1}k_i+s_n(d_n+|\lambda|)+1)}{\prod\limits_{i=1}^{n-1}\Gamma(k_i+1)\Gamma(s_n(d_n+|\lambda|)+1)}\\
&\times\prod_{i=1}^{n-1}(\omega_i^{j_i}x_i^{\frac{1}{s_i}})^{k_i}(p_n)_\lambda^{(n)}K_\lambda^{(n)}(z_n,\overline{u_n})\\
=\ &\partial_x\sum_{j_1=0}^{s_1-1}\cdots\sum_{j_{n-1}=0}^{s_{n-1}-1}\sum_{\ell(\lambda)\le r_n}\frac{1}{(1-\sum\limits_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i})^{s_n(d_n+|\lambda|)+1}}(p_n)_\lambda^{(n)}K_\lambda^{(n)}(z_n,\overline{u_n})\\
=\ &\partial_x\sum_{j_1=0}^{s_1-1}\cdots\sum_{j_{n-1}=0}^{s_{n-1}-1}\frac{1}{(1-\sum\limits_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i})^{s_nd_n+1}}\{\frac{1}{N_n(\frac{z_n}{(1-\sum_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i})^{sn}},\overline{u_n})}\}^{p_n}.
\end{aligned}$$

It completes the proof of Lemma 4.2.

By Lemma 4.2, we give the Szegö kernel of D_ψ in explicit form.

Corollary 4.3 For $s_i, n \in \mathbb{N}_+$, $n > 1$, and ranks of Cartan domains Ω_i equal to 1 $(1 \le i \le n-1)$, the Szegö kernel of D_ψ may be written as

$$\begin{aligned}
&S(z_0,z_1,\cdots,z_n;\overline{u_0},\overline{u_1},\cdots,\overline{u_n})\\
=\ &\frac{1}{4\pi}\prod_{i=1}^{n}\frac{1}{V(\Omega_i)}\frac{1}{A^{1+\sum_{i=1}^{n}s_id_i}}\prod_{i=1}^{n-1}\frac{1}{s_id_i!}\frac{\partial^{d_1+\cdots+d_{n-1}}}{\partial x_1^{d_1}\cdots\partial x_n^{d_{n-1}}}\\
&\sum_{j_1=0}^{s_1-1}\cdots\sum_{j_{n-1}=0}^{s_{n-1}-1}\frac{1}{(1-\sum\limits_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i})^{s_nd_n+1}}\{\frac{1}{N_n(\frac{z_n}{A^{sn}(1-\sum_{i=1}^{n-1}\omega_i^{j_i}x_i^{1/s_i})^{sn}},\overline{u_n})}\}^{p_n},
\end{aligned}$$

where $A = -\frac{\sqrt{-1}}{2}(z_0-\overline{u_0})$, $x_i = \frac{z_iu_i^\dagger}{A^{s_i}}$, $\omega_i = \exp\{\frac{2\pi\sqrt{-1}}{s_i}\}$ and symbols $u_i^\dagger$ denote the conjugation transposition of the row vectors u_i.

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一类无界域的Szegö核

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摘要: 本文研究了由任意不可约有界齐次圆域构造的一类无界域D_ψ的Szegö核. 利用Cartan域上一类积分的明显表达式, 获得了无界域D_ψ的Szegö核的明显公式.

关键词: Szegö 核; Cartan 域; 对称Jack 多项式; 广义Selberg 积分

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