ALGEBRAIC SOLUTIONS OF SYSTEMS OF COMPLEX DIFFERENTIAL EQUATIONS

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Abstract: This paper investigates the problem of existence of algebraic solutions of system of complex differential equations. Using maximum modulus principle and the Nevanlinna theory of the value distribution of meromorphic functions, a new result is obtained, and some existing results are improved and generalized. Examples show that our result is precise.

Keywords: differential equation; algebraic solution; maximum modulus principle

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1 Introduction and Main Results

In this paper, we assume that the reader is familiar with the standard notation and basic results of the Nevanlinna theory of meromorphic functions, for example, see [1, 2].

In 1988, Toda N considered the existence of algebroid solutions of algebraic differential equation of the form

\[(w')^n = \sum_{i=0}^{p} a_i w^i + \sum_{j=0}^{q} b_j w^j, \quad a_p b_q \neq 0.\]

He obtained the following result:

Theorem A (see [3]) Let \(w(z)\) be a nonconstant \(\nu\)-valued algebroid solution of the above differential equation and all \(\{a_i\}, \{b_j\}\) are polynomials. If \(p < n + q\), then \(w(z)\) is algebraic.

Since the 1990s, many authors, such as Tu Zhenhan, Xiao Xiuzhi, Song Shugang, Li Kamshun, Gao Lingyun, using the Nevanlinna theory of the value distribution of meromorphic functions, studied the problem of the existence and the growth of solutions of systems of complex differential equations and obtained many new and interesting results, for example, see [4–11].

The purpose of this paper is to study the system of complex algebraic differential equations on the base of Toda N’s paper with the aid of maximum modulus principle and Nevanlinna theory.
We will study the existence of algebraic solutions of system of complex differential equations of the following form

\[
\begin{align*}
(w_1')^{n_1} &= P_1(z, w_1, w_2) Q_1(z, w_1), \\
(w_2')^{n_2} &= P_2(z, w_1, w_2) Q_2(z, w_2),
\end{align*}
\tag{1.1}
\]

where

\[
P_1(z, w_1, w_2) = \sum_{i=0}^{p_{11}} \sum_{j=0}^{p_{12}} a_{ij} w_1^i w_2^j, \quad Q_1(z, w_1) = \sum_{i=0}^{q_1} b_i w_1^i,
\]

\[
P_2(z, w_1, w_2) = \sum_{i=0}^{p_{21}} \sum_{j=0}^{p_{22}} c_{ij} w_1^i w_2^j, \quad Q_2(z, w_2) = \sum_{i=0}^{q_2} d_i w_2^i,
\]

\(\{a_{ij}\}, \{b_i\}, \{c_{ij}\}, \{d_i\}\) are entire functions, \(n_1, n_2\) are positive integer numbers, \(p_{11}, p_{12}, q_1, p_{21}, p_{22}, q_2\) are non-negative integer numbers.

We will prove

**Theorem 1.1** Let \((w_1(z), w_2(z))\) be a nonconstant transcendental meromorphic solution of system (1.1) and all \(\{a_{ij}\}, \{b_i\}, \{c_{ij}\}, \{d_i\}\) are polynomials. If there exists a positive constant \(K\) such that \(\min\{n_1, n_1 + q_1 - p_{11}\} \cdot \min\{n_2, n_2 + q_2 - p_{22}\} > K p_{12} p_{21}\), and one of the following conditions is satisfied

(i) \(n_1 + q_1 > p_{11} + p_{12}, n_2 + q_2 > p_{21} + p_{22}\),

(ii) \(n_1 + q_1 > p_{11} + p_{21}, n_2 + q_2 > p_{22} + p_{12}\),

then \((w_1(z), w_2(z))\) is algebraic.

In this paper, we denote by \(E\) is a subset of \([0, \infty)\) for which \(m(E) < \infty\) and by \(K\) is a positive constant, where \(m(E)\) denotes the linear measure of \(E\). \(E\) or \(K\) does not always mean the same one when they appear in the following.

2 Some Lemmas

**Lemma 2.1** (see [2]) Let \(w(z)\) be a transcendental meromorphic function such that \(w(z), w'(z)\) has only finite number of poles. Then, for some constants \(C_i > 0, i = 1, 2\), it holds

\[M(r, w) \leq C_1 + C_2 r M(r, w'),\]

where \(M(r, w) = \max_{|z|=r} |w(z)|\).

**Lemma 2.2** (see [2]) Let \(w(z)\) be a transcendental meromorphic function such that \(w(z)\) has only finite number of poles. Then, for \(\alpha > 0\), it holds

\[M(r, w') \leq 2^\alpha [M(r, w)]^{\alpha+1}, (r \not\in E).\]

3 Proof of Theorem 1.1

Let \(S\) be the set of zeros of \(\{a_{ij}\}, \{b_i\}, \{c_{ij}\}, \{d_i\}\).
First, we will prove the poles of \( w_1 \) and \( w_2 \) are contained in \( S \).

By the conditions of Theorem 1.1, if \( z_0 \) is a pole of \( w_1 \) or \( w_2 \), \( z_0 \notin S \), then \( z_0 \) is also a pole of \( w_2 \) or \( w_1 \). Suppose that \( z_0 \) is a pole of \( w_1 \) of order \( \tau_1 \), a pole of \( w_2 \) of order \( \tau_2 \), but \( z_0 \notin S \). Then it follows from (1.1), we have

\[
\begin{align*}
&n_1(\tau_1 + 1) \leq p_{11}\tau_1 + p_{12}\tau_2 - q_1\tau_1, \\
&n_2(\tau_2 + 1) \leq p_{21}\tau_1 + p_{22}\tau_2 - q_2\tau_2,
\end{align*}
\]

that is

\[
\begin{align*}
&n_1(\tau_1 + 1) \leq (p_{11} - q_1)\tau_1 + p_{12}\tau_2, \\
&n_2(\tau_2 + 1) \leq p_{21}\tau_1 + (p_{22} - q_2)\tau_2.
\end{align*}
\]

Case (i): Noting that \( p_{11} + p_{12} < n_1 + q_1, \ p_{21} + p_{22} < n_2 + q_2 \), we get

\[
\begin{align*}
&n_1(\tau_1 + 1) < (n_1 - p_{12})\tau_1 + p_{12}\tau_2, \\
&n_2(\tau_2 + 1) < p_{21}\tau_1 + (n_2 - p_{21})\tau_2.
\end{align*}
\]

That is

\[
\begin{align*}
n_1 < p_{12}(\tau_2 - \tau_1), \ n_2 < p_{21}(\tau_1 - \tau_2).
\end{align*}
\]

This is a contradiction because \( p_{12} \geq 0, p_{21} \geq 0 \).

Case (ii): Noting that \( p_{11} + p_{21} < n_1 + q_1, \ p_{12} + p_{22} < n_2 + q_2 \), we obtain

\[
\begin{align*}
n_1(\tau_1 + 1) + n_2(\tau_2 + 1) \leq (p_{11} + p_{21} - q_1)\tau_1 + (p_{22} + p_{12} - q_2)\tau_2 < n_1\tau_1 + n_2\tau_2.
\end{align*}
\]

Then, we have \( n_1 + n_2 < 0 \). This is a contradiction.

Combining Case (i) and Case (ii), we obtain the poles of \( w_1 \) and \( w_2 \) are contained in \( S \), that is

\[
\begin{align*}
N(r, w_1) &\leq K_1[N(r, \frac{1}{a_{ij}}) + N(r, \frac{1}{b_i}) + N(r, \frac{1}{c_{ij}}) + N(r, \frac{1}{d_i})], \\
N(r, w_2) &\leq K_2[N(r, \frac{1}{a_{ij}}) + N(r, \frac{1}{b_i}) + N(r, \frac{1}{c_{ij}}) + N(r, \frac{1}{d_i})],
\end{align*}
\]

where \( K_1, K_2 \) are positive constants.

Next, we will estimate \( m(r, w_1), m(r, w_2) \).

We rewrite the system of complex differential equation (1.1) as follows:

\[
\begin{align*}
&b_{q_1}^n Q_1(w_1) w_1^n = P_1(z, w_1, w_2) Q_1(z, w_1)^{n_1-1}, \\
d_{q_2}^n Q_2(w_2) w_2^n = P_2(z, w_1, w_2) Q_2(z, w_2)^{n_2-1},
\end{align*}
\]

where \( Q_1(w_1) = \frac{Q_1(z, w_1)}{b_{q_1}}, \ Q_2(w_2) = \frac{Q_2(z, w_2)}{d_{q_2}} \).

Let

\[
\begin{align*}
U_1(z) &= \frac{w_1^{q_1+1}}{q_1 + 1} + \sum_{i=0}^{q_1-1} \frac{b_i w_i^{i+1}}{b_{q_1} i + 1}, \\
V_1(z) &= \sum_{i=0}^{q_1-1} \frac{b_i w_i^{i+1}}{b_{q_1} i + 1}, \quad V_1(z) = \sum_{i=0}^{q_1-1} \frac{d_i w_i^{i+1}}{d_{q_2} i + 1},
\end{align*}
\]

\[
\begin{align*}
U_2(z) &= \frac{w_2^{q_2+1}}{q_2 + 1} + \sum_{i=0}^{q_2-1} \frac{d_i w_i^{i+1}}{d_{q_2} i + 1}, \\
V_2(z) &= \sum_{i=0}^{q_1-1} \frac{d_i w_i^{i+1}}{d_{q_2} i + 1}.
\end{align*}
\]
Then system (3.3) becomes

\[
\begin{aligned}
\frac{b_{q_1}^{n_1}(U'_1(z) - V'_1(z))^{n_1}}{n_1} &= P_1(z, w_1, w_2)Q_1(z, w_1)^{n_1-1}, \\
\frac{d_{q_2}^{n_2}(U'_2(z) - V'_2(z))^{n_2}}{n_2} &= P_2(z, w_1, w_2)Q_2(z, w_2)^{n_2-1}.
\end{aligned}
\] (3.4)

We can easily prove the poles of \(U_i(z), U'_i(z)\) \((i = 1, 2)\) are contained in \(S\).

Because \(S\) is a finite set, it follows from Lemma 2.1 that

\[
M(r, U_1) \leq C_1 + C_2 r M(r, U'_1), r \notin E,
\]

\[
M(r, U_2) \leq C_3 + C_4 r M(r, U'_2), r \notin E.
\]

According to the definitions of \(U_i, i = 1, 2\), we obtain

\[
M(r, U_1) \geq \frac{M(r, w_1)^{q_1+1}}{q_1+1} - \frac{K_3 M(r, w_1)^{q_1} \sum_{i=0}^{q_1-1} M(r, b_i)}{r^d_1},
\] (3.5)

\[
M(r, U_2) \geq \frac{M(r, w_2)^{q_2+1}}{q_2+1} - \frac{K_4 M(r, w_2)^{q_2} \sum_{i=0}^{q_2-1} M(r, d_i)}{r^d_2},
\] (3.6)

where \(d_1, d_2\) are respectively the degrees of polynomials \(b_{q_1}, d_{q_2}\) and \(K_3, K_4\) are positive constants.

From Lemma 2.2, we obtain

\[
M(r, \left(\frac{b_i}{b_{q_1}}\right)') \leq K_5 \frac{r[M(r, b_i)]^2 + M(r, b_i)}{r^d_1+1}, r \notin E,
\]

\[
M(r, \left(\frac{d_i}{d_{q_2}}\right)') \leq K_6 \frac{r[M(r, d_i)]^2 + M(r, d_i)}{r^d_2+1}, r \notin E,
\]

where \(K_5, K_6\) are positive constants.

Therefore

\[
M(r, V_1) \leq \frac{K_7 M(r, w_1)^{q_1} \sum_{i=0}^{q_1-1} \{r[M(r, b_i)]^2 + M(r, b_i)\}}{r^d_1+1},
\] (3.7)

\[
M(r, V_2) \leq \frac{K_8 M(r, w_2)^{q_2} \sum_{i=0}^{q_2-1} \{r[M(r, d_i)]^2 + M(r, d_i)\}}{r^d_2+1},
\] (3.8)

where \(K_7, K_8\) are positive constants.
By (3.5) and (3.7), we have

\[
M(r, U'_1) - M(r, V_1) \geq \frac{M(r, U_1) - C_1}{C_2 r} - M(r, V_1)
\]

\[
\geq K_9 M(r, w_1)^{q_1+1} r^{-q_1+1} - K_{10} M(r, w_1)^{q_1} \left\{ r [M(r, b_1)]^2 + M(r, b_1) \right\},
\]

\[
M(r, U'_2) - M(r, V_2) \geq \frac{M(r, U_2) - C_3}{C_4 r} - M(r, V_2)
\]

\[
\geq K_{11} M(r, w_2)^{q_2+1} r^{-q_2+1} - K_{12} M(r, w_2)^{q_2} \left\{ r [M(r, d_1)]^2 + M(r, d_1) \right\},
\]

(3.9)

where \(K_9, K_{10}, K_{11}, K_{12}\) are positive constants.

Let \(z_r\) be a point such that \(M(r, U''(z)) = |U''(z_r)|, |z_r| = r, (r \not\in E)\). Then

\[
(M(r, U'_1) - M(r, V_1))^{n_1} \leq \left| U'_1(z_r) - V_1(z_r) \right|^{n_1} \leq M(r, P_1(z, w_1, w_2) Q_1(z, w_1)^{n_1-1}),
\]

(3.11)

\[
(M(r, U'_2) - M(r, V_2))^{n_2} \leq \left| U'_2(z_r) - V_2(z_r) \right|^{n_2} \leq M(r, P_2(z, w_1, w_2) Q_2(z, w_2)^{n_2-1}).
\]

(3.12)

Since

\[
M(r, P_1(z, w_1, w_2) Q_1(z, w_1)^{n_1-1}) \leq M(r, P_1(z, w_1, w_2) Q_1(z, w_1)^{n_1-1})
\]

\[
\leq K_{13} M(r, w_2)^{p_2} M(r, w_1)^{p_2+q_1(n_1-1)} \left\{ \sum_{i=0}^{p_2} \sum_{j=0}^{q_1} M(r, a_{ij}) \right\} \left\{ \sum_{i=0}^{q_1} M(r, b_{1i}) \right\}^{n_1-1} r^{-n_1 d_1},
\]

(3.13)

\[
K_{14} M(r, w_2)^{p_2} M(r, w_1)^{p_2+q_2(n_2-1)} \left\{ \sum_{i=0}^{p_2} \sum_{j=0}^{q_2} M(r, c_{ij}) \right\} \left\{ \sum_{i=0}^{q_2} M(r, d_{1i}) \right\}^{n_2-1} r^{-n_2 d_2},
\]

(3.14)

where \(K_{13}, K_{14}\) are positive constants.

Combining (3.9)–(3.14), we obtain

\[
\left\{ \begin{array}{l}
\frac{M(r, w_1)^{q_1+1}}{r} \leq \frac{K_{15} M(r, w_2)^{1/2} M(r, w_1)^{(p_2+q_1(n_1-1)/n_1)} \left\{ \sum_{i=0}^{p_2} \sum_{j=0}^{q_1} M(r, a_{ij}) \right\}^{1/n_1} \left\{ \sum_{i=0}^{q_1} M(r, b_{1i}) \right\}^{(n_1-1)/n_1}}{r^{q_1+1}}, r \not\in E, \\
\frac{M(r, w_2)^{q_2+1}}{r} \leq \frac{K_{16} M(r, w_1)^{q_1} \sum_{i=0}^{q_1} r [M(r, b_{1i})]^2 + M(r, b_{1i})}{r^{q_1+1}}, r \not\in E,
\end{array} \right.
\]

\[
+ \frac{K_{17} M(r, w_2)^{p_2} M(r, w_2)^{(p_2+q_2(n_2-1)/n_2)} \left\{ \sum_{i=0}^{p_2} \sum_{j=0}^{q_2} M(r, c_{ij}) \right\}^{1/n_2} \left\{ \sum_{i=0}^{q_2} M(r, d_{1i}) \right\}^{(n_2-1)/n_2}}{r^{q_2+1}}, r \not\in E,
\]

\[
+ \frac{K_{18} M(r, w_2)^{q_2} \sum_{i=0}^{q_2} r [M(r, d_{1i})]^2 + M(r, d_{1i})}{r^{q_2+1}}, r \not\in E,
\]
where $K_{15}, K_{16}, K_{17}, K_{18}$ are positive constants.

Further, we get

$$
\begin{align*}
M(r, w_1) & \leq \frac{K_{19}[M(r, w_2)^{p_{12}/n_1} \left\{ \sum_{i=0}^{p_{11}} \sum_{j=0}^{p_{12}} M(r, a_{ij}) \right\}^{1/n_1} \{ \sum_{i=0}^{q_1} M(r, b_i) \}]^{(n_1-1)/n_1}}{r^{n_1}} + \frac{\sum_{i=0}^{q_1-1} \{ r[M(r, b_i)^2 + M(r, b_i)] \}}{r^{n_1}}, \\
M(r, w_2) & \leq \frac{K_{20}[M(r, w_1)^{p_{21}/n_2} \left\{ \sum_{i=0}^{p_{21}} \sum_{j=0}^{p_{22}} M(r, c_{ij}) \right\}^{1/n_2} \{ \sum_{i=0}^{q_2} M(r, d_i) \}]^{(n_2-1)/n_2}}{r^{n_2}} + \frac{\sum_{i=0}^{q_2-1} \{ r[M(r, d_i)^2 + M(r, d_i)] \}}{r^{n_2}},
\end{align*}
$$

where $K_{19}, K_{20}$ are positive constants.

By calculating $\log^+$ of the both sides of the above inequalities, we have

$$
\begin{align*}
\min\{n_1, n_1 + q_1 - p_{11}\} \log^+ M(r, w_1) & \leq K_{19}p_{12} \log^+ M(r, w_2) + S_1(r), \\
\min\{n_2, n_2 + q_2 - p_{22}\} \log^+ M(r, w_2) & \leq K_{20}p_{21} \log^+ M(r, w_1) + S_2(r),
\end{align*}
$$

where

$$
\begin{align*}
S_1(r) & = K_{19} \sum_{i=0}^{p_{11}} \sum_{j=0}^{p_{12}} \log^+ M(r, a_{ij}) + \sum_{i=0}^{q_1} \log^+ M(r, b_i) + O(\log r), \\
S_2(r) & = K_{20} \sum_{i=0}^{p_{21}} \sum_{j=0}^{p_{22}} \log^+ M(r, c_{ij}) + \sum_{i=0}^{q_2} \log^+ M(r, d_i) + O(\log r).
\end{align*}
$$

Further, we get

$$(\min\{n_1, n_1 + q_1 - p_{11}\} \min\{n_2, n_2 + q_2 - p_{22}\} - K_{19}K_{20}p_{12}p_{21}) \log^+ M(r, w_i) \leq S(r), r \notin E,$$

where $i = 1, 2$,

$$
S(r) = K_{21} \sum_{i=0}^{p_{11}} \sum_{j=0}^{p_{12}} \log^+ M(r, a_{ij}) + \sum_{i=0}^{q_1} \log^+ M(r, b_i) + \sum_{i=0}^{p_{21}} \sum_{j=0}^{p_{22}} \log^+ M(r, c_{ij}) + \sum_{i=0}^{q_2} \log^+ M(r, d_i) + O(\log r),
$$

where $K_{21}$ is a positive constant.

Since there exists a positive constant $K$ such that $\min\{n_1, n_1 + q_1 - p_{11}\} \min\{n_2, n_2 + q_2 - p_{22}\} > Kp_{12}p_{21}$, then we obtain

$$
m(r, w_1) \leq S(r),
$$

(3.15)

$$
m(r, w_2) \leq S(r).
$$

(3.16)

Combining the inequalities (3.1), (3.2), (3.15) and (3.16), we have

$$
T(r, w_i) = O(\log r) \quad (r \notin E), \quad i = 1, 2,
$$
which shows that \((w_1(z), w_2(z))\) is an algebraic solution of (1.1).

This completes the proof of Theorem 1.1.

4 Some Examples

Example 4.1 and example 4.2 show that the conditions in Theorem 1.1 are sharp. Example 4.3 and example 4.4 show that Theorem 1.1 holds.

**Example 4.1** \((w_1(z), w_2(z)) = (e^{3z}, 2e^z)\) is a nonconstant transcendental meromorphic solution of the following system of differential equations

\[
\begin{align*}
(w_1')^3 &= \frac{8w_1^3 + w_1w_2^2 + 2w_1^3w_2^2 - 4w_1^2}{w_1 + 1}, \\
(w_2')^3 &= \frac{4w_1^3 - 3w_1w_2 - 4w_1^3w_2 - w_1^3w_2^2 - \frac{1}{4}w_1^3 + 5w_1^2w_2^2}{w_2^3 - 1}.
\end{align*}
\]

It is easy to know that

\[
n_1 = 3, q_1 = 1, p_{11} = 3, p_{12} = 2; n_2 = 3, q_2 = 2, p_{21} = 2, p_{22} = 3.
\]

Thus

\[
n_1 + q_1 = 4 < 5 = p_{11} + p_{12}, n_2 + q_2 = 5 = p_{21} + p_{22},
\]

or

\[
n_1 + q_1 = 4 < 5 = p_{11} + p_{21}, n_2 + q_2 = 5 = p_{12} + p_{22}.
\]

**Example 4.2** \((w_1(z), w_2(z)) = (\sin z, \cos^2 z)\) is a nonconstant transcendental meromorphic solution of the following system of differential equations

\[
\begin{align*}
(w_1')^2 &= \frac{(w_2^2)}{1 - w_1^2}, \\
(w_2')^2 &= \frac{5w_1w_2 + 4w_1^3w_2^2 - w_2 + w_2^2}{w_2^2 + 1}.
\end{align*}
\]

In this case,

\[
n_1 = 2, q_1 = 2, p_{11} = 0, p_{12} = 2; n_2 = 2, q_2 = 1, p_{21} = 2, p_{22} = 2.
\]

Thus

\[
n_1 + q_1 = 4 > 2 = p_{11} + p_{12}, n_2 + q_2 = 3 < 4 = p_{21} + p_{22},
\]

or

\[
n_1 + q_1 = 4 > 2 = p_{11} + p_{21}, n_2 + q_2 = 3 < 4 = p_{12} + p_{22}.
\]

**Example 4.3** \((w_1(z), w_2(z)) = (z^2, 2z)\) is a nonconstant algebraic solution of the following system of differential equations

\[
\begin{align*}
(w_1')^2 &= \frac{6zw_1 + 2zw_2 + 2z^2w_1^2 - 3w_1w_2 - 3zw_1^2w_2}{1 - w_1^2}, \\
(w_2')^3 &= \frac{z^2 - 2z + 8 + 3w_1^2w_2^2 + (9z - 7)w_2 - 3z^3w_1w_2 + 7zw_1w_2 + 13w_1 - 5z^2w_1w_2^2}{w_2^3 - w_2 + 1}.
\end{align*}
\]
Clearly, we get
\[ n_1 = 2, q_1 = 2, p_{11} = 2, p_{12} = 1; \quad n_2 = 3, q_2 = 2, p_{21} = 2, p_{22} = 2. \]

In this case, Case (i) holds, that is
\[ n_1 + q_1 = 4 > 3 = p_{11} + p_{12}, n_2 + q_2 = 5 > 4 = p_{21} + p_{22}, \]
but Case (ii) does not hold, that is
\[ n_1 + q_1 = 4 = p_{11} + p_{21}, n_2 + q_2 = 5 > 3 = p_{12} + p_{22}. \]

There exists a positive constant \( K = 2 \) such that
\[ \min\{n_1, n_1 + q_1 - p_{11}\} \min\{n_2, n_2 + q_2 - p_{22}\} = 2 \times 3 = 6 > 4 = Kp_{12}p_{21}. \]

**Example 4.4** \((w_1(z), w_2(z)) = \left(\frac{1}{z}, 3z\right)\) is a nonconstant algebraic solution of the following system of differential equations
\[
\begin{align*}
(w'_1)^3 &= \frac{2w_2^3 - 11z^2w_1^2 + zw_1^3 w_2 - z^2w_1^2 w_2 - 3zw_1^2 w_2^2 - 2zw_1^3 w_2 - 2zw_1}{(4z^2 + 1)w_1^2 w_2 - 5z^2w_1^2 w_2 - w_1 w_2 - 2w_2 + (5z^2 + 7z - 3)w_1 - 2z + 1}, \\
(w'_2)^2 &= \frac{1}{9}w_2^3 - z^2 w_2^2 + 2z^2 w_2 + z
\end{align*}
\]

Easily, we obtain
\[ n_1 = 3, q_1 = 2, p_{11} = 2, p_{12} = 3; n_2 = 2, q_2 = 3, p_{21} = 2, p_{22} = 1. \]

In this case, Case (i) does not hold, that is
\[ n_1 + q_1 = 5 > 4 = p_{11} + p_{21}, n_2 + q_2 = 5 > 3 = p_{12} + p_{22}. \]
but Case (ii) holds, that is
\[ n_1 + q_1 = 5 > 4 = p_{11} + p_{21}, n_2 + q_2 = 5 > 4 = p_{12} + p_{22}. \]

There exists a positive constant \( K = 1 \) such that
\[ \min\{n_1, n_1 + q_1 - p_{11}\} \min\{n_2, n_2 + q_2 - p_{22}\} = 3 \times 4 = 12 > 6 = Kp_{12}p_{21}. \]
References