

非线性互补约束规划问题的一个新的QP-free算法^{*†}

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摘要: 本文针对非线性规划一些经典的算法一般不能直接应用到均衡问题上的缺点, 利用一个互补函数以及光滑近似法的思想, 把非线性互补约束均衡问题转化为一光滑非线性规划问题, 该光滑非线性规划问题通过一个新的QP-free算法求解. 特别地, 在不要求Hessian 阵估计正定的假设条件, 算法仍具有全局收敛性. 且在一些适当的假设条件下, 得到了超线性收敛速度. 数值实验结果表明本文提出的算法是可行的.

关键词: 均衡问题, 非线性互补, QP-free算法, 全局收敛性, 超线性收敛性

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§1. 引言

本文考虑如下非线性互补约束均衡问题(MPEC):

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s.t.} \quad & g_j(x, y) \geq 0, \quad w_j = F_j(x, y), \quad 0 \leq w \perp y \geq 0, \end{aligned} \quad (1.1)$$

其中 $f: R^{n+m_2} \rightarrow R$, $g_j: R^{n+m_2} \rightarrow R$, $j = 1, \dots, m_1$, $F_j: R^{n+m_2} \rightarrow R$, $j = 1, \dots, m_2$, 二阶连续可微, $w \perp y$ 表示向量 w 与 y 垂直. 该问题在经济模型, 工程设计等领域有着广泛的应用, 近年来关于该类问题的研究及应用见文献[1]-[3].

显然, 若将互补约束条件 $w \perp y$ 视为 $w^T y = 0$, 则均衡问题(MPEC)(1.1)等价于一个光滑非线性规划问题. 然而, 文献[4]指出: 即使没有约束 $g(x, y) \geq 0$, 且函数 $F(x, y)$ 具有很好的性质, 对于该光滑非线性规划问题, 较弱的MFCQ条件在任意可行点处都不满足, 故非线性规划一些经典的算法一般不能直接应用到均衡问题上来. 关于均衡问题的研究, 文献[5]-[7]利用带扰动参数的互补函数, 通过求解一系列光滑扰动问题来成功逼近原均衡问题的解. 在渐近弱退化条件下, 这类扰动问题所产生序列的聚点称为原问题的稳定点.

本文通过构造新的互补函数, 利用逐次逼近思想, 为问题(1.1)提出新的光滑化技巧.

易知

$$y \geq 0, w \geq 0, y^T w = 0 \Leftrightarrow \min\{y_j, w_j\} = 0, j = 1, \dots, m_2. \quad (1.2)$$

如文献[2], 定义函数 $\phi(y_j, w_j, u) = -u \ln(e^{-\frac{y_j}{u}} + e^{-\frac{w_j}{u}})$, $j = 1, \dots, m_2$, 有

$$\lim_{u \rightarrow 0} \phi(y_j, w_j, u) = \lim_{u \rightarrow 0} [-u \ln(e^{-\frac{y_j}{u}} + e^{-\frac{w_j}{u}})] = \min\{y_j, w_j\} = 0, j = 1, \dots, m_2. \quad (1.3)$$

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由(1.3), 很自然定义

$$\lim_{u \rightarrow 0} \phi(y_j, w_j, u) = \phi(y_j, w_j, 0), j = 1, \dots, m_2. \quad (1.4)$$

由 ϕ 的定义, 有

$$\begin{aligned} \nabla \phi(y, w, u) &= \begin{pmatrix} \mathbf{0} \\ \nabla_y \phi(y, w, u) \\ \nabla_w \phi(y, w, u) \end{pmatrix}_{(n+2m_2) \times 1} \\ &= \left(\mathbf{0}, \frac{\partial \phi(y_1, w_1, u)}{\partial y_1}, \dots, \frac{\partial \phi(y_{m_2}, w_{m_2}, u)}{\partial y_{m_2}}, \frac{\partial \phi(y_1, w_1, u)}{\partial w_1}, \dots, \frac{\partial \phi(y_{m_2}, w_{m_2}, u)}{\partial w_{m_2}} \right)^T \end{aligned} \quad (1.5)$$

借助于互补函数 ϕ , 下面我们提出如下非线性规划(NLP_u):

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s.t.} \quad & g_j(x, y) \geq 0, \quad j = 1, \dots, m_1, \\ & c_j(x, y, w) = 0, \quad j = 1, \dots, m_2, \\ & \phi(y_j, w_j, u) = 0, \quad j = 1, \dots, m_2, \end{aligned} \quad (1.6)$$

其中 $c_j(x, y, w) = w_j - F_j(x, y)$, $j = 1, \dots, m_2$. 当 $u \rightarrow 0$, 由(1.2)及(1.3), 易知问题(NLP_u)(1.6)是问题(MPEC)(1.1)的光滑近似.

本文通过一个光滑函数 $\phi(y_j, w_j, u) = -u \ln(e^{-\frac{y_j}{u}} + e^{-\frac{w_j}{u}}) \in R$, 把问题(MPEC)(1.1)转化为一带参数的一般优化问题, 基于文献[8]的思想, 提出了一个新的光滑QP-free算法. 该算法具有如下优点:

- (i) 罚参数更新比文献[10]简单;
- (ii) 在不要求Hessian阵估计正定的假设条件下, 算法仍具有超线性收敛性.

§2. 算法的描述

令 $z = (x, y, w)$, $p = (x, y)$, $q = (y, w)$, $L_1 = \{1, \dots, m_1\}$, $L_2 = \{1, \dots, m_2\}$, 记 X 为问题(1.6)的可行集, 即 $X := \{z : g_j(p) \geq 0, j \in L_1, c_j(z) = 0, j \in L_2, \phi(q_j, u) = 0, j \in L_2\}$.

类似文献[8], 将问题(NLP_u)(1.6)转化为序列不等式约束优化问题(NLP_ρ):

$$\begin{aligned} \min \quad & f_\rho(x, y, w) \\ \text{s.t.} \quad & g_j(x, y) \geq 0, \quad j \in L_1, \\ & c_j(x, y, w) \geq 0, \quad j \in L_2, \\ & \phi(y_j, w_j, u) \geq 0, \quad j \in L_2, \end{aligned} \quad (2.1)$$

其中 $\rho = (\rho_1, \rho_2)$, $f_\rho(x, y, w) = f(x, y) + \rho_1 \sum_{j \in L_2} c_j(x, y, w) + \rho_2 \sum_{j \in L_2} \phi(y_j, w_j, u)$, $c_j(x, y, w) = w_j - F_j(x, y)$, $j \in L_2$.

研究问题(NLP_ρ)(2.1)易知, ρ 惩罚满足 $c_j(x, y, w) > 0$, $j \in L_2$, $\phi(y_j, w_j, u) > 0$, $j \in L_2$ 的迭代, 而问题(NLP_ρ)(2.1)的可行性能保证 $c_j(x, y, w) \geq 0$, $j \in L_2$, $\phi(y_j, w_j, u) \geq 0$, $j \in L_2$. 故 ρ 充分大, 能保证问题(NLP_u)(1.6)的可行性. 事实上, 当采用精确罚函数时, 无需 ρ 趋向无穷, 就能收敛到问题(NLP_u)(1.6)的一解.

记 $\tilde{X} := \{z : g_j(p) \geq 0, j \in L_1, c_j(z) \geq 0, j \in L_2, \phi(q_j, u) \geq 0, j \in L_2\}$, 同时, 记 $I^1(p)$, $I^2(z)$, $I^3(q, u)$ 分别对应于 g , c , ϕ 的积极集, 即

$$I^1(p) = \{j \in L_1 : g_j(p) = 0\}; I^2(z) = \{j \in L_2 : c_j(z) = 0\}; I^3(q, u) = \{j \in L_2 : \phi(q_j, u) = 0\}.$$

为简化叙述, 记

$$\nabla g_j(p) = \left(\frac{\partial g_j(p)}{\partial y_1}, \dots, \frac{\partial g_j(p)}{\partial y_{m_2}}, \frac{\partial g_j(p)}{\partial w_1}, \dots, \frac{\partial g_j(p)}{\partial w_{m_2}}, \mathbf{0} \right)_{(n+2m_2) \times 1}^T, \quad j \in L_1,$$

$$\nabla c_j(z) = \left(\frac{\partial c_j(z)}{\partial x_1}, \dots, \frac{\partial c_j(z)}{\partial x_n}, \frac{\partial c_j(z)}{\partial y_1}, \dots, \frac{\partial c_j(z)}{\partial y_{m_2}}, \frac{\partial c_j(z)}{\partial w_1}, \dots, \frac{\partial c_j(z)}{\partial w_{m_2}} \right)_{(n+2m_2) \times 1}^T, \quad j \in L_2,$$

$$\nabla \phi(q_j, u) = \left(\mathbf{0}, \frac{\partial \phi(q_j, u)}{\partial y_1}, \dots, \frac{\partial \phi(q_j, u)}{\partial y_{m_2}}, \frac{\partial \phi(q_j, u)}{\partial w_1}, \dots, \frac{\partial \phi(q_j, u)}{\partial w_{m_2}} \right)_{(n+2m_2) \times 1}^T, \quad j \in L_2.$$

首先, 给出如下三个基本假设:

H 2.1 问题(1.1)的可行域非空.

H 2.2 函数 $f(p)$, $g_j(p)$, $j \in L_1$, $c_j(z)$, $j \in L_2$, $\phi(q_j, u)$, $j \in L_2$ 连续可微.

H 2.3 对任意的 $z \in X$, (i) 向量 $\{\nabla g_j(p) \mid j \in I^1(p)\} \cup \{\nabla c_j(z) \mid j \in I^2(z)\} \cup \{\nabla \phi(q_j, u) \mid j \in I^3(q, u)\}$ 线性无关; (ii) 若 $z \notin X$, 则不存在常数 $b_1 > 0$, $b_2 > 0$, 纯量 $\beta^{1,j} \geq 0$, $j \in I^1(p)$, $\beta^{2,j} \geq 0$, $j \in I^2(z)$, $\beta^{3,j} \geq 0$, $j \in I^3(q, u)$ 使得

$$\sum_{j \in L_2} b_1 \nabla c_j(z) + \sum_{j \in L_2} b_2 \nabla \phi(q_j, u) = \sum_{j \in I^1(p)} \beta^{1,j} \nabla g_j(p) + \sum_{j \in I^2(z)} \beta^{2,j} \nabla c_j(z) + \sum_{j \in I^3(q, u)} \beta^{3,j} \nabla \phi(q_j, u). \quad (2.2)$$

假设条件H 2.3 (ii) 用于证明(2.14)方程中系数矩阵非奇异.

在算法前, 先研究问题 (NLP_u) (1.6) 与问题 (NLP_ρ) (2.1) 的关系.

z 称为问题 (NLP_u) (1.6)的一K-T 点, 若存在乘子 $\lambda^1 \in R^{m_1}$, $\lambda^2, \lambda^3 \in R^{m_2}$ 满足

$$h(p) - A(p)^T \lambda^1 - B(z)^T \lambda^2 - T(q, u)^T \lambda^3 = \mathbf{0}, \quad (2.3)$$

$$g(p) \geq 0, \quad c(z) = 0, \quad \phi(q, u) = 0, \quad (2.4)$$

$$\lambda^{1,j} g_j(p) = 0, \quad j \in L_1, \quad (2.5)$$

$$\lambda^1 \geq 0, \quad (2.6)$$

其中

$$h(p) = \left(\nabla_x f(x, y), \nabla_y f(x, y), \mathbf{0} \right)_{(n+2m_2) \times 1}^T,$$

$$A(p)^T = \begin{pmatrix} \frac{\partial g_1(p)}{\partial x_1} & \dots & \frac{\partial g_{m_1}(p)}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial g_1(p)}{\partial x_n} & & \frac{\partial g_{m_1}(p)}{\partial x_n} \\ \frac{\partial g_1(p)}{\partial y_1} & \dots & \frac{\partial g_{m_1}(p)}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial g_1(p)}{\partial y_{m_2}} & \dots & \frac{\partial g_{m_1}(p)}{\partial y_{m_2}} \\ \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \quad B(z)^T = \begin{pmatrix} \frac{\partial c_1(z)}{\partial x_1} & \dots & \frac{\partial c_{m_2}(z)}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial c_1(z)}{\partial x_n} & & \frac{\partial c_{m_2}(z)}{\partial x_n} \\ \frac{\partial c_1(z)}{\partial y_1} & \dots & \frac{\partial c_{m_2}(z)}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial c_1(z)}{\partial y_{m_2}} & \dots & \frac{\partial c_{m_2}(z)}{\partial y_{m_2}} \\ \frac{\partial c_1(z)}{\partial w_1} & \dots & \frac{\partial c_{m_2}(z)}{\partial w_1} \\ \vdots & & \vdots \\ \frac{\partial c_1(z)}{\partial w_{m_2}} & \dots & \frac{\partial c_{m_2}(z)}{\partial w_{m_2}} \end{pmatrix},$$

$$T(q, u)^T = \left(\mathbf{0}, \text{diag} \left(\frac{\partial \phi(q_j, u)}{\partial y_j}, j \in L_2 \right), \text{diag} \left(\frac{\partial \phi(q_j, u)}{\partial w_j}, j \in L_2 \right) \right)_{(n+2m_2) \times m_2}^T.$$

若存在乘子 $\lambda^1 \in R^{m_1}$, $\lambda^2, \lambda^3 \in R^{m_2}$ 使得(2.3)-(2.5) 成立, 则 z 称为问题 (NLP_u) (1.6)的稳定点.

下面, 对给定的 $\rho = (\rho_1, \rho_1)$, $z \in \tilde{X}$ 称为问题 $(NLP_\rho)(2.1)$ 的一K-T点, 若存在乘子 $\lambda^1 \in R^{m_1}$, $\lambda^2, \lambda^3 \in R^{m_2}$, 使得

$$h(p) + B(z)^T(\rho_1 e) + T(q, u)^T(\rho_2 e) - A(p)^T \lambda^1 - B(z)^T \lambda^2 - T(q, u)^T \lambda^3 = \mathbf{0}, \quad (2.7)$$

$$g(p) \geq 0, \quad c(z) \geq 0, \quad \phi(q, u) \geq 0, \quad (2.8)$$

$$\lambda^{1,j} g_j(p) = 0, j \in L_1, \quad \lambda^{2,j} c_j(z) = 0, j \in L_2, \quad \lambda^{3,j} \phi(q_j, u) = 0, j \in L_2, \quad (2.9)$$

$$\lambda^1 \geq 0, \quad \lambda^2 \geq 0, \quad \lambda^3 \geq 0, \quad (2.10)$$

其中 $e = (1, \dots, 1)^T \in R^{m_2}$. 若存在乘子 $\lambda^1 \in R^{m_1}$, $\lambda^2, \lambda^3 \in R^{m_2}$ 使得(2.7)-(2.9)成立, 则 z 称为问题 $(NLP_\rho)(2.1)$ 的一稳定点.

记 $\mu = (\mu^1, \mu^2, \mu^3)$, 则 $(NLP_\rho)(2.1)$ 的对数障碍函数如下:

$$\begin{aligned} \beta(z, u, \rho, \mu) = & f(p) + \rho_1 \sum_{j \in L_2} c_j(z) + \rho_2 \sum_{j \in L_2} \phi(q_j, u) \\ & - \sum_{j \in L_1} \mu^{1,j} \ln(g_j(p)) - \sum_{j \in L_2} \mu^{2,j} \ln(c_j(z)) - \sum_{j \in L_2} \mu^{3,j} \ln(\phi(q_j, u)), \end{aligned}$$

其中 $\mu^1 > 0 \in R^{m_1}$, $\mu^2, \mu^3 > 0 \in R^{m_2}$, 其梯度为:

$$\begin{aligned} \nabla_z \beta(z, u, \rho, \mu) = & h(p) + B(z)^T(\rho_1 e) + T(q, u)^T(\rho_2 e) \\ & - A(p)^T G(p)^{-1} \mu^1 - B(z)^T C(z)^{-1} \mu^2 - T(q, u)^T \Phi(q, u)^{-1} \mu^3, \end{aligned} \quad (2.11)$$

其中 $G(p) = \text{diag}(g_j(p), j \in L_1)$, $C(z) = \text{diag}(c_j(z), j \in L_2)$, $\Phi(q, u) = \text{diag}(\phi(q_j, u), j \in L_2)$.

问题 $(NLP_\rho)(2.1)$ 能通过 $\min \beta(z, u, \rho, \mu)$, $\mu \rightarrow 0$ 求解. 由(2.11), 令 $\lambda = (\lambda^1, \lambda^2, \lambda^3)$, 定义 $\lambda^1 = G(p)^{-1} \mu^1$, $\lambda^2 = C(z)^{-1} \mu^2$, $\lambda^3 = \Phi(q, u)^{-1} \mu^3$, 能视为与 $(NLP_\rho)(2.1)$ 的解有关的K-T乘子, 等式(2.11)的右边为 $(NLP_\rho)(2.1)$ 拉格朗日函数的梯度在 $(z, \lambda^1, \lambda^2, \lambda^3)$ 处的值.

由对偶内点法的思想, 求解下列关于 (z, λ) 的非线性系统, $\lambda = (\lambda^1, \lambda^2, \lambda^3)$:

$$h(p) - A(p)^T \lambda^1 - B(z)^T(\lambda^2 - \rho_1 e) - T(q, u)^T(\lambda^3 - \rho_2 e) = \mathbf{0}, \quad (2.12)$$

$$\mu^1 - G(p) \lambda^1 = 0, \quad \mu^2 - C(z) \lambda^2 = 0, \quad \mu^3 - \Phi(q, u) \lambda^3 = 0, \quad (2.13)$$

该非线性系统由拟牛顿迭代求解, $\mu = (\mu^1, \mu^2, \mu^3)$:

$$\begin{aligned} \mathbf{N}(z, \lambda, \mathbf{H}, \mathbf{u}, \rho, \mu): & \begin{pmatrix} -H & A(p)^T & B(z)^T & T(q, u)^T \\ Y^1 A(p) & G(p) & \mathbf{0} & \mathbf{0} \\ Y^2 B(z) & \mathbf{0} & C(z) & \mathbf{0} \\ Y^3 T(q, u) & \mathbf{0} & \mathbf{0} & \Phi(q, u) \end{pmatrix} \begin{pmatrix} \Delta z \\ \Delta \lambda^1 \\ \Delta \lambda^2 \\ \Delta \lambda^3 \end{pmatrix} \\ & = \begin{pmatrix} h(p) - A(p)^T \lambda^1 - B(z)^T(\lambda^2 - \rho_1 e) - T(q, u)^T(\lambda^3 - \rho_2 e) \\ \mu^1 - G(p) \lambda^1 \\ \mu^2 - C(z) \lambda^2 \\ \mu^3 - \Phi(q, u) \lambda^3 \end{pmatrix}, \end{aligned} \quad (2.14)$$

其中 $Y^1 = \text{diag}(\lambda^{1,j}, j \in L_1)$, $Y^2 = \text{diag}(\lambda^{2,j}, j \in L_2)$, $Y^3 = \text{diag}(\lambda^{3,j}, j \in L_2)$, H 为 $(NLP_\rho)(2.1)$ 拉格朗日函数Hessian阵的近似.

算法基于文献[9]的内点法思想求解问题 $(NLP_\rho)(2.1)$, 对固定的 $\rho > 0$. 关键技术是如何调整 ρ , 使得迭代渐近满足 $c(z) = 0$, $\phi(q, u) = 0$, 算法用了一个比文献[10]更为简单的更新准则. 当等式约束违反时, $\rho > 0$ 应增加, 但当 $\rho > 0$ 不受控制地增加时, 可能达不到问题 $(NLP_\rho)(2.1)$ 的K-T点. 为防止这种现象, 要求下列三个条件成立(r_1, r_2, r_3 为正数):

- (i) $\|\Delta z_0^k\| \leq r_1$, 表明对问题 $(NLP_{\rho_k})(2.1)$ 稳定点的逼近;
- (ii) $\lambda_k^2 + \Delta\lambda_{0,k}^2 \not\geq r_2 e$, $\lambda_k^3 + \Delta\lambda_{0,k}^3 \not\geq r_2 e$, 即在极限点处, 并不是所有的 $c_j(z)$, $\phi(q_j, u)$ 都成为积极约束;
- (iii) $\lambda_k^1 + \Delta\lambda_{0,k}^1 \geq r_3 e$, $\lambda_k^2 + \Delta\lambda_{0,k}^2 \geq r_3 e$, $\lambda_k^3 + \Delta\lambda_{0,k}^3 \geq r_3 e$, 即当 ρ_k 增加得非常快时, λ_k^1 , λ_k^2 或 λ_k^3 没有分量趋向于 $-\infty$.

算法2.1

步骤0 初始化:

给定 $z^0 \in \tilde{X}_0$, $u_0 > 0$, $\rho_0 > 0$, $\lambda_0^{1,j} \in (0, t_{\max}]$, $j \in L_1$, $\lambda_0^{2,j}, \lambda_0^{3,j} \in (0, t_{\max}]$, $j \in L_2$, $H_0 \in R^{(n+2m_2) \times (n+2m_2)}$ 为对称正定阵. 选取参数 $\xi \in (0, \frac{1}{2})$, $\eta_1 \in (0, 1]$, $\eta_2 \in (0, 1]$, $\eta_3 \in (0, 1]$, $r_1 > 0$, $r_2 > 0$, $r_3 > 0$, $\nu > 2$, $\theta \in (0, 1)$, $\delta > 1$, $\varrho \in (0, 1)$, $\tau \in (2, 3)$, $\kappa \in (0, 1)$, $l \in (0, 1)$, $t_{\max} > 0$. 令 $k = 0$.

步骤1 通过求解 $N(z^k, \lambda_k, H_k, u_k, \rho_k, \theta)$ 计算 $(\Delta z_0^k, \Delta\lambda_{0,k}^1, \Delta\lambda_{0,k}^2, \Delta\lambda_{0,k}^3)$.

若 $\Delta z_0^k = 0$, $u_k < \varepsilon$, 算法停止! 否则令 $u_k = \frac{1}{2}u_k$, 转步骤2.

步骤2 考察上面提到的三个条件:

若三个条件同时成立, 则令 $\rho_{k+1} = \delta\rho_k$, $z^{k+1} = z^k$, $\lambda_{k+1}^1 = \lambda_k^1$, $\lambda_{k+1}^2 = \lambda_k^2$, $\lambda_{k+1}^3 = \lambda_k^3$, $H_{k+1} = H_k$, 置 $k = k + 1$, 转步骤1. 否则, 转步骤3.

步骤3 通过求解 $N(z^k, \lambda_k, H_k, u_k, \rho_k, \mu_k)$ 计算 $(\Delta z_1^k, \Delta\lambda_{1,k}^1, \Delta\lambda_{1,k}^2, \Delta\lambda_{1,k}^3)$, 其中 $\mu_k = (\mu_k^1, \mu_k^2, \mu_k^3) = (\|\Delta z_0^k\|^\nu \lambda_k^1, \|\Delta z_0^k\|^\nu \lambda_k^2, \|\Delta z_0^k\|^\nu \lambda_k^3)$.

步骤4 令

$$\begin{aligned} \Delta z^k &= (1 - \varphi_k)\Delta z_0^k + \varphi_k\Delta z_1^k, \Delta\lambda_k^1 = (1 - \varphi_k)\Delta\lambda_{0,k}^1 + \varphi_k\Delta\lambda_{1,k}^1, \\ \Delta\lambda_k^2 &= (1 - \varphi_k)\Delta\lambda_{0,k}^2 + \varphi_k\Delta\lambda_{1,k}^2, \Delta\lambda_k^3 = (1 - \varphi_k)\Delta\lambda_{0,k}^3 + \varphi_k\Delta\lambda_{1,k}^3, \end{aligned} \quad (2.15)$$

$$\varphi^k = \begin{cases} 1, & \text{若 } \nabla f_{\rho_k}(z^k)^T \Delta z_1^k \leq \theta \nabla f_{\rho_k}(z^k)^T \Delta z_0^k, \\ (1 - \theta) \frac{\nabla f_{\rho_k}(z^k)^T \Delta z_0^k}{\nabla f_{\rho_k}(z^k)^T (\Delta z_0^k - \Delta z_1^k)}, & \text{否则.} \end{cases} \quad (2.16)$$

步骤5 令

$$\begin{aligned} I_k^1 &= \{j \in L_1 : g_j(p^k) \leq \lambda_k^{1,j} + \Delta\lambda_k^{1,j}\}, I_k^2 = \{j \in L_2 : c_j(z^k) \leq \lambda_k^{2,j} + \Delta\lambda_k^{2,j}\}, \\ I_k^3 &= \{j \in L_2 : \phi(q_j^k, u_k) \leq \lambda_k^{3,j} + \Delta\lambda_k^{3,j}\}, \end{aligned} \quad (2.17)$$

$$\begin{aligned} J_k^1 &= \{j \in L_1 : \lambda_k^{1,j} + \Delta\lambda_k^{1,j} \leq -g_j(p^k)\}, J_k^2 = \{j \in L_2 : \lambda_k^{2,j} + \Delta\lambda_k^{2,j} \leq -c_j(z^k)\}, \\ J_k^3 &= \{j \in L_2 : \lambda_k^{3,j} + \Delta\lambda_k^{3,j} \leq -\phi(q_j^k, u_k)\}. \end{aligned} \quad (2.18)$$

若 $J_k^1 \cup J_k^2 \cup J_k^3 = \emptyset$, 通过求解下列最小二乘问题计算修正方向 $\Delta \tilde{z}^k$:

$$\begin{aligned} \min \quad & \frac{1}{2} \Delta \tilde{z}^k T H_k \Delta \tilde{z}^k \\ \text{s.t.} \quad & g_j(p^k + \Delta p^k) + \nabla g_j(p^k)^T \Delta \tilde{p}^k = \psi_k, \quad \forall j \in I_k^1, \\ & c_j(z^k + \Delta z^k) + \nabla c_j(z^k)^T \Delta \tilde{z}^k = \psi_k, \quad \forall j \in I_k^2, \\ & \phi(q_j^k + \Delta q_j^k, u_k) + \nabla \phi(q_j^k, u_k)^T \Delta \tilde{q}_j^k = \psi_k, \quad \forall j \in I_k^3, \end{aligned} \quad (2.19)$$

其中

$$\begin{aligned} p^k &= (x^k, y^k), \Delta p^k = (\Delta x^k, \Delta y^k), \Delta \tilde{p}^k = (\Delta \tilde{x}^k, \Delta \tilde{y}^k), \\ q_j^k &= (y_j^k, w_j^k), \Delta q_j^k = (\Delta y_j^k, \Delta w_j^k), \Delta \tilde{q}_j^k = (\Delta \tilde{y}_j^k, \Delta \tilde{w}_j^k), \\ z^k &= (x^k, y^k, w^k), \Delta z^k = (\Delta x^k, \Delta y^k, \Delta w^k), \Delta \tilde{z}^k = (\Delta \tilde{x}^k, \Delta \tilde{y}^k, \Delta \tilde{w}^k), \end{aligned}$$

$$\psi_k = \max \left\{ \|\Delta z^k\|^\tau, \max_{j \in I_k^1} \left| \frac{\lambda_k^{1,j}}{\lambda_k^{1,j} + \Delta \lambda_k^{1,j}} - 1 \right|^\kappa \|\Delta z^k\|^2, \max_{j \in I_k^2} \left| \frac{\lambda_k^{2,j}}{\lambda_k^{2,j} + \Delta \lambda_k^{2,j}} - 1 \right|^\kappa \|\Delta z^k\|^2, \right. \\ \left. \max_{j \in I_k^3} \left| \frac{\lambda_k^{3,j}}{\lambda_k^{3,j} + \Delta \lambda_k^{3,j}} - 1 \right|^\kappa \|\Delta z^k\|^2 \right\}, \quad (2.20)$$

若 $J_k^1 \cup J_k^2 \cup J_k^3 \neq \emptyset$ 或 (2.19) 不可行或无界或 $\|\Delta \tilde{z}^k\| > \|\Delta z^k\|$, 置 $\Delta \tilde{z}^k = 0$.

步骤6 曲线搜索. 计算 $\{1, \varrho, \varrho^2, \dots\}$ 中满足以下不等式

$$f_{\rho_k}(z^k + \alpha \Delta z^k + \alpha^2 \Delta \tilde{z}^k) \leq f_{\rho_k}(z^k) + \xi \alpha \nabla f_{\rho_k}(z^k)^T \Delta z^k, \quad (2.21)$$

$$g_j(p^k + \alpha \Delta p^k + \alpha^2 \Delta \tilde{p}^k) \geq \eta_1 g_j(p^k), \quad \forall j \in J_k^1, \quad (2.22)$$

$$c_j(z^k + \alpha \Delta z^k + \alpha^2 \Delta \tilde{z}^k) \geq \eta_2 c_j(z^k), \quad \forall j \in J_k^2, \quad (2.23)$$

$$\phi(q_j^k + \alpha \Delta q_j^k + \alpha^2 \Delta \tilde{q}_j^k, u_k) \geq \eta_3 \phi(q_j^k, u_k), \quad \forall j \in J_k^3, \quad (2.24)$$

$$g_j(p^k + \alpha \Delta p^k + \alpha^2 \Delta \tilde{p}^k) > 0, \quad \forall j \in L_1, \quad (2.25)$$

$$c_j(z^k + \alpha \Delta z^k + \alpha^2 \Delta \tilde{z}^k) > 0, \quad \forall j \in L_2, \quad (2.26)$$

$$\phi(q_j^k + \alpha \Delta q_j^k + \alpha^2 \Delta \tilde{q}_j^k, u_k) > 0, \quad \forall j \in L_2. \quad (2.27)$$

的最大值 α_k .

步骤7 更新. 令 $z^{k+1} = z^k + \alpha \Delta z^k + \alpha^2 \Delta \tilde{z}^k$. 若 $J_k^1 \cup J_k^2 \cup J_k^3 = \emptyset$, 置

$$\lambda_{k+1}^{1,j} = \min\{\max\{\|\Delta z^k\|^2, \lambda_k^{1,j} + \Delta \lambda_k^{1,j}\}, t_{\max}\}, \quad j \in L_1, \quad (2.28)$$

$$\lambda_{k+1}^{2,j} = \min\{\max\{\|\Delta z^k\|^2, \lambda_k^{2,j} + \Delta \lambda_k^{2,j}\}, t_{\max}\}, \quad j \in L_2, \quad (2.29)$$

$$\lambda_{k+1}^{3,j} = \min\{\max\{\|\Delta z^k\|^2, \lambda_k^{3,j} + \Delta \lambda_k^{3,j}\}, t_{\max}\}, \quad j \in L_2; \quad (2.30)$$

否则, 置 $\lambda_{k+1}^{1,j} = \lambda_0^{1,j}$, $j \in L_1$, $\lambda_{k+1}^{2,j} = \lambda_0^{2,j}$, $j \in L_2$, $\lambda_{k+1}^{3,j} = \lambda_0^{3,j}$, $j \in L_2$. 令 $\rho_{k+1} = \rho_k$, 更新 H_k 为对称正定阵. 置 $k = k + 1$, 转步骤1.

§3. 收敛性分析

本节首先讨论算法的全局收敛性, 先给出如下一较弱假设条件.

H 3.1 给定任意指标集 K 使得序列 $\{z^k\}$ 有界, 且存在常数 $\sigma_1, \sigma_2 > 0$, 对任意 $k \in K$ 满足: $\|H_k\| \leq \sigma_2$,

$$v^T \left(H_k + \sum_{j \in L_1} \frac{\lambda_k^{1,j}}{g_j(p^k)} \nabla g_j(p^k) \nabla g_j(p^k)^T + \sum_{j \in L_2} \frac{\lambda_k^{2,j}}{c_j(z^k)} \nabla c_j(z^k) \nabla c_j(z^k)^T \right. \\ \left. + \sum_{j \in L_2} \frac{\lambda_k^{3,j}}{\phi(q_j^k, u_k)} \nabla \phi(q_j^k, u_k) \nabla \phi(q_j^k, u_k)^T \right) v \geq \sigma_1 \|v\|^2, \quad \forall v \in R^n.$$

引理 3.1 搜索方向 Δz^k 满足

$$\nabla f_{\rho_k}(z^k)^T \Delta z^k \leq \theta \nabla f_{\rho_k}(z^k)^T \Delta z_0^k \leq -\sigma_1 \theta \|\Delta z_0^k\|^2, \quad (3.1)$$

且

$$\nabla g_j(p^k)^T \Delta z^k > 0, \forall j \in J_k^1, \nabla c_j(z^k)^T \Delta z^k > 0, \forall j \in J_k^2, \nabla \phi(q_j^k, u_k)^T \Delta z^k > 0, \forall j \in J_k^3. \quad (3.2)$$

证明 由步骤1, 有

$$\begin{aligned} -H_k \Delta z_0^k + A(p^k)^T \Delta \lambda_{0,k}^1 + B(z^k)^T \Delta \lambda_{0,k}^2 + T(q^k, u_k)^T \Delta \lambda_{0,k}^3 &= h(p^k) - A(p^k)^T \lambda_k^1 \\ &\quad - B(z^k)^T (\lambda_k^2 - \rho_{1,k} e) - T(q^k, u_k)^T (\lambda_k^3 - \rho_{2,k} e), \end{aligned} \quad (3.3)$$

$$Y_k^1 A(p^k) \Delta z_0^k + G(p^k) \Delta \lambda_{0,k}^1 = -G(p^k) \lambda_k^1, \quad (3.4)$$

$$Y_k^2 B(z^k) \Delta z_0^k + C(z^k) \Delta \lambda_{0,k}^2 = -C(z^k) \lambda_k^2, \quad (3.5)$$

$$Y_k^3 T(q^k, u_k) \Delta z_0^k + \Phi(q^k, u_k) \Delta \lambda_{0,k}^3 = -\Phi(q^k, u_k) \lambda_k^3, \quad (3.6)$$

令 $S_k := H_k + A(p^k)^T G(p^k)^{-1} Y_k^1 A(p^k) + B(z^k)^T C(z^k)^{-1} Y_k^2 B(z^k) + T(q^k, u_k)^T \Phi(q^k, u_k)^{-1} Y_k^3 T(q^k, u_k)$, 故有

$$\nabla f_{\rho_k}(z^k) = -S_k \Delta z_0^k.$$

而由假设H 3.1, 有 $\nabla f_{\rho_k}(z^k)^T \Delta z_0^k = -(\Delta z_0^k)^T S_k \Delta z_0^k \leq -\sigma_1 \|\Delta z_0^k\|^2$.

根据(2.16), 若 $\nabla f_{\rho_k}(z^k)^T \Delta z_1^k \leq \theta \nabla f_{\rho_k}(z^k)^T \Delta z_0^k$, $\varphi_k = 1$, 故

$$\Delta z^k = \Delta z_1^k, \quad \nabla f_{\rho_k}(z^k)^T \Delta z^k \leq \theta \nabla f_{\rho_k}(z^k)^T \Delta z_0^k.$$

若 $\nabla f_{\rho_k}(z^k)^T \Delta z_1^k > \theta \nabla f_{\rho_k}(z^k)^T \Delta z_0^k$, 有

$$\nabla f_{\rho_k}(z^k)^T \Delta z^k = (1 - \varphi_k) \nabla f_{\rho_k}(z^k)^T \Delta z_0^k + \varphi_k \nabla f_{\rho_k}(z^k)^T \Delta z_1^k = \theta \nabla f_{\rho_k}(z^k)^T \Delta z_0^k,$$

故(3.1)式成立. 而由步骤3, 有

$$\begin{aligned} -H_k \Delta z_1^k + A(p^k)^T \Delta \lambda_{1,k}^1 + B(z^k)^T \Delta \lambda_{1,k}^2 + T(q^k, u_k)^T \Delta \lambda_{1,k}^3 &= h(p^k) - A(p^k)^T \lambda_k^1 \\ &\quad - B(z^k)^T (\lambda_k^2 - \rho_{1,k} e) - T(q^k, u_k)^T (\lambda_k^3 - \rho_{2,k} e), \end{aligned} \quad (3.7)$$

$$Y_k^1 A(p^k) \Delta z_1^k + G(p^k) \Delta \lambda_{1,k}^1 = \|\Delta z_0^k\|^\nu \lambda_k^1 - G(p^k) \lambda_k^1, \quad (3.8)$$

$$Y_k^2 B(z^k) \Delta z_1^k + C(z^k) \Delta \lambda_{1,k}^2 = \|\Delta z_0^k\|^\nu \lambda_k^2 - C(z^k) \lambda_k^2, \quad (3.9)$$

$$Y_k^3 T(q^k, u_k) \Delta z_1^k + \Phi(q^k, u_k) \Delta \lambda_{1,k}^3 = \|\Delta z_0^k\|^\nu \lambda_k^3 - \Phi(q^k, u_k) \lambda_k^3, \quad (3.10)$$

又由(2.18), $\forall j \in J_k^1$, $\lambda_k^{1,j} + \Delta \lambda_k^{1,j} \leq -g_j(p^k)$, 则有

$$\nabla g_j(p^k)^T \Delta z^k = -\frac{(\Delta \lambda_k^{1,j} + \lambda_k^{1,j}) g_j(p^k)}{\lambda_k^{1,j}} + \|\Delta z_0^k\|^\nu \varphi_k \geq \frac{g_j^2(p^k)}{\lambda_k^{1,j}} + \|\Delta z_0^k\|^\nu \varphi_k > 0.$$

类似如上分析, 有 $\nabla c_j(z^k)^T \Delta z^k > 0, \forall j \in J_k^2$, $\nabla \phi(q_j^k, u_k)^T \Delta z^k > 0, \forall j \in J_k^3$, 故(3.2) 式成立.

H 3.2 序列 $\{z^k\}$ 的一聚点 z^* 为问题 $(NLP_{\bar{\rho}})(2.1)$ 的一孤立 K - T 点, 且在 z^* 处:

(i) 严格互补条件成立; (ii) 乘子 $\lambda_*^1, \lambda_*^2, \lambda_*^3$ 满足 $\lambda_*^{1,j} \leq t_{\max}, j \in L_1, \lambda_*^{2,j}, \lambda_*^{3,j} \leq t_{\max}, j \in L_2$.

命题 3.1 若假设H 2.1- H 3.2 成立, 由算法产生的无穷序列 $\{z^k\}$ 有界, 则 $\{z^k\}$ 收敛到问题 $(NLP_{\bar{\rho}})(2.1)$ 的一 K - T 点 z^* . 且对 $k \rightarrow \infty$, 有

(i) $\{\Delta z^k\} \rightarrow 0, \{\lambda_k^1 + \Delta \lambda_k^1\} \rightarrow \lambda_*^1, \{\lambda_k^2 + \Delta \lambda_k^2\} \rightarrow \lambda_*^2, \{\lambda_k^3 + \Delta \lambda_k^3\} \rightarrow \lambda_*^3$;

(ii) $J_k^1 = J_k^2 = J_k^3 = \emptyset, I_k^1 = I^1(p^*), I_k^2 = I^2(z^*), I_k^3 = I^3(q^*, 0)$;

(iii) $\{\lambda_k^1\} \rightarrow \lambda_*^1, \{\lambda_k^2\} \rightarrow \lambda_*^2, \{\lambda_k^3\} \rightarrow \lambda_*^3$.

证明 证明类似文献[9]中命题4.2 的证明. ■

定理 3.1 若假设H 2.1- H 3.2成立, 算法产生一无穷序列 $\{z^k\}$, 在 $\{z^k\}$ 的任一聚点 z^* 处非退化条件成立: $(y_j^*, w_j^*) \neq (0, 0)$, $j \in L_2$, 则 z^* 为问题(MPEC)(1.1)的一稳定点.

证明 由命题3.1及(1.3), 有

$$\begin{aligned} h(p^*) - A(p^*)^T \lambda_*^1 - B(z^*)^T (\lambda_*^2 - \bar{\rho}_1 e) - T(q_j^*, 0)^T (\lambda_*^3 - \bar{\rho}_2 e) &= 0, \\ \lambda_*^{1,j} g_j(p^*) &= 0, \quad j \in L_1, \quad \phi(q_j^*, 0) = \min\{y_j^*, w_j^*\} = 0, \quad j \in L_2. \end{aligned} \quad (3.11)$$

记 $I_{y^*}(z^*) = \{j \in L_2 | w_j^* > 0 = y_j^*\}$, $I_{w^*}(z^*) = \{j \in L_2 | y_j^* > 0 = w_j^*\}$, 令

$$\bar{\pi}^* = \begin{cases} \frac{\lambda_*^3 - \bar{\rho}_2 e}{w_j^*}, & j \in I_{y^*}(z^*), \\ \frac{\lambda_*^3 - \bar{\rho}_2 e}{y_j^*}, & j \in I_{w^*}(z^*). \end{cases}$$

根据(3.11) 及(1.2), 有

$$\begin{aligned} h(p^*) - A(p^*)^T \lambda_*^1 - B(z^*)^T s^* - V(q^*)^T \bar{\pi}^* &= 0, \\ \lambda_*^{1,j} g_j(p^*) &= 0, \quad j \in L_1, \quad 0 \leq y_j^* \perp w_j^* \geq 0, \quad j \in L_2, \end{aligned}$$

其中 $V(q^*) = (0, \text{diag}(w_j^*), \text{diag}(y_j^*))$, $j \in L_2$, $w_j^* = F_j(x^*, y^*)$, $j \in L_2$, 这表明 z^* 是问题(MPEC)(1.1)的一稳定点. ■

记 λ_*^1, s^*, π^* 是与 z^* 有关的问题(NLP_u)(1.6)的K-T 乘子, 其中 $s^* = \lambda_*^2 - \bar{\rho}_1 e$, $\pi^* = \lambda_*^3 - \bar{\rho}_2 e$. (NLP_u)(1.6) 的拉格朗日函数如下:

$$L(z, \lambda^1, s, \pi) = f(x, y) - \lambda^{1T} g(p) - s^T c(z) - \pi^T \phi(q, u),$$

对应的乘子为 λ^1 , $s = \lambda^2 - \rho_1 e$, $\pi = \lambda^3 - \rho_2 e$, 与(NLP _{ρ})(2.1)的拉格朗日函数相同, 即,

$$L_\rho(z, \lambda^1, \lambda^2, \lambda^3) = f(x, y) + \rho_1 \sum_{j \in L_2} c_j(z) + \rho_2 \sum_{j \in L_2} \phi(q_j, u) - \lambda^{1T} g(p) - \lambda^{2T} c(z) - \lambda^{3T} \phi(q, u).$$

为证明算法的超线性收敛性, 另作如下假设:

H 3.3 函数 $f(p)$, $g_j(p)$, $j \in L_1$, $c_j(z)$, $\phi(q_j, u)$, $j \in L_2$, 三阶连续可微; 在 z^* 处二阶充分性条件成立, 即 $\nabla_{zz}^2 L(z^*, \lambda_*^1, s^*, \pi^*)$ 在如下空间上正定:

$$\{v \in R^{n+2m_2} | \nabla g_j(p^*)^T v = 0, j \in L_1, \nabla c_j(z^*)^T v = 0, j \in L_2, \nabla \phi(q_j^*, 0)^T v = 0, j \in L_2\}.$$

H 3.4 矩阵序列 $\{H_k\}$ 满足

$$\|P_k(H_k - \nabla_{zz}^2 L(z^*, \lambda_*^1, s^*, \pi^*)) \Delta z^k\| = o(\|\Delta z^k\|),$$

其中 $P_k = I_{n+2m_2} - R_k(R_k^T R_k)^{-1} R_k^T$, $R_k = [\nabla g_j(p^k), j \in I^1(p^*), \nabla c_j(z^k), j \in I^2(z^*), \nabla \phi(q_j^k, u_k), j \in I^3(q^*, 0)] \in R^{(n+2m_2) \times (|I^1(p^*)| + |I^2(z^*)| + |I^3(q^*, 0)|)}$.

引理 3.2 当 k 充分大时, $z^{k+1} = z^k + \Delta z^k + \Delta \tilde{z}^k$, 即步长 $\alpha_k \equiv 1$.

证明 当 k 充分大时, 由命题3.1(ii)知, $J_k^1 = J_k^2 = J_k^3 = \emptyset$, 即只需证

- (i) $f_{\rho_k}(z^k + \Delta z^k + \Delta \tilde{z}^k) \leq f_{\rho_k}(z^k) + \xi \nabla f_{\rho_k}(z^k)^T \Delta z^k$,
- (ii) $g_j(p^k + \Delta p^k + \Delta \tilde{p}^k) > 0, \forall j \in L_1, \quad c_j(z^k + \Delta z^k + \Delta \tilde{z}^k) > 0, \forall j \in L_2$,

$$\phi(q_j^k + \Delta q_j^k + \Delta \tilde{q}_j^k, u_k) > 0, \forall j \in L_2.$$

首先证(i) 成立. 根据假设H 3.3, 有

$$f_{\rho_k}(z^k + \Delta z^k + \Delta \tilde{z}^k) = f_{\rho_k}(z^k) + \nabla f_{\rho_k}(z^k)^T (\Delta z^k + \Delta \tilde{z}^k) + \frac{1}{2} \Delta z^k T \nabla_{zz}^2 f_{\rho_k}(z^k) \Delta z^k + o(\|\Delta z^k\|^2). \quad (3.12)$$

$$\text{定义 } \lambda_k^{1,j} = \lambda_k^{1,j} + \Delta \lambda_k^{1,j}, j \in L_1, \lambda_k^{2,j} = \lambda_k^{2,j} + \Delta \lambda_k^{2,j}, j \in L_2, \lambda_k^{3,j} = \lambda_k^{3,j} + \Delta \lambda_k^{3,j}, j \in L_2,$$

$$\begin{aligned} \nabla f_{\rho_k}(z^k)^T \Delta z^k &= -\Delta z^k T H_k \Delta z^k + \sum_{j \in L_1} \lambda_k^{1,j} \nabla g_j(p^k)^T \Delta p^k + \sum_{j \in L_2} \lambda_k^{2,j} \nabla c_j(z^k)^T \Delta z^k \\ &\quad + \sum_{j \in L_2} \lambda_k^{3,j} \nabla \phi(q_j^k, u_k)^T \Delta q_j^k, \end{aligned} \quad (3.13)$$

$$\nabla f_{\rho_k}(z^k)^T \Delta \tilde{z}^k = \sum_{j \in L_1} \lambda_k^{1,j} \nabla g_j(p^k)^T \Delta \tilde{p}^k + \sum_{j \in L_2} \lambda_k^{2,j} \nabla c_j(z^k)^T \Delta \tilde{z}^k + \sum_{j \in L_2} \lambda_k^{3,j} \nabla \phi(q_j^k, u_k)^T \Delta \tilde{q}_j^k. \quad (3.14)$$

由(3.13) 及(3.14), (3.12) 可写为:

$$\begin{aligned} f_{\rho_k}(z^k + \Delta z^k + \Delta \tilde{z}^k) &= f_{\rho_k}(z^k) - \frac{1}{2} \Delta z^k T H_k \Delta z^k + \frac{1}{2} \sum_{j \in L_1} \lambda_k^{1,j} \nabla g_j(p^k)^T \Delta p^k \\ &\quad + \frac{1}{2} \sum_{j \in L_2} \lambda_k^{2,j} \nabla c_j(z^k)^T \Delta z^k + \frac{1}{2} \sum_{j \in L_2} \lambda_k^{3,j} \nabla \phi(q_j^k, u_k)^T \Delta q_j^k \\ &\quad + \frac{1}{2} \Delta z^k T (\nabla_{zz}^2 f_{\rho_k}(z^k) - H_k) \Delta z^k + o(\|\Delta z^k\|^2) + E_k^1 + E_k^2 + E_k^3, \end{aligned} \quad (3.15)$$

$$\text{其中 } E_k^1 = \frac{1}{2} \sum_{j \in L_1} \lambda_k^{1,j} \nabla g_j(p^k)^T \Delta p^k + \sum_{j \in L_1} \lambda_k^{1,j} \nabla g_j(p^k)^T \Delta \tilde{p}^k, E_k^2 = \frac{1}{2} \sum_{j \in L_2} \lambda_k^{2,j} \nabla c_j(z^k)^T \Delta z^k + \sum_{j \in L_2} \lambda_k^{2,j} \nabla c_j(z^k)^T \Delta \tilde{z}^k, E_k^3 = \frac{1}{2} \sum_{j \in L_2} \lambda_k^{3,j} \nabla \phi(q_j^k, u_k)^T \Delta q_j^k + \sum_{j \in L_2} \lambda_k^{3,j} \nabla \phi(q_j^k, u_k)^T \Delta \tilde{q}_j^k.$$

$$\begin{aligned} E_k^1 &= \sum_{j \in I^1(p^*)} \left[\frac{\frac{1}{2}(\lambda_k^{1,j})^2}{\lambda_k^{1,j}} - \lambda_k^{1,j} \right] g_j(p^k) - \frac{1}{2} \sum_{j \notin I^1(p^*)} \frac{(\lambda_k^{1,j})^2}{\lambda_k^{1,j}} g_j(p^k) \\ &\quad - \frac{1}{2} \sum_{j \in L_1} \lambda_k^{1,j} \Delta p^k T \nabla_{zz}^2 g_j(p^k) \Delta p^k + o(\|\Delta z^k\|^2). \end{aligned} \quad (3.16)$$

E_k^2, E_k^3 形式类似 E_k^1 .

又由于假设H 3.4, 有

$$f_{\rho_k}(z^k + \Delta z^k + \Delta \tilde{z}^k) \leq f_{\rho_k}(z^k) + \frac{1}{2} \nabla f_{\rho_k}(z^k)^T \Delta z^k + o(\|\Delta z^k\|^2).$$

又 $\nabla f_{\rho_k}(z^k)^T \Delta z^k \leq -\sigma_1 \theta \|\Delta z_0^k\|^2 + o(\|\Delta z^k\|^2)$, 因 $0 < \xi < \frac{1}{2}$, 故(i) 成立.

下证(ii) 成立. 对 $j \notin I^1(p^*)$, 因 $\{\Delta z^k\}, \{\Delta \tilde{z}^k\}$ 都收敛到0, 故 $g_j(p^k + \Delta p^k + \Delta \tilde{p}^k) > 0$. 考虑 $j \in I^1(p^*)$, 有

$$\begin{aligned} g_j(p^k + \Delta p^k + \Delta \tilde{p}^k) &= g_j(p^k + \Delta p^k) + \nabla g_j(p^k + \Delta p^k)^T \Delta \tilde{p}^k + O(\|\Delta z^k\|^2) \\ &= g_j(p^k + \Delta p^k) + \nabla g_j(p^k)^T \Delta \tilde{p}^k + O(\|\Delta z^k\| \|\Delta \tilde{z}^k\|), \end{aligned}$$

故

$$\begin{aligned} g_j(p^k + \Delta p^k + \Delta \tilde{p}^k) &= \psi_k + O\left(\max \left\{ \|\Delta z^k\|^3, \max_{j \in I^1(p^*)} \left| \frac{\lambda_k^{1,j}}{\lambda_k^{1,j} + \Delta \lambda_k^{1,j}} - 1 \right| \|\Delta z^k\|^2, \right. \right. \\ &\quad \left. \left. \max_{j \in I^2(z^*)} \left| \frac{\lambda_k^{2,j}}{\lambda_k^{2,j} + \Delta \lambda_k^{2,j}} - 1 \right| \|\Delta z^k\|^2, \max_{j \in I^3(q^*, 0)} \left| \frac{\lambda_k^{3,j}}{\lambda_k^{3,j} + \Delta \lambda_k^{3,j}} - 1 \right| \|\Delta z^k\|^2 \right\} \right). \end{aligned} \quad (3.17)$$

因 $2 < \tau < 3, 0 < \kappa < 1$, 由(3.17)及(2.20) 有 $g_j(p^k + \Delta p^k + \Delta \tilde{p}^k) > 0$. 类似的分析, 有 $c_j(z^k + \Delta z^k + \Delta \tilde{z}^k) > 0, \phi(q_j^k + \Delta q_j^k + \Delta \tilde{q}_j^k, u_k) > 0, \forall j \in L_2$. (ii) 得证. ■

定理 3.2 在上述假设条件下, 若 $u_k = o(\|\Delta z^k\|)$, 则无穷序列 $\{z^k\}$ 超线性收敛到 (MPEC) (1.1) 的稳定点 z^* , 即

$$\|z^{k+1} - z^*\| = o(\|z^k - z^*\|).$$

证明 证明类似文献[9]中定理4.6 的证明. ■

§4. 数值实验

利用Lingo 软件建立模型求得算法对应问题的内点, 算法中的参数选取为 $\xi = 0.1$, $\eta_1 = 0.8$, $\eta_2 = 0.8$, $\eta_3 = 0.8$, $r_1 = 1$, $r_2 = 1$, $r_3 = 1$, $\nu = 3$, $\delta = 2$, $\tau = 2.5$, $\kappa = 0.5$, H_0 为 $(n + 2m_2) \times (n + 2m_2)$ 的单位阵. 数值实验问题来源于文[5], 文[6], 文[11]及文[12].

问题1 (见[5] 问题2).

$$\begin{aligned} \min \quad & f(x, y) = \frac{1}{2}((x_1 + x_2 + y_1 - 15)^2 + (x_1 + x_2 + y_2 - 15)^2) \\ \text{s.t.} \quad & 0 \leq x \leq 10, \quad w = N^T x + M^T y + q, \quad 0 \leq y \perp w \geq 0, \\ & q = \begin{pmatrix} -36 \\ -25 \end{pmatrix}, \quad N = \begin{pmatrix} 8/3 & 2 \\ 2 & 5/4 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 5/4 \\ 8/3 & 2 \end{pmatrix}. \end{aligned}$$

问题2 (见[6]).

$$\begin{aligned} \min \quad & f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + x - y \\ \text{s.t.} \quad & 0 \leq (y - x) \perp y \geq 0. \end{aligned}$$

问题3 (见[11] 问题4).

$$\begin{aligned} \min \quad & f(x, y) = x^4 + 8y \\ \text{s.t.} \quad & g_1(x, y) = x + y - 50 \leq 0, \quad g_2(x, y) = x^2 + y^2 - 100 \leq 0, \\ & F(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + xy + 10, \quad 0 \leq F(x, y) \perp y \geq 0. \end{aligned}$$

问题4 (见[12] outrata33).

$$\begin{aligned} \min \quad & f(x, y) = \frac{1}{2}((y_1 - 3)^2 + (y_2 - 4)^2 + 10y_4^2) \\ \text{s.t.} \quad & 0 \leq x \leq 10, \\ & 0 \leq [(1 + 0.2x)y_1 - (3 + 1.333x) - 0.333y_3 + 2y_1y_4] \perp y_1 \geq 0, \\ & 0 \leq [(1 + 0.1x)y_2 - x + y_3 + 2y_2y_4] \perp y_2 \geq 0, \\ & 0 \leq (0.333y_1 - y_2 + 1 - 0.1x) \perp y_3 \geq 0, \\ & 0 \leq (9 + 0.1x - y_1^2 - y_2^2) \perp y_4 \geq 0. \end{aligned}$$

问题5 (见[12] qpec2).

$$\begin{aligned} \min \quad & f(x, y) = \sum_{i=1}^{10} (x_i - 1)^2 + \sum_{j=1}^{20} (y_j - 2)^2 \\ \text{s.t.} \quad & 0 \leq (y_1 - x_1) \perp y_1 \geq 0, \\ & \vdots \\ & 0 \leq (y_{10} - x_{10}) \perp y_{10} \geq 0, \\ & 0 \leq y_{11} \perp y_{11} \geq 0, \\ & \vdots \\ & 0 \leq y_{20} \perp y_{20} \geq 0. \end{aligned}$$

表1 算法的数值实验结果

问题	(x^0, y^0)	k	(x^*, y^*)	$f(x^*, y^*)$	$\ dz^k\ $
1	(5, 2, 5, 5)	39	(9.13002012259, 5.86999325127, 0.00000042824, 0.00000006214)	0.00000000018	$7.461607899988e - 006$
2	(0, 1)	43	(-0.99999632970, 0.00000000000)	-0.49999999999	$9.387893085805e - 007$
3	(1, 10)	54	(1.02094641227, 9.99734915712)	8.10652483728	$8.303011633663e - 006$
4	(2, 2, 1, 1, 1)	18	(1.65735513638, 2.92333348007, 0.78660125922, 0.01031229962, 0.46430322506)	6.24379203527	$4.9956466412e - 006$
5	(0, : 0, 1, : 1, 1, : 1)	27	(1.54128255550, : 1.54128255550, 3.34185228257, : 3.34185228257, 0.54090639949, : 0.54090639949)	42.22508488179 45.00000[12]	$3.5764151842e - 006$

通过表1可以看出本文提出的算法是可行的.

§5. 结论

对非线性互补约束均衡问题, 通过引入罚函数思想, 将非线性互补约束均衡问题转化为只含不等式约束的参数规划问题, 且当参数充分大时, 转化问题与原问题等价, 从而减少了问题研究的复杂性. 基于内点法的思想, 提出了一个光滑QP-free算法求解转化后的光滑非线性规划问题. 该QP-free算法具有如下优点:

(1) 罚参数更新比文献[10]更简单;

(2) 减弱了Hessian阵估计正定的假设条件, 算法仍具有超线性收敛速度.

且数值实验结果表明本文提出的算法是可行的.

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A new QP-free algorithm for mathematical programs with nonlinear complementarity constraints

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Abstract: Against the shortcomings that many existing algorithms for solving the standard smoothing nonlinear programming would fail if they were used directly to solve the mathematical programs with equilibrium constraints (MPEC). By using a complementarity function and the idea of smoothing approximation method, the mathematical program with equilibrium constraints (MPEC) problem was transformed into a nonlinear programming, and a QP-free algorithm is proposed for the solution of (MPEC) problem. In particular, without the positive definiteness assumption on the Hessian estimate, the proposed algorithm is still global convergent. And superlinear convergence is obtained under some suitable assumptions. Numerical experiment results show that the proposed algorithm is feasible.

Keywords: Program with equilibrium constraints, Nonlinear complementarity, QP-free algorithm, global convergence, superlinear convergence

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