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ON CERTAIN SUBCLASS OF MEROMORPHIC MULTIVALENT STARLIKE FUNCTIONS

ZHENG Ying-hui, SHI Lei

(*School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China*)

Abstract: In this paper, we study a subclass $\mathcal{H}_p(\beta, \lambda)$ of meromorphic multivalent starlike functions. By using the analytical methods and techniques, we obtain the coefficient estimates, neighborhoods, partial sums and inclusion relationships of the class $\mathcal{H}_p(\beta, \lambda)$, which generalize the related works of some authors.

Keywords: meromorphic functions; Hadamard product; neighborhoods; partial sums

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1 Introduction

Let Σ_p denote the class of function f of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^k, \ \ p \in \mathbb{N} = \{1, 2, 3, \cdots\}, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

A function $f \in \Sigma_p$ is said to be in the class $\mathcal{MS}_p^*(\alpha)$ of meromorphic p-valent starlike function of order α if it satisfies the inequality $\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \ \ (z \in \mathbb{U};\ 0 \leqq \alpha < p)$.

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \ \ (z \in \mathbb{U}), \tag{1.2}$$

which are analytic in $\mathbb{U}$ and satisfy the condition $\Re\big(p(z)\big) > 0 \ \ (z \in \mathbb{U})$.

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Biography: Zheng Yinghui (1982–), female, born at Anyang, Henan, lecturer, master, major in complex analysis.

Given two functions $f,\ g \in \Sigma_p$, where f is given by (1.1) and g is given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_k z^k,$$

the Hadamard product $f * g$ is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g * f)(z).$$

A functions $f \in \Sigma_p$ is said to be in the class $\mathcal{H}_p(\beta, \lambda)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta\frac{z^2 f''(z)}{f(z)}\right) < \beta\lambda\left(\lambda + \frac{1}{2}\right) + \frac{1}{2}p\beta - \lambda \ \ (z \in \mathbb{U}), \tag{1.3}$$

where (and throughout this paper unless otherwise mentioned) the parameters β and λ are constrained as follows:

$$\beta \geqq 0 \qquad \text{and} \qquad p - \frac{1}{2} \leqq \lambda < p. \tag{1.4}$$

Clearly, we have $\mathcal{H}_p(0, \lambda) = \mathcal{MS}_p^*(\lambda)$.

2 Preliminary Results

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [2]) If the function $p \in \mathcal{P}$ is given by (1.2), then $|p_k| \leqq 2\ \ (k \in \mathbb{N})$.

The following lemma shows that the class $\mathcal{H}_p(\beta, \lambda)$ is a subclass of the class $\mathcal{MS}_p^*(\lambda)$ of meromorphic p-valent starlike functions of order λ.

Lemma 2.2 (see [1]) If $f \in \mathcal{H}_p(\beta, \lambda)$, then $f \in \mathcal{MS}_p^*(\lambda)$.

Lemma 2.3 Let $\beta > 0$ and $p - \beta p(p+1) - \gamma > 0$. Suppose also that the sequence $\{A_k\}_{k=1}^{\infty}$ is defined by

$$A_1 = \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + 1}, \tag{2.1}$$

and

$$A_{k+1} = \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + (\beta k + 1)(k+1)}\left(1 + \sum_{l=1}^{k} |A_l|\right) \qquad (k \in \mathbb{N}). \tag{2.2}$$

Then

$$A_k = \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + 1}\prod_{j=1}^{k-1}\frac{3p - 3\beta p(p+1) - 2\gamma + (\beta j + 1 - \beta)j}{p - \beta p(p+1) + (\beta j + 1)(j+1)} \qquad (k \in \mathbb{N} \setminus \{1\}). \tag{2.3}$$

Proof By virtue of (2.1), we easily get

$$[p - \beta p(p+1) + (\beta k + 1)(k+1)]A_{k+1} = 2(p - \beta p(p+1) - \gamma)\left(1 + \sum_{l=1}^{k} |A_l|\right), \tag{2.4}$$

and

$$[p - \beta p(p+1) + (\beta k + 1 - \beta)k]A_k = 2(p - \beta p(p+1) - \gamma)\left(1 + \sum_{l=1}^{k-1} |A_l|\right). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\frac{A_{k+1}}{A_k} = \frac{3p - 3\beta p(p+1) - 2\gamma + (\beta k + 1 - \beta)k}{p - \beta p(p+1) + (\beta k + 1)(k+1)}. \tag{2.6}$$

Thus, for $k \geqq 2$, we deduce from (2.6) that

$$\begin{aligned} A_k &= \frac{A_k}{A_{k-1}} \cdots \frac{A_3}{A_2} \cdot \frac{A_2}{A_1} \cdot A_1 \\ &= \frac{2(p - \beta p(p+1) - \gamma)}{p - \beta p(p+1) + 1} \prod_{j=1}^{k-1} \frac{3p - 3\beta p(p+1) - 2\gamma + (\beta j + 1 - \beta)j}{p - \beta p(p+1) + (\beta j + 1)(j+1)}. \end{aligned}$$

3 Properties of the Function Class $\mathcal{H}_p(\beta, \lambda)$

Theorem 3.1 Let

$$p - \beta p(p+1) + \beta\lambda\left(\lambda + \frac{1}{2}\right) + \frac{1}{2}p\beta - \lambda > 0. \tag{3.1}$$

Suppose also that $f \in \Sigma_p$ is given by (1.1) . If

$$\sum_{k=1}^{\infty} [k + \beta k(k-1) + \gamma]|a_k| \leqq p - \beta p(p+1) - \gamma, \tag{3.2}$$

where (and throughout this paper unless otherwise mentioned) the parameter γ is constrained as follows:

$$\gamma := \lambda - \beta\lambda\left(\lambda + \frac{1}{2}\right) - \frac{1}{2}p\beta, \tag{3.3}$$

then $f \in \mathcal{H}_p(\beta, \lambda)$.

Proof To prove $f \in \mathcal{H}_p(\beta, \lambda)$, it suffices to show that

$$\left|\frac{\frac{zf'(z)}{f(z)} + \beta\frac{z^2 f''(z)}{f(z)} + 1}{\frac{zf'(z)}{f(z)} + \beta\frac{z^2 f''(z)}{f(z)} + 2\gamma - 1}\right| < 1 \qquad (z \in \mathbb{U}). \tag{3.4}$$

Combining (3.1), (3.2) and (3.3), we know that

$$\begin{aligned} &p - \beta p(p+1) - 2\gamma + 1 - \sum_{k=1}^{\infty} [k + \beta k(k-1) + 2\gamma - 1]|a_k| \\ \geqq &1 - p + \beta p(p+1) + \sum_{k=1}^{\infty} [k + 1 + \beta k(k-1)]|a_k| > 0. \end{aligned} \tag{3.5}$$

Now, by the maximum modulus principle, we deduce from (1.1) and (3.5) that

$$\begin{aligned}
&\left|\frac{\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}+2\gamma-1}\right| \\
=&\left|\frac{1-p+\beta p(p+1)+\sum_{k=1}^{\infty}[k+1+\beta k(k-1)]a_kz^{k+p}}{-p+\beta p(p+1)+2\gamma-1+\sum_{k=1}^{\infty}[k+\beta k(k-1)+2\gamma-1]a_kz^{k+p}}\right| \\
<&\frac{1-p+\beta p(p+1)+\sum_{k=1}^{\infty}[k+1+\beta k(k-1)]|a_k|}{p-\beta p(p+1)-2\gamma+1-\sum_{k=1}^{\infty}[k+\beta k(k-1)+2\gamma-1]|a_k|} \\
\leqq&1.
\end{aligned}$$

Theorem 3.2 Let γ be defined by (3.3). If $f\in\mathcal{H}_p(\beta,\lambda)$ with $0<\beta<\frac{p-\lambda}{p(p+1)}$, then

$$|a_1|\leqq\frac{2[p-\beta p(p+1)-\gamma]}{p-\beta p(p+1)+1},$$

and

$$|a_k|\leqq\frac{2[p-\beta p(p+1)-\gamma]}{p-\beta p(p+1)+1}\prod_{j=1}^{k-1}\frac{3p-3\beta p(p+1)-2\gamma+(\beta j+1-\beta)j}{p-\beta p(p+1)+(\beta j+1)(j+1)}\qquad(k\in\mathbb{N}\setminus\{1\}).$$

Proof Suppose that

$$q(z):=-\frac{zf'(z)}{f(z)}-\beta\frac{z^2f''(z)}{f(z)}+\beta\lambda(\lambda+\frac{1}{2})+\frac{1}{2}p\beta-\lambda.\tag{3.6}$$

Then, q is analytic in $\mathbb{U}$ and $\Re(q(z))>0$ $(z\in\mathbb{U})$ with $q(0)=p-\beta p(p+1)-\gamma>0$. It follows from (3.3) and (3.6) that

$$q(z)f(z)=-zf'(z)-\beta z^2f''(z)-\gamma f(z).\tag{3.7}$$

By noting that

$$h(z)=\frac{q(z)}{p-\beta p(p+1)-\gamma}\in\mathcal{P},$$

if we put

$$q(z)=c_0+\sum_{k=1}^{\infty}c_kz^k\qquad(c_0=p-\beta p(p+1)-\gamma).$$

by Lemma 2.1, we know that $|c_k|\leqq 2[p-\beta p(p+1)-\gamma]$ $(k\in\mathbb{N})$. It follows from (3.7) that

$$\begin{aligned}
&\left(c_0+\sum_{k=1}^{\infty}c_kz^k\right)\left(\frac{1}{z^p}+\sum_{k=1}^{\infty}a_kz^k\right)\\
=&\left(\frac{p}{z^p}-\sum_{k=1}^{\infty}ka_kz^k\right)-\left[\beta\frac{p(p+1)}{z^p}+\beta\sum_{k=1}^{\infty}k(k-1)a_kz^k\right]-\gamma\left(\frac{1}{z^p}+\sum_{k=1}^{\infty}a_kz^k\right).
\end{aligned}\tag{3.8}$$

In view of (3.8), we get

$$c_0 a_1 + c_{p+1} = -a_1 - \gamma a_1, \tag{3.9}$$

and

$$c_0 a_{k+1} + c_{p+k+1} + \sum_{l=1}^{k} a_l c_{k+1-l} = -(k+1)a_{k+1} - \beta k(k+1)a_{k+1} - \gamma a_{k+1}. \tag{3.10}$$

From (3.9), we obtain

$$|a_1| \leqq \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + 1}, \tag{3.11}$$

and

$$|a_{k+1}| \leqq \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + (\beta k + 1)(k+1)} \left(1 + \sum_{l=1}^{k} |A_l|\right) \qquad (k \in \mathbb{N}). \tag{3.12}$$

Next, we define the sequence $\{A_k\}_{k=1}^{\infty}$ as follows

$$A_1 = \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + 1}, \tag{3.13}$$

and

$$A_{k+1} = \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + (\beta k + 1)(k+1)} \left(1 + \sum_{l=1}^{k} |A_l|\right) \qquad (k \in \mathbb{N}). \tag{3.14}$$

In order to prove that

$$|a_k| \leqq |A_k| \qquad (k \in \mathbb{N}),$$

we make use of the principle of mathematical induction. By noting that

$$|a_1| \leqq |A_1| = \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + 1}.$$

Therefore, assuming that

$$|a_l| \leqq |A_l| \qquad (l = 1, 2, 3, \cdots, k;\ k \in \mathbb{N}).$$

Combining (3.12) and (3.13), we get

$$\begin{aligned} |a_{k+1}| &\leqq \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + (\beta k + 1)(k+1)} \left(1 + \sum_{l=1}^{k} |a_l|\right) \\ &\leqq \frac{2[p - \beta p(p+1) - \gamma]}{p - \beta p(p+1) + (\beta k + 1)(k+1)} \left(1 + \sum_{l=1}^{k} |A_l|\right) \\ &= A_{k+1} \qquad (k \in \mathbb{N}). \end{aligned}$$

Hence, by the principle of mathematical induction, we have

$$|a_k| \leqq |A_k| \qquad (k \in \mathbb{N}) \tag{3.15}$$

as desired.

By means of Lemma 2.3 and (3.13), we know that (2.3) holds true. Combining (3.15) and (2.3), we readily get the coefficient estimates asserted by Theorem 3.2. Assuming that γ is given by (3.3) and that the condition (3.1) holds true, we here introduce the δ-neighborhood of a function $f \in \Sigma_p$ of the form (1.1) by means of the following definition:

$$\mathcal{N}_\delta(f) := \left\{ g \in \Sigma_p : g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_k z^k \text{ and } \sum_{k=1}^{\infty} \frac{k + \beta k(k-1) + \gamma}{p - \beta p(p+1) - \gamma} |a_k - b_k| \leqq \delta \right\}, \tag{3.16}$$

where $\delta \geqq 0$.

Theorem 3.3 Let the condition (3.1) hold true. If $f \in \Sigma_p$ satisfies the condition

$$\frac{f(z) + \varepsilon z^{-p}}{1 + \varepsilon} \in \mathcal{H}_p(\beta, \lambda) \qquad (\varepsilon \in \mathbb{C};\ |\varepsilon| < \delta;\ \delta > 0), \tag{3.17}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{H}_p(\beta, \lambda). \tag{3.18}$$

Proof By noting that the condition (1.3) is equivalent to (3.4), we easily find from (3.4) that a function $g \in \mathcal{H}_p(\beta, \lambda)$ if and only if

$$\frac{zg'(z) + \beta z^2 g''(z) + g(z)}{zg'(z) + \beta z^2 g''(z) + (2\gamma - 1)g(z)} \neq \sigma \qquad (z \in \mathbb{U};\ \sigma \in \mathbb{C};\ |\sigma| = 1),$$

which is equivalent to

$$\frac{(g * \mathfrak{h})(z)}{z^{-p}} \neq 0 \qquad (z \in \mathbb{U}), \tag{3.19}$$

where

$$\mathfrak{h}(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} c_k z^k \left(c_k := \frac{k + \beta k(k-1) + 1 - [k + \beta k(k-1) + (2\gamma - 1)]\sigma}{-p + \beta p(p+1) + 1 + [p - \beta p(p+1) - (2\gamma - 1)]\sigma} \right). \tag{3.20}$$

It follows from (3.20) that

$$\begin{aligned} |c_k| &= \left| \frac{k + \beta k(k-1) + 1 - [k + \beta k(k-1) + (2\gamma - 1)]\sigma}{-p + \beta p(p+1) + 1 + [p - \beta p(p+1) - (2\gamma - 1)]\sigma} \right| \\ &\leqq \frac{k + \beta k(k-1) + \gamma}{p - \beta p(p+1) - \gamma} \qquad (|\sigma| = 1). \end{aligned}$$

If $f \in \Sigma_p$ given by (1.1) satisfies the condition (3.17) , we deduce from (3.19) that

$$\frac{(f * \mathfrak{h})(z)}{z^{-p}} \neq -\varepsilon \qquad (|\varepsilon| < \delta;\ \delta > 0),$$

or equivalently,

$$\left| \frac{(f * \mathfrak{h})(z)}{z^{-p}} \right| \geqq \delta \qquad (z \in \mathbb{U}; \delta > 0). \tag{3.21}$$

We now suppose that

$$q(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}d_kz^k\in\mathcal{N}_\delta(f).$$

It follows from (3.16) that

$$\left|\frac{[(q-f)*\mathfrak{h}])(z)}{z^{-p}}\right|=\left|\sum_{k=1}^{\infty}(d_k-a_k)c_kz^{k+p}\right|\leqq|z|\sum_{k=1}^{\infty}\frac{k+\beta k(k-1)+\gamma}{p-\beta p(p+1)-\gamma}|d_k-a_k|<\delta. \quad (3.22)$$

Combining (3.21) and (3.22) , we easily find that

$$\left|\frac{(q*\mathfrak{h})(z)}{z^{-p}}\right|=\left|\frac{([f+(q-f)]*\mathfrak{h})(z)}{z^{-p}}\right|\geqq\left|\frac{(f*\mathfrak{h})(z)}{z^{-p}}\right|-\left|\frac{[(q-f)*\mathfrak{h}](z)}{z^{-p}}\right|>0,$$

which implies that

$$\frac{(q*\mathfrak{h})(z)}{z^{-p}}\neq 0 \qquad (z\in\mathbb{U}).$$

Therefore, we have

$$q(z)\in\mathcal{N}_\delta(f)\subset\mathcal{H}_p(\beta,\lambda).$$

Next, we derive the partial sums of the class $\mathcal{H}_p(\beta,\lambda)$. For some recent investigations involving the partial sums in analytic function theory, one can refer to [3–7].

Theorem 3.4 Let $f\in\Sigma_p$ be given by (1.1) and define the partial sums $f_n(z)$ of f by

$$f_n(z)=\frac{1}{z^p}+\sum_{k=1}^{n}a_kz^k(n\in\mathbb{N}). \quad (3.23)$$

If

$$\sum_{k=1}^{\infty}\frac{k+\beta k(k-1)+\gamma}{p-\beta p(p+1)-\gamma}|a_k|\leqq 1, \quad (3.24)$$

where γ given by (3.3) and the condition (3.1) holds true, then

(i) $f\in\mathcal{H}_p(\beta,\lambda)$;

(ii)

$$\Re\left(\frac{f(z)}{f_n(z)}\right)\geqq\frac{n+\beta n(n+1)+\beta p(p+1)-p+1+2\gamma}{n+\beta n(n+1)+1+\gamma} \qquad (n\in\mathbb{N};z\in\mathbb{U}), \quad (3.25)$$

and

$$\Re\left(\frac{f_n(z)}{f(z)}\right)\geqq\frac{n+\beta n(n+1)+\beta p(p+1)-p+2+\gamma-2\beta}{n+\beta n(n+1)+2-2\beta} \qquad (n\in\mathbb{N};z\in\mathbb{U}). \quad (3.26)$$

The bounds in (3.25) and (3.26) are sharp.

Proof First of all, we suppose that $f_1(z)=\frac{1}{z^p}$. We know that

$$\frac{f(z)+\varepsilon z^{-p}}{1+\varepsilon}=\frac{1}{z^p}\in\mathcal{H}_p(\beta,\lambda).$$

From (3.24), we easily find that

$$\sum_{k=1}^{\infty} \frac{k+\beta k(k-1)+\gamma}{p-\beta p(p+1)-\gamma}|a_k-0| \leqq 1,$$

which implies that $f \in N_1(z^{-p})$. By virtue of Theorem 3.3, we deduce that

$$f \in \mathcal{N}_1(z^{-p}) \subset \mathcal{H}_p(\beta,\lambda).$$

Next, it is easy to see that

$$\frac{n+1+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma} > \frac{n+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma} > 1 \qquad (n \in \mathbb{N}).$$

Therefore, we have

$$\sum_{k=1}^{n}|a_k| + \frac{n+1+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma}\sum_{k=n+1}^{\infty}|a_k| \leqq \sum_{k=1}^{\infty}\frac{k+\beta k(k-1)+\gamma}{p-\beta p(p+1)-\gamma}|a_k| \leqq 1. \tag{3.27}$$

We now suppose that

$$\begin{aligned} h_1(z) &= \frac{n+1+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma}\left(\frac{f(z)}{f_n(z)} - \frac{n+\beta n(n+1)+\beta p(p+1)-p+1+2\gamma}{n+1+\beta n(n+1)+\gamma}\right) \\ &= 1 + \frac{\frac{n+1+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma}\sum\limits_{k=n+1}^{\infty} a_k z^{k+p}}{1+\sum\limits_{k=1}^{n} a_k z^{k+p}}. \end{aligned} \tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \leqq \frac{\frac{n+1+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma}\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=1}^{n}|a_k| - \frac{n+1+\beta n(n+1)+\gamma}{p-\beta p(p+1)-\gamma}\sum\limits_{k=n+1}^{\infty}|a_k|} \leqq 1 \qquad (z \in \mathbb{U}),$$

which shows that

$$\Re(h_1(z)) \geqq 0 \qquad (z \in \mathbb{U}). \tag{3.29}$$

Combining (3.28) and (3.29), we deduce that the assertion (3.25) holds true.

Furthermore, if we put

$$f(z) = \frac{1}{z^p} - \frac{p-\beta p(p+1)-\gamma}{n+1+\beta n(n+1)+\gamma}z^{n+1}, \tag{3.30}$$

then

$$\begin{aligned} \frac{f(z)}{f_n(z)} &= 1 - \frac{p-\beta p(p+1)-\gamma}{n+1+\beta n(n+1)+\gamma}z^{n+1+p} \\ &\to \frac{n+\beta n(n+1)+\beta p(p+1)-p+1+2\gamma}{n+\beta n(n+1)+1+\gamma} \qquad (z \to 1^-), \end{aligned} \tag{3.31}$$

which implies that the bound in (3.25) is the best possible for each $n \in \mathbb{N}$.

Similarly, we suppose that

$$h_2(z) = \frac{n+\beta n(n+1)+2-2\beta}{p-\beta p(p+1)-\gamma}\left(\frac{f_n(z)}{f(z)} - \frac{n+\beta n(n+1)+\beta p(p+1)-p+\gamma+2-2\beta}{n+\beta n(n+1)+2-2\beta}\right)$$
$$= 1 - \frac{\frac{n+\beta n(n+1)+2-2\beta}{p-\beta p(p+1)-\gamma}\sum\limits_{k=n+1}^{\infty} a_k z^{k+p}}{1+\sum\limits_{k=1}^{\infty} a_k z^{k+p}}. \tag{3.32}$$

In view of (3.27) and (3.32), we conclude that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \leqq \frac{\frac{n+\beta n(n+1)+2-2\beta}{p-\beta p(p+1)-\gamma}\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=1}^{n}|a_k| - \frac{n+\beta n(n+1)+2\gamma+2\beta}{p-\beta p(p+1)-\gamma}\sum_{k=n+1}^{\infty}|a_k|} \leqq 1 \qquad (z \in \mathbb{U}),$$

which implies that

$$\Re(h_2(z)) \geqq 0 \qquad (z \in \mathbb{U}). \tag{3.33}$$

Combining (3.32) and (3.33) , we readily get the assertion (3.26) of Theorem 3.4. The bound in (3.26) is sharp with the extremal function f given by (3.30).

Finally, we prove the following inclusion relationship for the function class $\mathcal{H}_p(\beta, \lambda)$.

Theorem 3.5 Let $\beta_1 \geqq \beta_2 \geqq 1$ and $p - \frac{1}{2} \leqq \lambda_1 \leqq \lambda_2 < p$. Then

$$\mathcal{H}_p(\beta_1, \lambda_1) \subset \mathcal{H}_p(\beta_2, \lambda_2).$$

Proof Suppose that $f \in \mathcal{H}_p(\beta_1, \lambda_1)$. Then

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta_1\frac{z^2f''(z)}{f(z)}\right) < \lambda_1\left[\beta_1\left(\lambda_1+\frac{1}{2}\right)-1\right] + \frac{p\beta_1}{2} \qquad (z \in \mathbb{U}).$$

Since $\beta_1 \geqq \beta_2 \geqq 1$ and $p - \frac{1}{2} \leqq \lambda_1 \leqq \lambda_2 < p$, we find that

$$\lambda_1\left[\beta_1\left(\lambda_1+\frac{1}{2}\right)-1\right] + \frac{p\beta_1}{2} \leqq \lambda_2\left[\beta_1\left(\lambda_2+\frac{1}{2}\right)-1\right] + \frac{p\beta_1}{2}.$$

Then we obtain

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta_1\frac{z^2f''(z)}{f(z)}\right) < \lambda_2\left[\beta_1\left(\lambda_2+\frac{1}{2}\right)-1\right] + \frac{p\beta_1}{2} \qquad (z \in \mathbb{U}),$$

which shows that $f \in \mathcal{H}_p(\beta_1, \lambda_2)$. By Lemma 2.2, we see that $f \in \mathcal{MS}_p^*(\lambda_2)$, that is

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\lambda_2 \qquad (z \in \mathbb{U}).$$

Now, by setting $\delta = \frac{\beta_2}{\beta_1}$, so that $0 < \delta \leqq 1$, we easily find that

$$\begin{aligned}
&\Re\left(\frac{zf^{'}(z)}{f(z)} + \beta_2\frac{z^2f^{''}(z)}{f(z)} - \lambda_2\left[\beta_2\left(\lambda_2 + \frac{1}{2}\right) - 1\right] - \frac{p\beta_2}{2}\right)\\
=&\delta\Re\left(\frac{zf^{'}(z)}{f(z)} + \beta_1\frac{z^2f^{''}(z)}{f(z)} - \lambda_2\left[\beta_1\left(\lambda_2 + \frac{1}{2}\right) - 1\right] - \frac{p\beta_1}{2}\right)\\
&+ (1-\delta)\Re\left(\frac{zf^{'}(z)}{f(z)} + \lambda_2\right) < 0 \qquad (z \in \mathbb{U}),
\end{aligned}$$

that is $f \in \mathcal{H}_p(\beta_2, \lambda_2)$. The proof of Theorem 3.5 is thus completed.

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一类亚纯多叶星象函数的子族

郑颖慧，石 磊

(安阳师范学院数学与统计学院, 河南 安阳 455002)

摘 要: 本文研究了一类亚纯多叶星象函数的子族$\mathcal{H}_p(\beta, \lambda)$. 利用分析的方法与技巧, 得到了函数族$\mathcal{H}_p(\beta, \lambda)$的系数估计、邻域与部分和性质及一些包含关系, 所得结果推广了相关作者的一些成果.

关键词: 亚纯函数; 哈达玛卷积; 邻域; 部分和

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