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STABILITY OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS OF RETARDED TYPE

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Abstract: In this paper, analytical stability and numerical stability are both studied for stochastic differential equations with piecewise constant arguments of retarded type. First, the condition under which the analytical solutions are mean-square stable is obtained by Itô formula. Second, some new results on the numerical stability including the mean-square stability and T-stability of the Euler-Maruyama method are established by using inequality technique and stochastic analysis method. It is proved that the Euler-Maruyama method is both mean-square stable and T-stable under some suitable conditions. Our results can be seen as the generalization of the corresponding exist ones on the numerical stability of stochastic delay differential equations.

Keywords: stochastic delay differential equations; piecewise constant arguments; Euler-Maruyama method; mean-square stability; T-stability

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1 Introduction

Stochastic differential equations with piecewise constant arguments (SEPCA) can be regarded as generalizations of both stochastic delay differential equations (SDDEs) and differential equations with piecewise constant arguments (EPCA). The general form of SEPCA of retarded type is

$$dX(t) = f(t, X(t), X([t-p]))dt + g(t, X(t), X([t-p]))dW(t), t \geq 0$$

with the initial function $X(-1) = X_{-1}$ and $X(0) = X_0$. Where $[\cdot]$ signifies the greatest integer function, $p \in \mathbb{R}^+$. One-dimensional standard Wiener process $W(t)$ satisfies $\mathbb{E}(W(t)) = 0, \mathbb{E}(W(t)W(s)) = \min\{t, s\}$ and the initial value X_{-1}, X_0 are random variables. In addition, we assume that $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0$.

EPCA describe hybrid dynamical systems and combine properties of both differential and difference equations. They are appeared in modeling of various problems in real life such as biology, mechanics and electronics. Several important properties of the analytical

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solution of EPCA as well as numerical methods have been studied by many authors [1–4]. For more details of EPCA, the reader can see Wiener's book [5].

In many scientific and applied areas, including finance, ecology, computational biology and population dynamics, SDDEs are often used to model the corresponding systems. In recent years, there has been increasingly interesting in studying such equations, and their numerical treatments have also received considerable attention. Tudor and Tudor [6] and Tudor [7] first studied numerical solutions of SDDEs. Cao [8] derived some stability properties of Euler-Maruyama method for linear SDDEs. Buckwar and Shardlow [9] considered weak approximation methods for SDDEs. Rathinasamy and Balachandran [10] analyzed the mean-square stability of the semi-implicit Euler method for linear stochastic differential equations with multiple delays and Markovian switching. Hu and Huang [11] studied mean-square stability of stochastic θ-methods for stochastic delay integro-differential equations. Xiao et al. [12] analyzed convergence and stability of semi-implicit Euler methods for a linear stochastic pantograph equations. Recently, Wang and Chen [13] have given results of mean-square stability of semi-implicit Euler methods for nonlinear neutral SDDEs. We note that most of the above numerical stability results are focused on the mean-square stability, few results have been found in the references that involve T (Trajectory)-stability of numerical method for SDDEs. The definition of T-stability of numerical schemes for stochastic differential equations was introduced by Saito and Mitsui [14]. Burrage et al. [15] extended this concept from weak approximation to strong approximation. Burrage and Tian [16] discussed the T-stability of the composite Euler method for stochastic ordinary differential equations (SODEs). Cao [17] studied the T-stability of the semi-implicit Euler method for delay differential equations with multiplicative noise. For linear stochastic delay integro-differential equations, Rathinasamy and Balachandran [18] considered the T-stability of the split-step θ-methods. Motivated by the work of Cao [17] and Rathinasamy and Balachandran [18], the present paper will focus on both the mean-square stability and the T-stability of the Euler-Maruyama method for SEPCA.

In this paper, we consider the following scalar SEPCA of retarded type

$$\begin{cases} dX(t) = (a_1X(t) + a_2X([t-1]))dt + (a_3X(t) + a_4X([t-1]))dW(t), t \geq 0, \\ X(-1) = X_{-1}, X(0) = X_0, \end{cases} \tag{1.1}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$. The major objective of this paper is to illustrate that the Euler-Maruyama method applied to (1.1) is both mean-square stable and T-stable under the condition which guarantees the stability of the analytical solution.

The structure of this paper is organized as follows. In Section 2 we will introduce some necessary notations and hypotheses of (1.1) and discuss the stability properties of its analytical solution. In Section 3, the Euler-Maruyama method will be used to produce the numerical solutions. Moreover, our main results will be shown and proved in this section. Conclusion is provided in Section 4.

2 The Stability of the Analytical Solution

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathcal{F}_{t\geq 0}$ satisfying the usual conditions. The Wiener process $W(t)$ in (1.1) be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$, $L^1([0,\infty),\mathbb{R})$ and $L^2([0,\infty),\mathbb{R})$ denote the family of all real valued measurable $\mathcal{F}_t$-adapted process $f(t)_{t\geq 0}$ such that for every $T>0$, $\int_0^T |f(t)|dt < \infty$ w.p.1 and $\int_0^T |f(t)|^2 dt < \infty$ w.p.1, respectively. Moreover, assume that the initial value X_{-1} and X_0 are $\mathcal{F}_0$-measurable and $\mathbb{E}(X_0)^2 < \infty$.

Definition 2.1 A stochastic process $X(t)$ is called a solution of (1.1) on $[0,\infty)$ if it has the following properties:

(i) $X(t)$ is continuous and $\mathcal{F}_{t\geq 0}$-adapted;

(ii) $f(t, X(t), X([t-1]))_{t\geq 0} \in L^1([0,\infty),\mathbb{R})$ and $g(t, X(t), X([t-1]))_{t\geq 0} \in L^2([0,\infty),\mathbb{R})$;

(iii) (1.1) is satisfied on every interval $[n, n+1) \subset [0,\infty)$ with integral end-points almost surely.

Definition 2.2 If any solution $X(t)$ of (1.1) satisfies

$$\lim_{t\to\infty} \mathbb{E}(X(t))^2 = 0,$$

then the zero solution of the (1.1) is called mean-square stable.

Lemma 2.3 [19] The solution $x=0$ of equation

$$x'(t) = ax(t) + a_0 x([t]) + a_1 x([t-1]) \tag{2.1}$$

is asymptotically stable (the solution $x(t)\to 0$ as $t\to\infty$) if and only if the inequalities

$$\begin{cases} |a_1| < \dfrac{a}{e^a - 1}, \\ a_1 - \dfrac{a(e^a+1)}{e^a-1} < a_0 < -a - a_1 \end{cases} \tag{2.2}$$

hold.

From Lemma 2.3, we can easily obtain the following result.

Corollary 2.4 The solution $x=0$ of equation

$$x'(t) = ax(t) + a_1 x([t-1])$$

is asymptotically stable if and only if the inequality

$$-\frac{a}{e^a-1} < a_1 < -a$$

holds.

According to Corollary 2.4 and Lemma 2.1 in [20], we have the following Result.

Theorem 2.5 If $A_2 \geq 0$ and $A_1 + A_2 < 0$, then any continuous and positive solution $x(t)$ of system

$$\begin{cases} x'(t) \leq A_1 x(t) + A_2 x([t-1]), \\ x(-1) = x_{-1}, x(0) = x_0 \end{cases} \tag{2.3}$$

is asymptotically stable.

Hence, the result on the stability of analytical solution is obtained.

Theorem 2.6 If condition

$$2a_1 + a_3^2 + a_4^2 + 2|a_2 + a_3a_4| < 0 \tag{2.4}$$

holds, then the solution of (1.1) is mean-square stable.

Proof By Itô formula, we have

$$\begin{aligned} dX^2(t) =&(2X(t)(a_1X(t) + a_2X([t-1]) + (a_3X(t) + a_4X([t-1]))^2)dt \\ &+ 2X(t)(a_3X(t) + a_4X([t-1]))dW(t) \\ \le&((2a_1 + a_3^2 + |a_2 + a_3a_4|)X^2(t) + (a_4^2 + |a_2 + a_3a_4|)X^2([t-1]))dt \\ &+ 2X(t)(a_3X(t) + a_4X([t-1]))dW(t), \end{aligned} \tag{2.5}$$

let

$$Y(t) = \mathbb{E}(X^2(t)),$$

then

$$dY(t) \le \{(2a_1 + a_3^2 + |a_2 + a_3a_4|)Y(t) + (a_4^2 + |a_2 + a_3a_4|)Y([t-1])\}dt,$$

that is

$$Y'(t) \le (2a_1 + a_3^2 + |a_2 + a_3a_4|)Y(t) + (a_4^2 + |a_2 + a_3a_4|)Y([t-1]).$$

By using Theorem 2.5 and condition (2.4), we have

$$\lim_{t\to\infty} \mathbb{E}(X(t))^2 = \lim_{t\to\infty} Y(t) = 0,$$

which completes the proof.

3 Numerical Stability Analysis

Let $h = 1/m$ be a given stepsize with integer $m \ge 1$, and the gridpoints t_n be defined by $t_n = nh$. Let $n = km + l$ $(l = 0, 1, \cdots, m-1)$, applying the Euler-Maruyama method to (1.1), we have

$$X_{n+1} = X_n + (a_1X_n + a_2X_{(k-1)m})h + (a_3X_n + a_4X_{(k-1)m})\triangle W_n, \tag{3.1}$$

where $X_n = X(t_n)$, the increments $\triangle W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0,h)-$distributed Gaussian random variables. We assume X_n to be $\mathcal{F}_{t_n}$-measurable at the mesh-points t_n. For the convergence of the Euler-Maruyama method, we refer the interested reader to [21–23]. In the next two subsections, we will focus on its stability property.

3.1 Mean-square Stability

Definition 3.1 Under the condition (2.4), a numerical method applied to (1.1) is said to be mean-square stable, if there exists a $h_0(a_1, a_2, a_3, a_4) > 0$, such that the numerical solution sequence X_n produced by this numerical scheme satisfies

$$\lim_{n\to\infty} \mathbb{E}(X_n)^2 = 0 \tag{3.2}$$

for every stepsize $h \in (0, h_0(a_1, a_2, a_3, a_4))$ with $h = 1/m$ for an integer m.

Let

$$P = (1 + a_1 h)^2 + a_3^2 h + |(1 + a_1 h)a_2 h + a_3 a_4 h|,$$

$$Q = a_2^2 h^2 + a_4^2 h + |(1 + a_1 h)a_2 h + a_3 a_4 h|,$$

$$h_0(a_1, a_2, a_3, a_4) = -\frac{a_3^2 + a_4^2 + 2a_1 + 2|a_2 + a_3 a_4|}{(|a_1| + |a_2|)^2},$$

we have the following Lemma.

Lemma 3.2 If condition (2.4) is satisfied, then inequality $P + Q < 1$ holds for any $h \in (0, h_0(a_1, a_2, a_3, a_4))$.

Proof It is obvious that $h_0(a_1, a_2, a_3, a_4) > 0$ from condition (2.4), and

$$\begin{aligned} P + Q &= (1 + a_1 h)^2 + a_3^2 h + a_2^2 h^2 + a_4^2 h + 2|(1 + a_1 h)a_2 h + a_3 a_4 h| \\ &\le (a_1^2 + a_2^2)h^2 + (a_3^2 + 2a_1 + a_4^2)h + 1 + 2h|a_2 + a_3 a_4| + 2h^2|a_1 a_2| \\ &= (|a_1| + |a_2|)^2 h^2 + (a_3^2 + a_4^2 + 2a_1 + 2|a_2 + a_3 a_4|)h + 1, \end{aligned} \tag{3.3}$$

therefore $P + Q < 1$ if and only if

$$0 < h < -\frac{a_3^2 + a_4^2 + 2a_1 + 2|a_2 + a_3 a_4|}{(|a_1| + |a_2|)^2} = h_0(a_1, a_2, a_3, a_4),$$

the proof is completed.

Then the first main Theorem of this paper is obtained.

Theorem 3.3 Assume the condition (2.4) holds, then the Euler-Maruyama method applied to (1.1) is mean-square stable with $h \in (0, h_0(a_1, a_2, a_3, a_4)))$.

Proof It follows from (3.1) that

$$X_{n+1} = (1 + a_1 h + a_3 \triangle W_n)X_n + (a_2 h + a_4 \triangle W_n)X_{(k-1)m}. \tag{3.4}$$

Squaring both sides of the above equality, yields

$$\begin{aligned} X_{n+1}^2 =&(1 + a_1 h + a_3 \triangle W_n)^2 X_n^2 + (a_2 h + a_4 \triangle W_n)^2 X_{(k-1)m}^2 \\ &+ 2(1 + a_1 h + a_3 \triangle W_n)(a_2 h + a_4 \triangle W_n)X_n X_{(k-1)m}. \end{aligned} \tag{3.5}$$

Using the elementary inequality $2xyab \le |xy|(a^2 + b^2)$ we have

$$\begin{aligned} X_{n+1}^2 \le&(1 + a_1 h + a_3 \triangle W_n)^2 X_n^2 + (a_2 h + a_4 \triangle W_n)^2 X_{(k-1)m}^2 \\ &+ |(1 + a_1 h + a_3 \triangle W_n)(a_2 h + a_4 \triangle W_n)|(X_n^2 + X_{(k-1)m}^2), \end{aligned} \tag{3.6}$$

that is

$$\begin{aligned} X_{n+1}^2 \le&\{(1 + a_1 h + a_3 \triangle W_n)^2 + |(1 + a_1 h + a_3 \triangle W_n)(a_2 h + a_4 \triangle W_n)|\}X_n^2 \\ &+ \{(a_2 h + a_4 \triangle W_n)^2 + |(1 + a_1 h + a_3 \triangle W_n)(a_2 h + a_4 \triangle W_n)|\}X_{(k-1)m}^2. \end{aligned} \tag{3.7}$$

Denote

$$Y_n = \mathbb{E}(X_n^2).$$

Note that $\mathbb{E}(\triangle W_n) = 0$ and $\mathbb{E}(\triangle W_n)^2 = h$, the inequality (3.7) reduces to

$$\begin{aligned} Y_{n+1} \leq & [(1+a_1h)^2 + a_3^2 h + |(1+a_1h)a_2h + a_3a_4h|]Y_n \\ & + [a_2^2h^2 + a_4^2h + |(1+a_1h)a_2h + a_3a_4h|]Y_{(k-1)m}, \end{aligned} \tag{3.8}$$

which is equivalent to

$$Y_{n+1} \leq PY_n + QY_{(k-1)m}. \tag{3.9}$$

Namely

$$Y_{n+1} \leq (P+Q)\max\left\{Y_n, Y_{(k-1)m}\right\}. \tag{3.10}$$

By virtue of Lemma 3.2, the iteration of inequality (3.10) implies

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \mathbb{E}(X_n)^2 = 0.$$

It is shown that the method is mean-square stable.

3.2 T-stability

The mean-square stability may still cause difficulty from the point of view of computer implementation. In order to learn more about the nature of numerical scheme, we often need to evaluate the value of the expectation $\mathbb{E}|X_n|^2$ where X_n is an approximating sequence of the solution sample path. In a certain probability, X_n may happen to overflow in computer simulations. This actually interferes with the evaluation of X_n. To overcome this difficulty, T-stability is introduced with respect to the approximate sequence of sample path (trajectory) by Saito and Mitsui [14].

Definition 3.4 [15] Under the condition (2.4), the numerical scheme equipped with a specified driving process is said to be T-stable if

$$\lim_{n\to\infty} |X_n| = 0 \tag{3.11}$$

holding for the driving process.

The so called specified driving process is to approximate $\triangle W_n$ by random variable with specified distribution. In this paper, we treat the Euler-Maruyama scheme with two-point random variables. The wiener increment $\triangle W_n$ is taken as $U_n\sqrt{h}$ whose probability distribution is given by $P(U_n = \pm 1) = 1/2$, where P indicates probability.

By formula (3.4) and $\triangle W_n = U_n\sqrt{h}$, we have

$$\begin{aligned} |X_{n+1}| &\leq (|1 + a_1h + a_3\triangle W_n| + |a_2h + a_4\triangle W_n|)\max\{|X_n|, |X_{(k-1)m}|\} \\ &= \left(\left|1 + a_1h + a_3\sqrt{h}U_n\right| + \left|a_2h + a_4\sqrt{h}U_n\right|\right)\max\{|X_n|, |X_{(k-1)m}|\}. \end{aligned} \tag{3.12}$$

Considering the average of $n+1$ steps by recursiving (3.4), we call it average stability function with

$$\begin{aligned}
&R^{(2)}(h;a_1,a_2,a_3,a_4)\\
=&\left(\left|1+a_1h+a_3\sqrt{h}\right|+\left|a_2h+a_4\sqrt{h}\right|\right)\left(\left|1+a_1h-a_3\sqrt{h}\right|+\left|a_2h-a_4\sqrt{h}\right|\right)\\
=&\left|(1+a_1h)^2-a_3^2h\right|+\left|a_2^2h^2-a_4^2h\right|\\
&+\left|[(1+a_1h)a_2h-a_3a_4h]+[a_2a_3h\sqrt{h}-(1+a_1h)a_4\sqrt{h}]\right|\\
&+\left|[(1+a_1h)a_2h-a_3a_4h]-[a_2a_3h\sqrt{h}-(1+a_1h)a_4\sqrt{h}]\right|,
\end{aligned} \tag{3.13}$$

if $R^{(2)}(h;a_1,a_2,a_3,a_4)<1$, then $\lim\limits_{n\to\infty}|X_n|=0$, so the numerical method is T-stable. Therefore, the second main theorem of this paper is given as follows.

Theorem 3.5 Assume condition (2.4) is satisfied, denote

$$\begin{aligned}
h_1&=\min\left\{\frac{1}{|a_1|},-\frac{2(a_1+|a_2|)-(|a_3|-|a_4|)^2}{(a_1+|a_2|)^2}\right\},\\
h_2&=\min\left\{\frac{1}{|a_1|},\frac{-k_0+\sqrt{k_0^2-8(a_1^2-a_2^2)|a_4|}}{2(a_1^2-a_2^2)}\right\},\\
h_3&=\frac{-k_0-\sqrt{k_0^2-8(a_1^2-a_2^2)|a_4|}}{2(a_1^2-a_2^2)},\\
h_4&=\min\left\{\frac{1}{|a_1|},\frac{-k_1-\sqrt{k_1^2-8(a_1-|a_2|)^2}}{2(a_1-|a_2|)^2}\right\},\\
h_5&=\frac{-k_1+\sqrt{k_1^2-8(a_1-|a_2|)^2}}{2(a_1-|a_2|)^2},\\
h_6&=\min\left\{\frac{1}{|a_1|},\frac{-k_2-\sqrt{k_2^2-8(a_1^2-a_2^2)(1-|a_4|)}}{2(a_1^2-a_2^2)}\right\},\\
h_7&=\frac{-k_2+\sqrt{k_2^2-8(a_1^2-a_2^2)(1-|a_4|)}}{2(a_1^2-a_2^2)},\\
S_1&=\left\{h\,\middle|\,h\ge\frac{a_4^2}{a_2^2}\right\}\bigcap\left\{h\,\middle|\,h\ge\frac{a_3^2-2a_1+\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2},h\le\frac{a_3^2-2a_1-\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2}\right\},\\
S_2&=\left\{h\,\middle|\,h<\frac{a_4^2}{a_2^2}\right\}\bigcap\left\{h\,\middle|\,h\ge\frac{a_3^2-2a_1+\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2},h\le\frac{a_3^2-2a_1-\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2}\right\},\\
S_3&=\left\{h\,\middle|\,h<\frac{a_4^2}{a_2^2}\right\}\bigcap\left\{h\,\middle|\,\frac{a_3^2-2a_1-\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2}<h<\frac{a_3^2-2a_1+\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2}\right\},\\
S_4&=\left\{h\,\middle|\,h\ge\frac{a_4^2}{a_2^2}\right\}\bigcap\left\{h\,\middle|\,\frac{a_3^2-2a_1-\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2}<h<\frac{a_3^2-2a_1+\sqrt{a_3^2-4a_1}|a_3|}{2a_1^2}\right\},
\end{aligned}$$

where $k_0=2a_1+a_4^2-a_3^2+2|a_2a_3|+2a_1|a_4|, k_1=2a_1-2|a_2|-(|a_3|+|a_4|)^2, k_2=2a_1-2|a_2a_3|-2a_1|a_4|-a_3^2+a_4^2$, then

(i) If $h \in S_1$ and satisfies $h < h_1$, the Euler-Maruyama method for (1.1) is T-stable;

(ii) If $h \in S_2$ and satisfies $h_3 < h < h_2$, the Euler-Maruyama method for (1.1) is T-stable;

(iii) If $h \in S_3$ and satisfies $h < h_4$ or $h > h_5$, the Euler-Maruyama method for (1.1) is T-stable;

(iv) If $h \in S_4$ and satisfies $h < h_6$ or $h > h_7$, the Euler-Maruyama method for (1.1) is T-stable.

Proof By virtue of (3.13), if the inequality

$$\begin{aligned}&\left|(1+a_1h)^2 - a_3^2h\right| + \left|a_2^2h^2 - a_4^2h\right| \\ &+ \left|[(1+a_1h)a_2h - a_3a_4h] + [a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}]\right| \\ &+ \left|[(1+a_1h)a_2h - a_3a_4h] - [a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}]\right| < 1\end{aligned} \tag{3.14}$$

holds for $h < 1/|a_1|$, then $R^{(2)}(h; a_1, a_2, a_3, a_4) < 1$ means that $|X_n| \to 0$ $(n \to \infty)$. Hence, the Euler-Maruyama method is T-stable. In addition, we notice that $h_i > 0, i = 1, 2, \cdots, 7$ and $k_j < 0, j = 0, 1, 2$ under the condition (2.4) and Remark 3.6. The proof will be considered in four cases as follows.

(i) If $h \in S_1$, then $a_2^2h^2 - a_4^2h \geq 0, (1+a_1h)^2 - a_3^2h \geq 0$. So we have

$$\begin{aligned}&\left|[(1+a_1h)a_2h - a_3a_4h] + [a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}]\right| \\ &+ \left|[(1+a_1h)a_2h - a_3a_4h] - [a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}]\right| \\ =&2\left|(1+a_1h)a_2h - a_3a_4h\right|,\end{aligned} \tag{3.15}$$

therefore, by inequality (3.14), we can obtain

$$\begin{aligned}&(1+a_1h)^2 - a_3^2h + a_2^2h^2 - a_4^2h + 2|(1+a_1h)a_2h - a_3a_4h| \\ \leq&(1+a_1h)^2 - a_3^2h + a_2^2h^2 - a_4^2h + 2(1+a_1h)|a_2|h + 2|a_3a_4|h \\ =&1 + [2(a_1+|a_2|) - (|a_3| - |a_4|)^2]h + (a_1+|a_2|)^2h^2 \\ <&1,\end{aligned} \tag{3.16}$$

namely

$$[2(a_1+|a_2|) - (|a_3| - |a_4|)^2]h + (a_1+|a_2|)^2h^2 < 0.$$

It is easy to find that the inequality (3.14) holds when $h < h_1$.

(ii) If $h \in S_2$, then $a_2^2h^2 - a_4^2h < 0, (1+a_1h)^2 - a_3^2h \geq 0$. So we have

$$\begin{aligned}&\left|[(1+a_1h)a_2h - a_3a_4h] + [a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}]\right| \\ &+ \left|[(1+a_1h)a_2h - a_3a_4h] - [a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}]\right| \\ =&2\left|a_2a_3h\sqrt{h} - (1+a_1h)a_4\sqrt{h}\right|.\end{aligned} \tag{3.17}$$

Hence, the inequality (3.14) can be written by

$$\begin{aligned}&(1+a_1h)^2-a_3^2h+a_4^2h-a_2^2h^2+2\left|a_2a_3h\sqrt{h}-(1+a_1h)a_4\sqrt{h}\right|\\ \leq&(1+a_1h)^2-a_3^2h+a_4^2h-a_2^2h^2+2|a_2a_3|h\sqrt{h}+2(1+a_1h)|a_4|\sqrt{h}\\ \leq&(1+a_1h)^2-a_3^2h+a_4^2h-a_2^2h^2+2|a_2a_3|h+2(1+a_1h)|a_4|\\ =&1+2|a_4|+[2a_1+a_4^2-a_3^2+2|a_2a_3|+2a_1|a_4|]h+(a_1^2-a_2^2)h^2\\ <&1,\end{aligned}\tag{3.18}$$

consequently

$$(a_1^2-a_2^2)h^2+k_0h+2|a_4|<0.$$

We can easily derive that $a_1^2-a_2^2>0, k_0^2-8(a_1^2-a_2^2)|a_4|>0$. So the inequality (3.14) holds when $h_3<h<h_2$.

(iii) If $h\in S_3$, then $a_2^2h^2-a_4^2h<0, (1+a_1h)^2-a_3^2h<0$. Hence (3.15) holds, (3.14) can be written as

$$\begin{aligned}&a_3^2h-(1+a_1h)^2+a_4^2h-a_2^2h^2+2|(1+a_1h)a_2h-a_3a_4h|\\ \leq&a_3^2h-(1+a_1h)^2+a_4^2h-a_2^2h^2+2(1+a_1h)|a_2|h+2|a_3a_4|h\\ =&-1-(a_1-|a_2|)^2h^2+[(|a_3|+|a_4|)^2-2a_1+2|a_2|]h\\ <&1,\end{aligned}\tag{3.19}$$

that is

$$(a_1-|a_2|)^2h^2+k_1h+2>0,$$

so inequality (3.14) holds when $h<h_4$ or $h>h_5$.

(iv) If $h\in S_4$, then $a_2^2h^2-a_4^2h\geq 0, (1+a_1h)^2-a_3^2h<0$. Hence (3.17) holds, (3.14) can be written as

$$\begin{aligned}&a_3^2h-(1+a_1h)^2+a_2^2h^2-a_4^2h+2\left|a_2a_3h\sqrt{h}-(1+a_1h)a_4\sqrt{h}\right|\\ \leq&a_3^2h-(1+a_1h)^2+a_2^2h^2-a_4^2h+2|a_2a_3|h+2(1+a_1h)|a_4|\\ =&-1+(a_2^2-a_1^2)^2h^2+[a_3^2-a_4^2-2a_1+2|a_2a_3|+2a_1|a_4|]h+2|a_4|\\ <&1,\end{aligned}\tag{3.20}$$

that is

$$(a_1^2-a_2^2)^2h^2+k_2h+2-2|a_4|>0,$$

so inequality (3.14) holds when $h<h_6$ or $h>h_7$.

Therefore, the proof is completed.

Remark 3.6 In Theorem 3.5, if anyone of the following inequalities holds

$$k_1^2-8(a_1-|a_2|)^2<0, k_2^2-8(a_1^2-a_2^2)(1-|a_4|)<0,$$

we can let the corresponding stepsize h equals 1.

4 Conclusion

In this paper, we discuss the mean-square stability and T-stability of Euler-Maruyama method for linear SEPCA of retarded type. It may be worthwhile to remark that the paper makes a meaningful exploratory for T-stability of numerical method. We believe that this topic will be gained more and more attention by scientists and engineers. T-stability of numerical methods for SEPCA of advanced type will be considered in the further work.

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References

[1] Cooke K L, Wiener J. Retarded differential equations with piecewise constant delays[J]. J. Math. Anal. Appl., 1984, 99(1): 265–297.

[2] Akhmet M U. Almost periodic solutions of differential equations with piecewise constant argument of generalized type[J]. Nonlinear Anal.: HS, 2008, 2(2): 456–467.

[3] Liu M Z, Song M H, Yang Z W. Stability of Runge-Kutta methods in the numerical solution of equation $u'(t) = au(t) + a_0u([t])$[J]. J. Comput. Appl. Math., 2004, 166(2): 361–370.

[4] Wang Q, Zhu Q Y, Liu M Z. Stability and oscillations of numerical solutions for differential equations with piecewise continuous arguments of alternately advanced and retarded type[J]. J. Comput. Appl. Math., 2011, 235(5): 1542–1552.

[5] Wiener J. Generalized solutions of functional differential equations[M]. Singapore: World Scientific, 1993.

[6] Tudor C, Tudor M. On approximation of solutions for stochastic delay equations[J]. Stud. Cerc. Mat., 1987, 39: 265–274.

[7] Tudor M. Approximation schemes for stochastic equations with hereditary argument[J]. Stud. Cerc. Mat., 1992, 44: 73–85.

[8] Cao W R, Liu M Z, Fan Z C. MS-stability of the Euler-Maruyama method for stochastic differential dalay equations[J]. Appl. Math. Comput., 2004, 159(1): 127–135.

[9] Buckwar E, Shardlow T. Weak approximation of stochastic differential delay equations[J]. IMA J. Numer. Anal., 2005, 25(1): 57–86.

[10] Rathinasamy A, Balachandran K. Mean-square stability of semi-implicit Euler method for linear stochastic differential equations with multiple delays and Markovian switching[J]. Appl. Math. Comput., 2008, 206(2): 968–979.

[11] Hu P, Huang C M. Stability of stochastic θ-methods for stochastic delay integro-differential equations[J]. Int. J. Comput. Math., 2011, 88(7): 1417–1429.

[12] Xiao Y, Song M H, Liu M Z. Convergence and stability of the semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equation[J]. Int. J. Numer. Model., 2011, 8(2): 214–225.

[13] Wang W Q, Chen Y P. Mean-square stability of semi-implicit Euler method for nonlinear neutral stochastic delay differential equations[J]. Appl. Numer. Math., 2011, 61(5): 696–701.

[14] Saito Y, Mitsui T. T-stability of numerical schemes for stochastic differential equations[J]. World Sci. Ser. Appl. Annl., 1993, 2: 333–344.

[15] Burrage K, Burrage P, Mitsui T. Numerical solutions of stochastic differential equations-implementation and stability issues[J]. J. Comput. Appl. Math., 2000, 125(1-2): 171–182.

[16] Burrage K, Tian T H. The composite Euler method for stiff stochastic differential equations[J]. J. Comput. Appl. Math., 2000, 131(1-2): 407–426.

[17] Cao W R. T-stability of the semi-implicit Euler method for delay differential equations with multiplicative noise[J]. Appl. Math. Comput., 2010, 216(3): 999–1006.

[18] Rathinasamy A, Balachandran K. T-stability of the split-step θ-methods for linear stochastic delay integro-differential equations[J]. Nonlinear Anal.: HS, 2011, 5(4): 639–646.

[19] Yang Z W, Liu M Z, Song M H. Stability of Runge-Kutta methods in the numerical solution of equation $u'(t) = au(t) + a_0u([t]) + a_1u([t-1])$[J]. Appl. Math. Comput., 2005, 162(1): 37–50.

[20] Dai H Y, Liu M Z. Mean-square stability of stochastic differential equations with piecewise continuous aiguments[J]. Heilongjiang Univ. J. Nat. Sci., 2008, 25(5): 625–629.

[21] Wu F K, Mao X R, Chen K. The Cox-Ingersoll-Ross model with delay and strong convergence of its Euler-Maruyama approximate solutions[J]. Appl. Numer. Math., 2009, 59(10): 2641–2658.

[22] Jiang F, Shen Y, Wu F K. A note on order of convergence of numerical method for neutral stochastic functional differential equations[J]. Commun. Nonlinear Sci. Numer. Simul., 2012, 17(3): 1194–1200.

[23] Liu W, Mao X R. Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations[J]. Appl. Math. Comput., 2013, 223(15): 389–400.

滞后型分段连续随机微分方程的稳定性

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摘要: 本文研究了滞后型分段连续随机微分方程的解析稳定性和数值稳定性问题. 首先, 利用伊藤公式等方法获得了解析解均方稳定的条件, 其次, 对于包括均方稳定和T-稳定在内的Euler-Maruyama方法的数值稳定性问题, 运用不等式技术和随机分析方法获得了一些新的结果, 证明了在一定条件下, Euler-Maruyama方法既是均方稳定又是T-稳定的, 推广了随机延迟微分方程的数值稳定性结论.

关键词: 随机延迟微分方程; 分段连续项; Euler-Maruyama方法; 均方稳定性; T-稳定性

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