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A NEW FAMILY OF ELEMENTS IN THE STABLE HOMOTOPY GROUPS OF SPHERES

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Abstract: In this paper, we study the non-triviality of the elements in the stable homotopy groups of spheres. Using the May spectral sequence, the authors show that there exists a new product in the E_2-term of the Adams spectral sequence, which converges to a family of homotopy elements with order p and higher filtration in the stable homotopy groups of spheres.

Keywords: stable homotopy groups of spheres; Toda-Smith spectrum; sphere spectrum; Adams spectral sequence; May spectral sequence

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1 Introduction

To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence(ASS).

Let A be the mod p Steenrod algebra, and S be the sphere spectrum localized at an odd prime p. For connected finite type spectra X, Y, there exists the ASS $\{E_r^{s,t}, d_r\}$ such that

(1) $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is the differential;

(2) $E_2^{s,t} \cong \mathrm{Ext}_A^{s,t}(H^*(X), H^*(Y)) \Rightarrow [\Sigma^{t-s}Y, X]_p$, where $E_2^{s,t}$ is the cohomology of A. When X is sphere spectrum S, Toda-Smith spectrum $V(n)(n = 1, 2, 3)$, respectively, $(\pi_{t-s}(X))_p$ is the stable homotopy groups of S, $V(n)$. So, for computing the stable homotopy groups of spheres with the ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$.

From [1], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has the $\mathbb{Z}_p$-base consisting of

$$a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p), h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p, \mathbb{Z}_p)$$

for all $i \geqslant 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has the $\mathbb{Z}_p$-base consisting of α_2, a_0^2, $a_0h_i(i > 0)$, $g_i(i \geqslant 0)$, $k_i(i \geqslant 0)$, $b_i(i \geqslant 0)$ and $h_ih_j(j \geqslant i+2, i \geqslant 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively. From [2, P.110,

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Table 8.1], the $\mathbb{Z}_p$-base of $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has been completely listed and there is a generator $\tilde{\gamma}_t \in \mathrm{Ext}_A^{t,tp^2q+(t-1)pq+(t-2)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)$ which is described in [3].

Our main theorems of this paper are as follows.

Theorem 1.1 Let $p \geqslant 11$, $3 \leqslant t < p-5$, then

$$\tilde{\gamma}_t\ h_0 b_1^5 \in \mathrm{Ext}_A^{t+11,(t+5)p^2q+(t-1)(p+1)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a non-trivial element in $\pi_*(S)$.

Based on a new homotopy element in $\pi_*(V(2))$, the above homotopy element in $\pi_*(S)$ will be constructed .

For the reader's convenience, let us firstly give some preliminaries on Toda-Smith spectrum $V(n)$.

The $\mathbb{Z}_p$ cohomology group of Toda-Smith spectrum $V(n)$ is $H^*V(n) \cong E[Q_0, Q_1, \cdots, Q_n] \cong Q(2^{n+1})$, where $Q_i (i \geqslant 0)$ is the Milnor's elements of Steenrod algebra A, and E[ ] is the exterior algebra. From [4], when $n = 1, 2, 3$ and $p > 2n$, we know that $V(n)$ is realized, and there exists a cofibre sequence ($V(-1) = S$):

$$\Sigma^{2(p^n-1)} V(n-1) \xrightarrow{\alpha_n} V(n-1) \xrightarrow{i_n} V(n) \xrightarrow{j_n} \Sigma^{2p^n-1} V(n-1),$$

where $\alpha_n (n = 0, 1, 2, 3)$ are p, α, β, γ, respectively. The cofibre sequence can induce a short exact sequence of $\mathbb{Z}_p$ cohomology groups. Thus, we get the following long exact sequence of Ext groups:

$$\cdots \xrightarrow{(j_n)_*} \mathrm{Ext}_A^{s-1,t-(2p^n-1)}(H^*V(n-1), \mathbb{Z}_p) \xrightarrow{(\alpha_n)_*} \mathrm{Ext}_A^{s,t}(H^*V(n-1), \mathbb{Z}_p)$$

$$\xrightarrow{(i_n)_*} \mathrm{Ext}_A^{s,t}(H^*V(n), \mathbb{Z}_p) \xrightarrow{(j_n)_*} \mathrm{Ext}_A^{s,t-(2p^n-1)}(H^*V(n-1), \mathbb{Z}_p) \xrightarrow{(\alpha_n)_*} \cdots.$$

The following theorem is a key step to prove Theorem 1.1.

Theorem 1.2 Let $p \geqslant 11$, then $h_0 b_1^5 \in \mathrm{Ext}_A^{11,5p^2q+q}(H^*V(2), \mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a non-trivial element in $\pi_* V(2)$.

It is very difficult to determine the stable homotopy groups of spheres. So far, not so many nontrivial elements in the stable homotopy groups of spheres were detected. See, for example [1, 5, 6].

The detection of the element $\tilde{\gamma}_t\ h_0 b_1^5$ is parallel to that of the element $\tilde{\gamma}_t\ h_0 b_1^2$ given in [7]. Actually, our results are more complicated, especially to Proposition 3.3 and Proposition 3.4.

This paper is organized as follows: after giving some preliminaries on the May spectral sequence (MSS) in Section 2, the proofs of the main theorems will be given in Section 3.

2 Some Preliminaries on the May Spectral Sequence

The most successful tool for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the MSS. From [8, Theorem 3.2.5], there exists the MSS $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$. The E_1-term and differential of the MSS are

$$E_1^{*,*,*} = E(h_{i,j}|i>0, j\geqslant 0)\otimes P(b_{i,j}|i>0, j\geqslant 0)\otimes P(a_i|i\geqslant 0),$$

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}, r\geqslant 1,$$

where E is the exterior algebra, P is the polynomial algebra and

$$h_{i,j}\in E_1^{1,2(p^i-1)p^j,2i-1},\ b_{i,j}\in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)},\ a_i\in E_1^{1,2p^i-1,2i+1}.$$

Lemma 2.1 (see [9]) Let $t=(c_np^n+c_{n-1}p^{n-1}+\cdots+c_1p+c_0)q+c_{-1}$, $c_i\in\mathbb{Z}$, and $p-1\geqslant c_n\geqslant c_{n-1}\geqslant\cdots\geqslant c_1\geqslant c_0\geqslant c_{-1}\geqslant 0$, then the number of $h_{n+1-i,i}$ in the generator of $E_1^{c_n,t,*}$ will be (c_i-c_{i-1}) $(0\leqslant i\leqslant n)$.

Corollary 2.2 (see [9]) If $p>a\geqslant b\geqslant c\geqslant d\geqslant 0$, then the number of $h_{1,2}$, $h_{2,1}$ and $h_{3,0}$ in the generator of $E_1^{a,ap^2q+bpq+cq+d,*}$ will be $(a-b)$, $(b-c)$ and $(c-d)$, respectively.

Corollary 2.3 (see [9]) Let $t\geqslant 3$, then $E_1^{t,tp^2q+(t-1)pq+(t-2)q+t-3}=\mathbb{Z}_p\{h_{2,1}h_{1,2}h_{3,0}a_3^{t-3}\}$.

Lemma 2.4 (see [9]) Let

$$t=(c_np^n+c_{n-1}p^{n-1}+\cdots+c_1p+c_0)q+c_{-1},$$

$c_i\in\mathbb{Z}_p(-1\leqslant i\leqslant n)$, for $c_i<c_{i-1}$, $0\leqslant i\leqslant n-1$, then $E_1^{c_n,t,*}=0$.

Lemma 2.5 (see [9]) Let $u>0$, $p>c_2,c_1,c_0,c_{-1}\geqslant 0$ and $c_2-c_{-1}\geqslant 4$, there don't exist u factors in the generator of $E_1^{u,c_2p^2q+c_1pq+c_0q+c_{-1}}$.

3 The Convergence of $\widetilde{\gamma}_t\, h_0b_1^5$ in the Adams Spectral Sequence

Let P be the subalgebra of A generated by the reduced power operations $P^i(i>0)$, then we have the following results.

Proposition 3.1 (see [9]) $\mathrm{Ext}_A^{s,t}(H^*V(3),\mathbb{Z}_p)\cong\mathrm{Ext}_P^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$, $t-s<2p^4-1$.

Corollary 3.2 Let $s\geqslant 2$, then

$$\mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(3),\mathbb{Z}_p)\cong\mathrm{Ext}_P^{s+11,5p^2q+q+s-1}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Proposition 3.3 Let $3\leqslant t<p-5, p\geqslant 11$, then

$$0\neq\widetilde{\gamma}_t\, h_0b_1^5\in\mathrm{Ext}_A^{t+11,(t+5)p^2q+(t-1)(p+1)q+t-3}(\mathbb{Z}_p,\mathbb{Z}_p).$$

The generators of $E_1^{s,t,u}$ and their first, second degrees satisfying $t<p^3q$ are listed in Table 1.

Table 1: The generators and degrees

$h_{1,0}$	$h_{1,1}$	$h_{2,0}$	$h_{2,1}$	$h_{1,2}$	$h_{3,0}$	$b_{1,0}$
$(1,q)$	$(1,pq)$	$(1,(p+1)q)$	$(1,p(p+1)q)$	$(1,p^2q)$	$(1,(p^2+p+1)q)$	$(2,pq)$

$b_{1,1}$	$b_{2,0}$	a_0	a_1	a_2	a_3
$(2,p^2q)$	$(2,(p^2+p)q)$	$(1,1)$	$(1,q+1)$	$(1,(p+1)q+1)$	$(1,(p^2+p+1)q+1)$

To compare the degrees, h_0, $b_1 \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ are represented by $h_{1,0} \in E_1^{1,q,*}$, $b_{1,1} \in E_1^{2,p^2q,*}$ in the MSS. From Corollary 2.3, we conclude that $\widetilde{\gamma}_t \in \mathrm{Ext}_A^{t,tp^2q+(t-1)pq+(t-2)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)$ is represented by $h_{2,1}h_{1,2}h_{3,0}a_3^{t-3} \in E_1^{t,tp^2q+(t-1)pq+(t-2)q+t-3,*}$ $(t \geqslant 3)$ in the MSS.

Thus, $\widetilde{\gamma}_t\ h_0b_1^5$ is represented by $h_{1,0}b_{1,1}^5h_{2,1}h_{1,2}h_{3,0}a_3^{t-3} \in E_1^{t+11,(t+5)p^2q+(t-1)pq+(t-1)q+t-3,*}$ in the MSS. If we want to prove that $0 \neq \widetilde{\gamma}_t\ h_0b_1^5 \in \mathrm{Ext}_A^{t+11,(t+5)p^2q+(t-1)(p+1)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)$, we must prove that $E_1^{t+10,(t+5)p^2q+(t-1)(p+1)q+t-3,*} = 0$. For any $\sigma \in E_1^{t+10,(t+5)p^2q+(t-1)(p+1)q+t-3,*}$, we have the following discussions.

Case 1 When $t \geqslant 6$, from Lemma 2.5, the number of the factors in σ will be $t+9$, $t+8$, $t+7$, $t+6$ or $t+5$.

Subcase 1.1 If σ has $t+9$ factors, there exists a factor $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible forms will be $\sigma = \sigma_{1.1}b_{1,0}$, $\sigma = \sigma_{1.2}b_{1,1}$, $\sigma = \sigma_{1.3}b_{2,0}$, where

$$\sigma_{1.1} \in E_1^{t+8,(t+5)p^2q+(t-2)pq+(t-1)q+t-3,*}, \sigma_{1.2} \in E_1^{t+8,(t+4)p^2q+(t-1)pq+(t-1)q+t-3,*},$$
$$\sigma_{1.3} \in E_1^{t+8,(t+4)p^2q+(t-2)pq+(t-1)q+t-3,*}.$$

By Lemma 2.5, the number of the factors in $\sigma_{1.1}$ is $t+7, t+6$ or $t+5$, thus the number of the factors in σ will be $t+8, t+7$ or $t+6$. It is in contradiction with that σ has $t+9$ factors, so $\sigma_{1.1} = 0$. Similarly, we conclude that $\sigma_{1.2} = 0$, $\sigma_{1.3} = 0$, so $\sigma = 0$.

Subcase 1.2 If σ has $t+8$ factors, there exist two factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible forms will be $\sigma = \sigma_{2.1}b_{1,0}^2$, $\sigma = \sigma_{2.2}b_{1,1}^2$, $\sigma = \sigma_{2.3}b_{2,0}^2$, $\sigma = \sigma_{2.4}b_{1,0}b_{1,1}$, $\sigma = \sigma_{2.5}b_{1,0}b_{2,0}$, $\sigma = \sigma_{2.6}b_{1,1}b_{2,0}$, where

$$\sigma_{2.1} \in E_1^{t+6,(t+5)p^2q+(t-3)pq+(t-1)q+t-3,*}, \sigma_{2.2} \in E_1^{t+6,(t+3)p^2q+(t-1)pq+(t-1)q+t-3,*},$$
$$\sigma_{2.3} \in E_1^{t+6,(t+3)p^2q+(t-3)pq+(t-1)q+t-3,*}, \sigma_{2.4} \in E_1^{t+6,(t+4)p^2q+(t-2)pq+(t-1)q+t-3,*},$$
$$\sigma_{2.5} \in E_1^{t+6,(t+4)p^2q+(t-3)pq+(t-1)q+t-3,*}, \sigma_{2.6} \in E_1^{t+6,(t+3)p^2q+(t-2)pq+(t-1)q+t-3,*}.$$

By the similar argument in Subcase1.1, we can get that $\sigma_{2.i} = 0 (i = 1, 2 \cdots 6)$, thus $\sigma = 0$.

Subcase 1.3 If σ has $t+7$ factors, there exist three factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible forms will be $\sigma = \sigma_{3.1}b_{1,0}^3$, $\sigma = \sigma_{3.2}b_{1,0}^2b_{1,1}$, $\sigma = \sigma_{3.3}b_{1,0}^2b_{2,0}$, $\sigma = \sigma_{3.4}b_{1,1}^3$, $\sigma = \sigma_{3.5}b_{2,0}^3$, $\sigma = \sigma_{3.6}b_{1,1}^2b_{2,0}$, $\sigma = \sigma_{3.7}b_{1,1}^2b_{1,0}$, $\sigma = \sigma_{3.8}b_{2,0}^2b_{1,0}$, $\sigma = \sigma_{3.9}b_{2,0}^2b_{1,1}$, $\sigma = \sigma_{3.10}b_{1,0}b_{1,1}b_{2,0}$, where

$$\sigma_{3.1} \in E_1^{t+4,(t+5)p^2q+(t-4)pq+(t-1)q+t-3,*}, \sigma_{3.2} \in E_1^{t+4,(t+4)p^2q+(t-3)pq+(t-1)q+t-3,*},$$
$$\sigma_{3.3} \in E_1^{t+4,(t+4)p^2q+(t-4)pq+(t-1)q+t-3,*}, \sigma_{3.4} \in E_1^{t+4,(t+2)p^2q+(t-1)pq+(t-1)q+t-3,*},$$
$$\sigma_{3.5} \in E_1^{t+4,(t+2)p^2q+(t-4)pq+(t-1)q+t-3,*}, \sigma_{3.6} \in E_1^{t+4,(t+2)p^2q+(t-2)pq+(t-1)q+t-3,*},$$
$$\sigma_{3.7} \in E_1^{t+4,(t+3)p^2q+(t-2)pq+(t-1)q+t-3,*}, \sigma_{3.8} \in E_1^{t+4,(t+3)p^2q+(t-4)pq+(t-1)q+t-3,*},$$
$$\sigma_{3.9} \in E_1^{t+4,(t+2)p^2q+(t-3)pq+(t-1)q+t-3,*}, \sigma_{3.10} \in E_1^{t+4,(t+3)p^2q+(t-3)pq+(t-1)q+t-3,*}.$$

It is obvious that $\sigma_{3.1} = 0$. By the similar argument in Subcase1.1, we can get that $\sigma_{3.i} = 0$ $(i = 4, 5, \cdots, 10)$. From Lemma 2.4, note that $t-3 < t-1$, $t-4 < t-1$, thus the remainder are all zero. Therefore, we can get $\sigma = 0$.

Subcase 1.4 If σ has $t+6$ factors, there exist four factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, $\sigma = \sigma' b_{1,0}^x b_{1,1}^y b_{2,0}^z$, where $x+y+z=4, x,y,z \geqslant 0$ and $\sigma' \in E_1^{t+2,T}$, $T = (t+5-y-z)p^2q + (t-1-x-z)pq + (t-1)q + (t-3)$. If $x \geqslant 2$, we have that $y+z<3$ and $t+5-y-z > t+2$. It is obvious that $\sigma' = 0$. Thus, the possible nontrivial forms will be $\sigma = \sigma_{4.1} b_{1,1}^4$, $\sigma = \sigma_{4.2} b_{2,0}^4$, $\sigma = \sigma_{4.3} b_{1,1}^3 b_{1,0}$, $\sigma = \sigma_{4.4} b_{1,1}^3 b_{2,0}$, $\sigma = \sigma_{4.5} b_{2,0}^3 b_{1,0}$, $\sigma = \sigma_{4.6} b_{2,0}^3 b_{1,1}$, $\sigma = \sigma_{4.7} b_{1,1}^2 b_{2,0}^2$, $\sigma = \sigma_{4.8} b_{1,0} b_{1,1}^2 b_{2,0}$, $\sigma = \sigma_{4.9} b_{1,0} b_{1,1} b_{2,0}^2$, where the first degrees of $\sigma_{4.i} (i = 1,2,3,\cdots,9)$ are all $t+2$ and the second degrees of them are listed in Table 2 ($M = t-1$, and $N = t-3$).

Table 2: The factors and second degrees

$\sigma_{4.i}$	the second degree	$\sigma_{4.i}$	the second degree
$\sigma_{4.1}$	$(t+1)p^2q + (t-1)pq + Mq + N$	$\sigma_{4.2}$	$(t+1)p^2q + (t-5)pq + Mq + N$
$\sigma_{4.3}$	$(t+2)p^2q + (t-2)pq + Mq + N$	$\sigma_{4.4}$	$(t+1)p^2q + (t-2)pq + Mq + N$
$\sigma_{4.5}$	$(t+2)p^2q + (t-5)pq + Mq + N$	$\sigma_{4.6}$	$(t+1)p^2q + (t-4)pq + Mq + N$
$\sigma_{4.7}$	$(t+1)p^2q + (t-3)pq + Mq + N$	$\sigma_{4.8}$	$(t+2)p^2q + (t-3)pq + Mq + N$
$\sigma_{4.9}$	$(t+2)p^2q + (t-4)pq + Mq + N$		

By the argument similar to Subcase 1.3, we get that $\sigma_{4.i} = 0 (i = 1, 2 \cdots 9)$, thus $\sigma = 0$.

Subcase 1.5 If σ has $t+5$ factors, there exist five factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{5.1} b_{1,1}^3 b_{2,0}^2$, $\sigma = \sigma_{5.2} b_{1,1}^2 b_{2,0}^3$, $\sigma = \sigma_{5.3} b_{2,0}^4 b_{1,1}$, $\sigma = \sigma_{5.4} b_{1,1}^4 b_{2,0}$, $\sigma = \sigma_{5.5} b_{2,0}^5$, $\sigma = \sigma_{5.6} b_{1,1}^5$, where the first degrees of $\sigma_{5.i} (i = 1,2,3,\cdots,6)$ are all t and the second degrees of them are listed in Table 3 ($M = t-1$, and $N = t-3$).

Table 3: The factors and second degrees

$\sigma_{5.i}$	the second degree	$\sigma_{5.i}$	the second degree
$\sigma_{5.1}$	$tp^2q + (t-3)pq + Mq + N$	$\sigma_{5.2}$	$tp^2q + (t-4)pq + Mq + N$
$\sigma_{5.3}$	$tp^2q + (t-5)pq + Mq + N$	$\sigma_{5.4}$	$tp^2q + (t-2)pq + Mq + N$
$\sigma_{5.5}$	$tp^2q + (t-6)pq + Mq + N$	$\sigma_{5.6}$	$tp^2q + (t-1)pq + Mq + N$

Similarly to $\sigma_{3.2}$, we can get that $\sigma_{5.i} = 0 (i = 1, 2 \cdots 5)$. As for $\sigma_{5.6}$, from the Corollary 2.2, there exist two factors $h_{3,0}$, so $\sigma_{5.6} = 0$. Thus, we can get $\sigma = 0$.

Case 2 When $t = 5$, $E_1^{t+10,(t+5)p^2q+(t-1)(p+1)q+t-3,*} = E_1^{15,10p^2q+4pq+3q+q+2,*}$, the generator contains $(q+2)$ factors a_i. Therefore, the first degree $\geqslant q+2 > 15$, it's a contradiction. So, we get $\sigma = 0$. When $t = 4$, $t = 3$, the proofs are the similar to $t = 5$. Summarize the above Case 1 and Case 2, $E_1^{t+10,(t+5)p^2q+(t-1)(p+1)q+t-3,*} = 0$. That is

$$0 \neq \widetilde{\gamma}_t\, h_0 b_1^5 \in \mathrm{Ext}_A^{t+11,(t+5)p^2q+(t-1)(p+1)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)(3 \leqslant t < p-5).$$

Proposition 3.4 Let $r \geqslant 2$, $3 \leqslant t < p-5$, then

$$\mathrm{Ext}_A^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*}(\mathbb{Z}_p, \mathbb{Z}_p) = 0.$$

It is sufficient if we can show that $E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = 0$.

Case 1 If $r > 6$, $t+11-r < t+5$, so we have

$$E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = 0.$$

Case 2 If $r = 6$, then

$$E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = E_1^{t+5,(t+5)p^2q+(t-1)pq+(t-1)q+t-8,*}.$$

Subcase 2.1 When $t \geqslant 8$, from the Corollary 2.2, there exist six factors $h_{1,2}$, so $\sigma = 0$.

Subcase 2.2 When $t = 7$, $E_1^{t+5,(t+5)p^2q+(t-1)pq+(t-1)q+t-8,*} = E_1^{12,12p^2q+6pq+5q+q-1,*}$. The generator contains $(q-1)$ factors a_i. Therefore, the first degree$\geqslant q-1 > 12$, it's a contradiction. So, the generator is impossible to exist.

Subcase 2.3 When $3 \leqslant t \leqslant 6$, by the similar argument in Subcase 2.2, the generator is impossible to exist.

Case 3 If $r = 5$, then

$$E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = E_1^{t+6,(t+5)p^2q+(t-1)pq+(t-1)q+t-7,*}.$$

Subcase 3.1 When $t \geqslant 7$, from the Lemma 2.5, we know that the generator contains $t+5$ factors, one of which must be the factor $b_{i,j}$. Thus, the possible nontrivial forms will be $\sigma = \sigma_{3.1}b_{1,1}$, $\sigma = \sigma_{3.2}b_{2,0}$, where $\sigma_{3.1} \in E_1^{t+4,(t+4)p^2q+(t-1)pq+(t-1)q+t-7,*}$, $\sigma_{3.2} \in E_1^{t+4,(t+4)p^2q+(t-2)pq+(t-1)q+t-7,*}$. By the similar argument in Subcase2.1, we can get that $\sigma_{3.1} = 0$, $\sigma_{3.2} = 0$.

Subcase 3.2 When $3 \leqslant t \leqslant 6$, by the similar argument in Subcase 2.2, the generator is impossible to exist.

Case 4 If $r = 4$, then

$$E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = E_1^{t+7,(t+5)p^2q+(t-1)pq+(t-1)q+t-6,*}.$$

Subcase 4.1 When $t \geqslant 6$, from Lemma 2.5, we know that the number of the factors in σ will be $t+5$ or $t+6$.

Subcase 4.1.1 If σ contains $t+6$ factors, then there exists a factor $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{4.1}b_{1,0}, \sigma = \sigma_{4.2}b_{1,1}$, $\sigma = \sigma_{4.3}b_{2,0}$, where

$$\sigma_{4.1} \in E_1^{t+5,(t+5)p^2q+(t-2)pq+(t-1)q+t-6,*}, \sigma_{4.2} \in E_1^{t+5,(t+4)p^2q+(t-1)pq+(t-1)q+t-6,*},$$
$$\sigma_{4.3} \in E_1^{t+5,(t+4)p^2q+(t-2)pq+(t-1)q+t-6,*}.$$

By the similar argument in Subcase 2.1, we know that $\sigma_{4.1} = 0$. As for $\sigma_{4.2}$, from Lemma 2.5, $\sigma_{4.2}$ must contain $t+4$ factors, thus $\sigma = \sigma_{4.2}b_{1,1}$ contains $t+5$ factors. It is a contradiction with that σ contains $t+6$ factors, then $\sigma_{4.2} = 0$. Similarly, we can get $\sigma_{4.3} = 0$.

Subcase 4.1.2 If σ contains $t+5$ factors, then there exist two factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{4.4}b_{1,1}^2$, $\sigma = \sigma_{4.5}b_{2,0}^2$,

$\sigma = \sigma_{4.6}b_{1,1}b_{2,0}$, where

$$\sigma_{4.4} \in E_1^{t+3,(t+3)p^2q+(t-1)pq+(t-1)q+t-6,*}, \sigma_{4.5} \in E_1^{t+3,(t+3)p^2q+(t-3)pq+(t-1)q+t-6,*},$$
$$\sigma_{4.6} \in E_1^{t+3,(t+3)p^2q+(t-2)pq+(t-1)q+t-6,*}.$$

By the similar argument in Subcase 2.1, we can get $\sigma_{4.4} = 0$. As for $\sigma_{4.5}$, from the Lemma 2.4 and $t-3 < t-1$, so $\sigma_{4.5} = 0$. Similarly, we can get $\sigma_{4.6} = 0$.

Subcase 4.2 When $3 \leqslant t \leqslant 5$, by the similar argument in Subcase 2.2, we know that the generator is impossible to exist. Thus, we have $\sigma = 0$.

Case 5 If $r = 3$, then

$$E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = E_1^{t+8,(t+5)p^2q+(t-1)pq+(t-1)q+t-5,*}.$$

Subcase 5.1 When $t \geqslant 5$, from Lemma 2.5, the number of the factors in σ will be $t+5$, $t+6$ or $t+7$.

Subcase 5.1.1 If σ contains $t+7$ factors, then there exists a factor $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{5.1}b_{1,0}, \sigma = \sigma_{5.2}b_{1,1}$, $\sigma = \sigma_{5.3}b_{2,0}$, where

$$\sigma_{5.1} \in E_1^{t+6,(t+5)p^2q+(t-2)pq+(t-1)q+t-5,*}, \sigma_{5.2} \in E_1^{t+6,(t+4)p^2q+(t-1)pq+(t-1)q+t-5,*},$$
$$\sigma_{5.3} \in E_1^{t+6,(t+4)p^2q+(t-2)pq+(t-1)q+t-5,*}.$$

Similarly to $\sigma_{4.2}$, we can get $\sigma_{5.i} = 0 (i = 1, 2, 3)$.

Subcase 5.1.2 If σ contains $t+6$ factors, then there exist two factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{5.4}b_{1,1}^2$, $\sigma = \sigma_{5.5}b_{2,0}^2$, $\sigma = \sigma_{5.6}b_{1,0}b_{1,1}$, $\sigma = \sigma_{5.7}b_{1,0}b_{2,0}$, $\sigma = \sigma_{5.8}b_{1,1}b_{2,0}$, where

$$\sigma_{5.4} \in E_1^{t+4,(t+3)p^2q+(t-1)pq+(t-1)q+t-5,*}, \sigma_{5.5} \in E_1^{t+4,(t+3)p^2q+(t-3)pq+(t-1)q+t-5,*},$$
$$\sigma_{5.6} \in E_1^{t+4,(t+4)p^2q+(t-2)pq+(t-1)q+t-5,*}, \sigma_{5.7} \in E_1^{t+4,(t+4)p^2q+(t-3)pq+(t-1)q+t-5,*},$$
$$\sigma_{5.8} \in E_1^{t+4,(t+3)p^2q+(t-2)pq+(t-1)q+t-5,*}.$$

Similarly to $\sigma_{4.2}$, we can get that $\sigma_{5.4} = 0$, $\sigma_{5.5} = 0$, $\sigma_{5.8} = 0$. Similarly to $\sigma_{4.5}$, we can get that $\sigma_{5.6} = 0$, $\sigma_{5.7} = 0$.

Subcase 5.1.3 If σ contains $t+5$ factors, then there exist three factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{5.9}b_{1,1}^3$, $\sigma = \sigma_{5.10}b_{2,0}^3$, $\sigma = \sigma_{5.11}b_{1,1}^2b_{2,0}$, $\sigma = \sigma_{5.12}b_{1,1}b_{2,0}^2$, where

$$\sigma_{5.9} \in E_1^{t+2,(t+2)p^2q+(t-1)pq+(t-1)q+t-5,*}, \sigma_{5.10} \in E_1^{t+2,(t+2)p^2q+(t-4)pq+(t-1)q+t-5,*},$$
$$\sigma_{5.11} \in E_1^{t+2,(t+2)p^2q+(t-2)pq+(t-1)q+t-5,*}, \sigma_{5.12} \in E_1^{t+2,(t+2)p^2q+(t-3)pq+(t-1)q+t-5,*},$$

By the similar argument in Subcase2.1, we can know that $\sigma_{5.9} = 0$. Similarly to $\sigma_{4.5}$, we can get that $\sigma_{5.10} = 0$, $\sigma_{5.11} = 0$, $\sigma_{5.12} = 0$.

Subcase 5.2 When $t = 4$ or $t = 3$, by the similar argument in Subcase2.2, we know that the generator is impossible to exist. Thus, in this case, we can get $\sigma = 0$.

Case 6 If $r = 2$, then

$$E_1^{t+11-r,(t+5)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = E_1^{t+9,(t+5)p^2q+(t-1)pq+(t-1)q+t-4,*}.$$

Subcase 6.1 When $t \geqslant 5$, from Lemma 2.5, the number of the factors in σ will be $t+5$, $t+6$, $t+7$ or $t+8$.

Subcase 6.1.1 If σ contains $t+8$ factors, then there exists a factor $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{6.1}b_{1,0}, \sigma = \sigma_{6.2}b_{1,1}$, $\sigma = \sigma_{6.3}b_{2,0}$, where

$$\begin{aligned}&\sigma_{6.1} \in E_1^{t+7,(t+5)p^2q+(t-2)pq+(t-1)q+t-4,*}, \sigma_{6.2} \in E_1^{t+7,(t+4)p^2q+(t-1)pq+(t-1)q+t-4,*},\\ &\sigma_{6.3} \in E_1^{t+7,(t+4)p^2q+(t-2)pq+(t-1)q+t-4,*}.\end{aligned}$$

Similarly to $\sigma_{4.2}$, we can get $\sigma_{6.1} = 0$, $\sigma_{6.2} = 0$, $\sigma_{6.3} = 0$.

Subcase 6.1.2 If σ contains $t+7$ factors, then there exist two factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{6.4}b_{1,0}^2$, $\sigma = \sigma_{6.5}b_{1,1}^2$, $\sigma = \sigma_{6.6}b_{2,0}^2$, $\sigma = \sigma_{6.7}b_{1,0}b_{1,1}$, $\sigma = \sigma_{6.8}b_{1,0}b_{2,0}$, $\sigma = \sigma_{6.9}b_{1,1}b_{2,0}$, where

$$\begin{aligned}&\sigma_{6.4} \in E_1^{t+5,(t+5)p^2q+(t-3)pq+(t-1)q+t-4,*}, \sigma_{6.5} \in E_1^{t+5,(t+3)p^2q+(t-1)pq+(t-1)q+t-4,*},\\ &\sigma_{6.6} \in E_1^{t+5,(t+3)p^2q+(t-3)pq+(t-1)q+t-4,*}, \sigma_{6.7} \in E_1^{t+5,(t+4)p^2q+(t-2)pq+(t-1)q+t-4,*},\\ &\sigma_{6.8} \in E_1^{t+5,(t+4)p^2q+(t-3)pq+(t-1)q+t-4,*}, \sigma_{6.9} \in E_1^{t+5,(t+3)p^2q+(t-2)pq+(t-1)q+t-4,*}.\end{aligned}$$

Similarly to $\sigma_{4.5}$, we can get that $\sigma_{6.4} = 0$. Similarly to $\sigma_{4.2}$, we can get that $\sigma_{6.i} = 0$ $(i = 5, 6 \cdots 9)$.

Subcase 6.1.3 If σ contains $t+6$ factors, then there exist three factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{6.10}b_{1,1}^3$, $\sigma = \sigma_{6.11}b_{2,0}^3$, $\sigma = \sigma_{6.12}b_{1,1}^2b_{1,0}$, $\sigma = \sigma_{6.13}b_{1,1}^2b_{2,0}$, $\sigma = \sigma_{6.14}b_{1,0}b_{2,0}^2$, $\sigma = \sigma_{6.15}b_{1,1}b_{2,0}^2$, $\sigma = \sigma_{6.16}b_{1,0}b_{1,1}b_{2,0}$, where the first degrees of $\sigma_{6.i}(i = 10, 11, \cdots, 16)$ are all $t+3$ and the second degrees of them are listed in Table 4 ($M = t-1$, and $N = t-4$).

Table 4: The factors and second degrees

$\sigma_{6.i}$	the second degree	$\sigma_{6.i}$	the second degree
$\sigma_{6.10}$	$(t+2)p^2q+(t-1)pq+Mq+N$	$\sigma_{6.11}$	$(t+2)p^2q+(t-4)pq+Mq+N$
$\sigma_{6.12}$	$(t+3)p^2q+(t-2)pq+Mq+N$	$\sigma_{6.13}$	$(t+2)p^2q+(t-2)pq+Mq+N$
$\sigma_{6.14}$	$(t+3)p^2q+(t-4)pq+Mq+N$	$\sigma_{6.15}$	$(t+2)p^2q+(t-3)pq+Mq+N$
$\sigma_{6.16}$	$(t+3)p^2q+(t-3)pq+Mq+N$		

Similarly to $\sigma_{4.2}$, we can get that $\sigma_{6.10} = 0$, $\sigma_{6.11} = 0$, $\sigma_{6.13} = 0$. Similarly to $\sigma_{4.5}$, we can get that $\sigma_{6.12} = 0$, $\sigma_{6.14} = 0$, $\sigma_{6.15} = 0$, $\sigma_{6.16} = 0$.

Subcase 6.1.4 If σ contains $t+5$ factors, then there exist four factors $b_{i,j}(b_{1,0}, b_{1,1}, b_{2,0})$. Due to the commutativity, the possible nontrivial forms will be $\sigma = \sigma_{6.17}b_{1,1}^4$, $\sigma = \sigma_{6.18}b_{2,0}^4$,

$\sigma = \sigma_{6.19}b_{1,1}^3 b_{2,0}$, $\sigma = \sigma_{6.20}b_{1,1}b_{2,0}^3$, $\sigma = \sigma_{6.21}b_{1,2}^2 b_{2,0}^2$, where the first degrees of $\sigma_{6.i}(i = 17, 18, \cdots, 21)$ are all $t+1$ and the second degrees of them are listed in Table 5 ($M = t-1$, and $N = t-4$).

Table 5: The factors and second degrees

$\sigma_{6.i}$	the second degree	$\sigma_{6.i}$	the second degree
$\sigma_{6.17}$	$(t+1)p^2q + (t-1)pq + Mq + N$	$\sigma_{6.18}$	$(t+1)p^2q + (t-5)pq + Mq + N$
$\sigma_{6.19}$	$(t+1)p^2q + (t-2)pq + Mq + N$	$\sigma_{6.20}$	$(t+1)p^2q + (t-4)pq + Mq + N$
$\sigma_{6.21}$	$(t+1)p^2q + (t-3)pq + Mq + N$		

Similarly to Subcase2.1, we can get that $\sigma_{6.17} = 0$. Similarly to $\sigma_{3.2}$, we can get that $\sigma_{6.i} = 0(i = 18, 19, 20, 21)$.

Subcase 6.2 When $t = 3, t = 4$, by the similar argument in Subcase 2.2, we know that the generator is impossible to exist. Thus, we have $\sigma = 0$.

Therefore, we can get that $E_1^{t+11-r,(5+t)p^2q+(t-1)pq+(t-1)q+t-r-2,*} = 0$.

That is $\mathrm{Ext}_A^{t+11-r,(5+t)p^2q+(t-1)pq+(t-1)q+t-r-2,*}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$.

Proposition 3.5 Let $s \geqslant 2$, $p \geqslant 11$, then $\mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(2), \mathbb{Z}_p) = 0$.

From Corollary 3.2, we have

$$\mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(3), \mathbb{Z}_p) \cong \mathrm{Ext}_P^{s+11,5p^2q+q+s-1}(\mathbb{Z}_p, \mathbb{Z}_p).$$

From [4, Lemma 2.2], we know that the rank of $\mathrm{Ext}_P^{s+11,5p^2q+q+s-1}(\mathbb{Z}_p, \mathbb{Z}_p)$ is less than or equal to that of $[P(b_j^i) \otimes H^{*,*}(U(L))]^{s+11,5p^2q+q+s-1}$, and $[P(b_j^i) \otimes H^{*,*}(U(L))]^{s,t}$ is the E_2-term of the MSS, where $P()$ is the polynomial algebra. Up to the total degree $t - s < (p^3 + 3p^2 + 2p + 1)q - 4$, $H^{s,t}(U(L))$ is multiplicative by the following cohomology classes

$$\begin{aligned}
h_i &= \{R_1^i\}, & g_i &= \{R_2^i R_1^i\}, & k_i &= \{R_2^i R_1^{i+1}\}(i \geqslant 0),\\
l_1 &= \{R_3^0 R_2^0 R_1^0\}, & l_2 &= \{R_2^1 R_2^0 R_1^1\}, & l_3 &= \{R_3^0 R_1^2 R_1^0\},\\
l_4 &= \{R_3^0 R_2^1 R_1^2\}, & l_5 &= \{R_3^1 R_2^1 R_1^1\}, & l_6 &= \{R_2^2 R_2^1 R_1^2\},\\
m_1 &= \{R_3^0 R_2^1 R_2^0 R_1^1\}, m_2 = \{R_4^0 R_3^0 R_2^0 R_1^0\}, & & & &\\
m_3 &= \{R_3^1 R_2^1 R_2^0 R_1^1\}, m_4 = \{R_2^2 R_3^0 R_1^2 R_1^0\}. & & & &
\end{aligned}$$

Moreover, we have additively

$$\begin{aligned}
&H^{*,*}(U(L)) \cong \{1, l_4, h_3\} \otimes \{1, h_0, h_1, g_0, k_0, k_0 h_0\}\\
&+\{h_2, h_2 h_0, g_1, l_1, l_2, l_1 h_1, k_1, l_3, k_1 h_1, l_1 h_2, m_1, m_1 h_0, g_2, g_2 h_0, l_5, m_2, m_3, l_6, m_4\},
\end{aligned}$$

and the bidegrees of R_j^i, b_j^i are $(1, 2(p^{i+j} - p^i))$, $(2, 2(p^{i+j-1} - p^{i+1}))$, respectively. In the MSS, b_1^0 converges to $b_0 \in \mathrm{Ext}_P^{2,pq}(\mathbb{Z}_p, \mathbb{Z}_p)$, and the total degree of b_1^0 is $|b_1^0| = pq - 2$. The generators whose total degrees are less than or equal to $5p^2q+q-12$ in $[P(b_j^i) \otimes H^{*,*}(U(L))]^{s,t}$ and the total degrees $|\lambda|$ mod $pq - 2$ are listed in Table 6 ($t = 1, 2, 3, 4, 5, t' = 1, 2, 3, 4$).

Table 6: The generators λ and total degrees $|\lambda|$ mod $pq-2$

λ	$(b_1^1)^t, (b_2^0)^{t'}$	$\otimes$	$h_0,$	$h_1,$	$g_0,$	$k_0,$	$k_0h_0,$	$h_2,$	$h_2h_0,$	$g_1,$
$\lvert\lambda\rvert$	$tq, t'(q+2)$	$+$	$q-1,$	$1,$	$2q,$	$q+2,$	$2q+1,$	$q+1,$	$2q,$	$q+4,$

$l_1,$	$l_2,$	$l_1h_1,$	$k_1,$	$l_3,$	$k_1h_1,$	$l_1h_2,$	$m_1,$	m_1h_0
$4q+3,$	$2q+5,$	$4q+4,$	$2q+4,$	$4q+3,$	$2q+5,$	$5q+4,$	$4q+8,$	$5q+7$

Let x be a generator of $\mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(2),\mathbb{Z}_p)$, then we have

$$(i_3)_*(x) \in \mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(3),\mathbb{Z}_p).$$

The total degree of $(i_3)_*(x)$ is $5p^2q+q+s-1-(s+11)=5p^2q+q-12 \equiv 6q-2 \pmod{pq-2}$. From the above Table, we know that the generator λ with total degree mod $pq-2$ being equal to $6q-2$ in $[P(b_j^i)\otimes H^{*,*}(U(L))]^{s,t}$ doesn't exist. So, we can get that $(i_3)_*(x)=0$. Consider the following exact sequence:

$$\cdots \xrightarrow{(j_3)_*} \mathrm{Ext}_A^{s+10,5p^2q+q+s-1-(2p^3-1)}(H^*V(2),\mathbb{Z}_p) \xrightarrow{(\alpha_3)_*} \mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(2),\mathbb{Z}_p)$$

$$\xrightarrow{(i_3)_*} \mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(3),\mathbb{Z}_p) \xrightarrow{(j_3)_*} \cdots,$$

there exists an element $x_1 \in \mathrm{Ext}_A^{s+10,5p^2q+q+s-1-(2p^3-1)}(H^*V(2),\mathbb{Z}_p)$ satisfying $(\alpha_3)_*(x_1)=x$. The total degree of $(i_3)_*(x_1)$ is $5p^2q+q+s-1-(2p^3-1)-(s+10)\equiv 4q-6$ (mod $pq-2$). From the above Table, we know that

$$0=(i_3)_*(x_1)\in \mathrm{Ext}_A^{s+10,5p^2q+q+s-1-(2p^3-1)}(H^*V(3),\mathbb{Z}_p).$$

Using the exactness repeatedly, there exists an element $x_k \in \mathrm{Ext}_A^{s+11-k,5p^2q+q+s-1-k(2p^3-1)}$ $(H^*V(2),\mathbb{Z}_p)$ satisfying $(\alpha_3)_*(x_k)=x_{k-1}$. But the total degree of $(i_3)_*(x_k)$ mod $pq-2$ is different from that in the above table, so we know that

$$0=(i_3)_*(x_k)\in \mathrm{Ext}_A^{s+11-k,5p^2q+q+s-1-k(2p^3-1)}(H^*V(3),\mathbb{Z}_p).$$

Let $k=5$, then

$$x_5 \in \mathrm{Ext}_A^{s+6,5p^2q+q+s-1-5(2p^3-1)}(H^*V(2),\mathbb{Z}_p)=\mathrm{Ext}_A^{s+6,-10p^2+q+s+4}(H^*V(2),\mathbb{Z}_p)=0.$$

Therefore, we have $x=\underbrace{(\alpha_3)_*\cdots(\alpha_3)_*}_{5}(x_5)=0$, that is

$$\mathrm{Ext}_A^{s+11,5p^2q+q+s-1}(H^*V(2),\mathbb{Z}_p)=0 (s\geqslant 2, p\geqslant 11).$$

Proposition 3.6 Let $r\geqslant 2$, $p\geqslant 11$, then

$$\mathrm{Ext}_A^{11-r,5p^2q+q-r+1}(H^*V(2),\mathbb{Z}_p)=0.$$

The proposition is evident for $r \geqslant 11$. Thus, we need only to consider the case of $2 \leqslant r < 11$.

For any $y \in \mathrm{Ext}_A^{11-r,5p^2q+q-r+1}(H^*V(2),\mathbb{Z}_p)$, we can know $0 < q-r+1 < q$ from $2 \leqslant r < 11$. Since $Sdim((i_3)_*(y)) = 5p^2q+q-r+1 = q-r+1 \neq 0(\mathrm{mod}\ q)$ and $\mathrm{Ext}_P^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p) = 0$ $(t \neq 0 \bmod q)$, from Proposition 3.1, we can get that

$$0 = (i_3)_*(y) \in \mathrm{Ext}_A^{11-r,5p^2q+q-r+1}(H^*V(3),\mathbb{Z}_p).$$

According to the exactness, there exists an element

$$y_1 \in \mathrm{Ext}_A^{10-r,5p^2q+q-r+1-(2p^3-1)}(H^*V(2),\mathbb{Z}_p)$$

such that $(\alpha_3)_*(y_1) = y$, and

$$Sdim((i_3)_*(y_1)) = 5p^2q+q-r+1-(2p^3-1) = 4p^2q-pq-r = q-r \neq 0\ (\mathrm{mod}\ q),$$

so $0 = (i_3)_*(y_1) \in \mathrm{Ext}_A^{10-r,5p^2q+q-r+1-(2p^3-1)}(H^*V(3),\mathbb{Z}_p)$.

Similarly, there exists an element $y_k \in \mathrm{Ext}_A^{11-k-r,5p^2q+q-r+1-k(2p^3-1)}(H^*V(2),\mathbb{Z}_p)$ satisfying $(\alpha_3)_*(y_k) = y_{k-1}$, and $Sdim((i_3)_*(y_k)) \neq 0$ (mod q). Let $k=5$, then

$$y_5 \in \mathrm{Ext}_A^{6-r,5p^2q+q-r+1-5(2p^3-1)}(H^*V(2),\mathbb{Z}_p) = \mathrm{Ext}_A^{6-r,-10p^2+q-r+6}(H^*V(2),\mathbb{Z}_p) = 0.$$

Thus, we get that $y = \underbrace{(\alpha_3)_*\cdots(\alpha_3)_*}_{5}(y_5) = 0$, that is

$$\mathrm{Ext}_A^{11-r,5p^2q+q-r+1}(H^*V(2),\mathbb{Z}_p) = 0\ \ (r \geqslant 2, p \geqslant 11).$$

The Proof of Theorem 1.2 First, we consider the ASS with E_2-term:

$$\mathrm{Ext}_A^{s,t}(H^*V(2),\mathbb{Z}_p) \Rightarrow \pi_{t-s}V(2),$$

and its differential is $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

From Proposition 3.5,

$$E_2^{r+11,5p^2q+q+r-1} = \mathrm{Ext}_A^{r+11,5p^2q+q+r-1}(H^*V(2),\mathbb{Z}_p),$$

we can get that $E_r^{r+11,5p^2q+q+r-1} = 0(r \geqslant 2)$. Let $h_0b_1^5$ be the image of $h_0b_1^5 \in \mathrm{Ext}_A^{11,5p^2q+q}(\mathbb{Z}_p,\mathbb{Z}_p)$ under the map

$$(i_2)_*(i_1)_*(i_0)_* : \mathrm{Ext}_A^{11,5p^2q+q}(\mathbb{Z}_p,\mathbb{Z}_p) \to \mathrm{Ext}_A^{11,5p^2q+q}(H^*V(2),\mathbb{Z}_p),$$

then $d_r(h_0b_1^5) \in E_r^{r+11,5p^2q+q+r-1} = 0(r \geqslant 2)$. Furthermore, we can get that

$$h_0b_1^5 \in E_2^{11,5p^2q+q} = \mathrm{Ext}_A^{11,5p^2q+q}(H^*V(2),\mathbb{Z}_p)$$

is a permanent cycle in the ASS. Moreover, from Proposition 3.6,

$$E_2^{11-r,5p^2q+q-r+1} = \mathrm{Ext}_A^{11-r,5p^2q+q-r+1}(H^*V(2),\mathbb{Z}_p) = 0(r \geqslant 2),$$

we have that $E_r^{11-r,5p^2q+q-r+1} = 0 (r \geqslant 2)$. So, $h_0b_1^5$ is impossible to be the d_r-boundary in the ASS, and $h_0b_1^5 \in \mathrm{Ext}_A^{11,5p^2q+q}(H^*V(2), \mathbb{Z}_p)$ converges to a nontrivial element in $\pi_*V(2)$.

The Proof of Theorem 1.1 From Theorem 1.2, we know that there exists a nontrivial element f in $\pi_*V(2)$, which is represented by $h_0b_1^5 \in \mathrm{Ext}_A^{11,5p^2q+q}(H^*V(2), \mathbb{Z}_p)$, where $h_0b_1^5$ denotes the image of $h_0b_1^5 \in \mathrm{Ext}_A^{11,5p^2q+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ under the homomorphism

$$(i_2)_*(i_1)_*(i_0)_* : \mathrm{Ext}_A^{11,5p^2q+q}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathrm{Ext}_A^{11,5p^2q+q}(H^*V(2), \mathbb{Z}_p).$$

Consider the following composition of maps

$$\tilde{f} : \Sigma^{5p^2q+q-11}S \xrightarrow{f} V(2) \xrightarrow{\gamma^t} \Sigma^{t(p^2+p+1)q}V(2) \xrightarrow{j_0j_1j_2} \Sigma^{-t(p^2+p+1)q+(p+1)q+q+3}S,$$

the composed map $\tilde{f} = j_0j_1j_2\gamma^t f$ is represented by

$$(j_0j_1j_2)_*(\gamma^t)_*(i_2i_1i_0)_*(h_0b_1^5) \in \mathrm{Ext}_A^{11+t,(5+t)p^2q+(t-1)(p+1)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p).$$

From [3], we have known that $\gamma_t = j_0j_1j_2 \in \pi_*(S)$ is represented by $\widetilde{\gamma_t}$ in the ASS. By the knowledge of Yoneda products, we know that the following composition:

$$\mathrm{Ext}_A^{0,0}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{(i_2i_1i_0)_*} \mathrm{Ext}_A^{0,0}(H^*V(2), \mathbb{Z}_p) \xrightarrow{(\gamma^t)_*} \mathrm{Ext}_A^{t,t(p^2+p+1)q+t}(H^*V(2), \mathbb{Z}_p)$$

$$\xrightarrow{(j_0j_1j_2)_*} \mathrm{Ext}_A^{t,tp^2q+(t-1)pq+(t-2)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a homomorphism which is multiplied by $\widetilde{\gamma_t}$. Hence, $\widetilde{f} \in \pi_*(S)$ is represented by

$$\widetilde{\gamma_t}\ h_0b_1^5 \in \mathrm{Ext}_A^{11+t,(5+t)p^2q+(t-1)(p+1)q+t-3}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS.

Moreover, from the Proposition 3.4 we know that $\widetilde{\gamma_t}\ h_0b_1^5$ can't be hit by the differentials in the ASS, then we get that $\widetilde{\gamma_t}\ h_0b_1^5$ converges to a nontrivial element $\widetilde{f}$ in $\pi_*(S)$.

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球面稳定同伦群中的一族新元素

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摘要: 本文研究了球面稳定同伦群中元素的非平凡性. 利用May谱序列, 证明了在Adams谱序列E_2项中存在乘积元素收敛到球面稳定同伦群的一族阶为p的非零元, 此非零元具有更高维数的滤子.

关键词: 稳定同伦群; Toda-Smith谱; 球谱; Adams谱序列; May谱序列

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