

HOPF BIFURCATION OF A DELAYED PREDATOR-PREY MODEL WITH STAGE STRUCTURE FOR THE PREDATOR

KE Yu-sheng, LI Bi-wen, CHEN Bo-shan

(*College of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China*)

Abstract: This paper investigate the stability and Hopf bifurcation of a delayed predator-prey model with Holling type-III response function and stage-structure for the predator. By applying the norm form of differential dynamic system and the center manifold theorem, we determined the stability of the interior equilibrium the direction of the period solution. Obviously, the result of Tian and Xu is promoted in this paper.

Keywords: stage structure; Hopf bifurcation; stability; Holling type-III

2010 MR Subject Classification: 34C25; 34D05; 34F10; 34K18

Document code: A

Article ID: 0255-7797(2015)02-0252-15

1 Introduction

The predator-prey systems play an important and fundamental role among the relationships between the biological population. There exists a strong interest during the last few decades to study predator-prey system in population dynamics by mathematicians and biologists. It has turned out that many researches in mathematical models contribute to understanding the interactions of predator and prey in these systems. Then various dynamics behaviors are achieved, such as steady states, oscillations, bifurcations and chaos depending on the model parameters, see [1–5].

Recently, the models governed by the ordinary and partial differential equations involving delays attracted people's attention and were successfully applied to many problems in many research areas (such as biology, population dynamics, neural networks, feedback controlled mechanical systems and so on, see [6–7]). In delayed differential equations, period solutions can arise through the Hopf bifurcation. Several methods for analyzing the nature of Hopf bifurcation had been proposed by many authors. Thereinto, the center manifold theory is one of the rigorous mathematical tools to study bifurcations of delay differential equations, see [13–20].

From the view of biology, many species have a life history that take them through an immature and mature stage, and the predator in the first stage often has no ability to attack

* **Received date:** 2013-11-14

Accepted date: 2014-02-28

Foundation item: Supported by National Natural Science Foundation of China (10671182).

Biography: Ke Yusheng (1987–), male, born at Huangshi, Hubei, master, major in differential equations and control theory.

prey and reproduce. Many author have investigated the models with stage-structure and obtained varies dynamics behaviors, see [4–12]. In [4], a predator-prey model with stage-structure and Holling type II response is considered as follow:

$$\begin{cases} \dot{x}(t) = x(t) \left(r - ax(t) - \frac{a_1 y_2(t)}{1+mx(t)} \right), \\ \dot{y}_1(t) = \frac{a_2 x(t) y_2(t)}{1+mx(t)} - r_1 y_1(t) - D y_1(t), \\ \dot{y}_2(t) = D y_1(t) - r_2 y_2(t), \end{cases} \tag{1.1}$$

where $x(t)$, $y_1(t)$, $y_2(t)$ represent the densities of prey, immature and mature predator, respectively, $ax(t)$, r_1 , r_2 represent the death rate of prey, immature and mature predator, respectively. D is the convert rate from immature predator to mature predator. $\frac{ax}{1+mx}$ is response function of the mature predators’ capture. The stability of model (1.1) was studied by Xu et al. [4] in detail.

Consider the time-delay due to the pregnant of the mature predator for advanced animals influences the stability of the equilibrium in delayed differential equations, we establish a delayed Holling Type III response predator-prey system as follows:

$$\begin{cases} \dot{x}(t) = x(t) \left(r - ax(t) - \frac{a_1 x(t) y_2(t)}{1+mx^2(t)} \right), \\ \dot{y}_1(t) = \frac{a_2 x^2(t-\tau) y_2(t-\tau)}{1+mx^2(t-\tau)} - r_1 y_1(t) - D y_1(t), \\ \dot{y}_2(t) = D y_1(t) - r_2 y_2(t). \end{cases} \tag{1.2}$$

The rest of this paper is organized as follows: in the next section, by analyzing the corresponding characteristic equations, the local stability of each of the feasible equilibrium of system (1.2) is discussed and the existence of a Hopf bifurcation at the coexistence equilibrium is established. The stability and the direction of periodic solutions bifurcating from Hopf bifurcations are investigated by using the normal form theory [23] and the center manifold theorem [24] in Section 3. The numerical simulations are carried out to support the theoretical analysis of the research in Section 4. Finally, a brief discussion is given in Section 5 .

2 Equilibrium Analysis

In this section, we analysis the local stability of each of feasible equilibria of system (1.2) and the existence of Hopf bifurcations at the coexistence equilibrium. Obviously, system (1.2) always has a trivial equilibrium $E_0(0, 0, 0)$ and a predator-extinction equilibrium $E_1(\frac{r}{a}, 0, 0)$. If the following condition holds:

$$H_1 : a_2 hr^2 > r_2(a^2 + mr^2)(D + r_1),$$

then system (1.2) has a unique coexistence equilibrium $E^*(x^*, y_1^*, y_2^*)$, where

$$x^* = \sqrt{\frac{r_2(D + r_1)}{a_2 D - mr_2(D + r_1)}}, \quad y_1^* = \frac{r_2}{D} y_2^*, \quad y_2^* = \frac{(r - ax^*)(1 + m(x^*)^2)}{a_1 x^*}. \tag{2.1}$$

The characteristic equation of system (1.2) at the trivial equilibrium $E_0(0, 0, 0)$ is of the form

$$(\lambda - r)(\lambda + D + r_1)(\lambda + r_2) = 0. \quad (2.2)$$

It is apparently to see that the trivial equilibrium $E_0(0, 0, 0)$ is always unstable.

The characteristic equation of system (1.2) at the trivial equilibrium $E_1(\frac{r}{a}, 0, 0)$ takes the form

$$(\lambda + r)(\lambda^2 + (r_2 + D + r_1)\lambda + r_2(D + r_1) - \frac{a_2 D(x_0)^2}{1 + m(x_0)^2} e^{-\lambda\tau}) = 0. \quad (2.3)$$

Clearly, (2.3) has a negative real root $\lambda = -r$, other roots are determined by the follow equation:

$$\lambda^2 + P_1\lambda + P_0 + Q_0 e^{-\lambda\tau} = 0, P_0 = r_2(D + r_1), P_1 = D + r_1 + r_2, Q_0 = -\frac{a_2 r^2 D}{a^2 + mr^2}.$$

Denote

$$f(\lambda) = \lambda^2 + (r_2 + h + r_1)\lambda + r_2(D + r_1) - \frac{a_2 D(x_0)^2}{1 + m(x_0)^2} e^{-\lambda\tau},$$

if (H1) holds, it is easy to show that, for λ real,

$$f(0) = -\frac{a_2 Dr^2 - r_2(a^2 + mr^2)(D + r_1)}{a^2 + mr^2} < 0, \lim_{\lambda \rightarrow \infty} f(\lambda) = \infty.$$

Hence, $f(\lambda) = 0$ has at least one positive real root. Therefore, if (H1) holds, the equilibrium $E_1(\frac{r}{a}, 0, 0)$ is unstable. If $a_2 Dr^2 < r_2(a^2 + mr^2)(D + r_1)$, it is readily seen from (2.3) that $E_1(\frac{r}{a}, 0, 0)$ is locally asymptotically stable when $\tau = 0$. In this case, it is easy to show that

$$\begin{aligned} P_1^2 - 2P_0 &= (D + r_1)^2 + r_2^2 > 0, \\ P_0^2 - Q_0^2 &= (r_2(r_1 + D) + \frac{a_2 Dr^2}{a^2 + mr^2})(\frac{r_2(r_1 + D)(a^2 + mr^2) - a_2 Dr^2}{a^2 + mr^2}) > 0. \end{aligned}$$

By Theorem 3.4.1 in Kuang [30], we see that, $E_1(\frac{r}{a}, 0, 0)$ is asymptotically stable for all $\tau > 0$.

The characteristic equation of system at the coexistence equilibrium $E^*(x^*, y_1^*, y_2^*)$ is of the form

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)e^{-\lambda\tau} = 0, \quad (2.4)$$

where

$$\begin{aligned} p_0 &= r_2(D + r_1)\left(\frac{2a_1 x^* y_2^*}{(1 + m(x^*)^2)^2} + 2ax^* - r\right), \\ p_1 &= r_2(D + r_1) + (D + r_1 + r_2)\left(\frac{2a_1 x^* y_2^*}{(1 + m(x^*)^2)^2} + 2ax^* - r\right), \\ p_2 &= D + r_1 + r_2 - r + \frac{2a_1 x^* y_2^*}{(1 + m(x^*)^2)^2} + 2ax^*, \\ q_0 &= \frac{r_2(D + r_1)(r - 2ax^*)}{1 + m(x^*)^2}, q_1 = -\frac{r_2(D + r_1)}{1 + m(x^*)^2}. \end{aligned}$$

When $\tau = 0$, (2.4) becomes $\lambda^3 + p_2\lambda^2 + (p_1 + q_1)\lambda + p_0 + q_0 = 0$. Clearly, by calculation we derive that

$$p_0 + q_0 = r_2(D + r_1)\left(\frac{3(r - 2ax^*)}{1 + m(x^*)^2} + 2ax^* - r\right),$$

$$p_1 + q_1 = (D + r_1 + r_2)\left(\frac{2a_1x^*y_2^*}{(1 + m(x^*)^2)^2} + 2ax^* - r\right) + r_2(D + r_1)\left(\frac{m(x^*)^2}{1 + m(x^*)^2}\right).$$

Hence, by the Routh-Hurwitz theorem, the equilibrium $E^*(x^*, y_1^*, y_2^*)$ is locally asymptotically stable when $\tau = 0$ if the following holds: $p_2(p_1 + q_1) - (p_0 + q_0) > 0$ and $E^*(x^*, y_1^*, y_2^*)$ is unstable if the inequality in (H2) is reversed.

If $i\omega (w > 0)$ is a solution of (2.4), separating real and imaginary parts, we have

$$-\omega^3 + p_1\omega = q_0 \sin \omega\tau - q_1\omega \cos \omega\tau, \quad p_2\omega^2 - p_0 = q_0 \cos \omega\tau + q_1\omega \sin \omega\tau. \tag{2.5}$$

Squaring and adding the two equations of (2.5), it follows that

$$\omega^6 + (p_2^2 - 2p_1)\omega^4 + (p_1^2 - 2p_0p_2 - q_1^2)\omega^2 + p_0^2 - q_0^2 = 0, \tag{2.6}$$

it is easy to show that

$$p_2^2 - 2p_1 = (D + r_1)^2 + r_2^2 + \left(\frac{2a_1x^*y_2^*}{(1 + m(x^*)^2)^2} + 2ax^* - r\right)^2,$$

$$p_1^2 - 2p_0p_2 - q_1^2 = r_2^2(D + r_1)^2\left(1 - \frac{1}{(1 + m(x^*)^2)^2}\right) + r_2^2\left(\frac{2a_1x^*y_2^*}{(1 + m(x^*)^2)^2} + 2ax^* - r\right)^2$$

$$+ (D + r_1)^2\left(\frac{2a_1x^*y_2^*}{(1 + m(x^*)^2)^2} + 2ax^* - r\right)^2,$$

$$p_0 - q_0 = r_2(D + r_1)\left(\frac{r - ax^*}{1 + m(x^*)^2} + 2ax^* - r\right).$$

Hence, if $p_0 > q_0$, (2.6) has no positive real roots. Accordingly, if (H2) and $p_0 > q_0$ holds, then the equilibrium $E^*(x^*, y_1^*, y_2^*)$ is locally asymptotically stable for all $\tau \geq 0$. If $p_0 < q_0$, then (2.6) has a unique positive root ω_0 , that is, (2.5) has a pair of purely imaginary roots of the form $\pm i\omega_0$. Denote

$$\tau_{0n} = \frac{1}{\omega_0} \arccos \frac{q_0(p_2\omega_0^2 - p_0) + q_1\omega_0(\omega_0^3 - p_1\omega_0)}{q_0^2 + q_1^2\omega_0^2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \dots \tag{2.7}$$

Noting that if (H2) holds, $E^*(x^*, y_1^*, y_2^*)$ is locally stable when $\tau = 0$, by the general theory on characteristic equations of delay differential equations from [21] (Theorem 3.4.1), $E^*(x^*, y_1^*, y_2^*)$ remains stable for $\tau < \tau_0$, where $\tau_0 = \tau_{00}$.

We claim that $\left.\frac{d(Re\lambda)}{d\tau}\right|_{\tau=\tau_0} > 0$, this will show that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [23] are then satisfied yielding a periodic solution. To this end, differentiating (2.4) with respect τ , it follows that

$$(3\lambda^2 + 2p_2\lambda + p_1)\frac{d\lambda}{d\tau} + q_1e^{-\lambda\tau}\frac{d\lambda}{d\tau} - \tau(q_1\lambda + q_0)e^{-\lambda\tau}\frac{d\lambda}{d\tau} = \lambda(q_1\lambda + q_0)e^{-\lambda\tau},$$

which yield

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{q_1}{\lambda(q_1\lambda + q_0)} - \frac{\tau}{\lambda}.$$

Hence, it can be calculated that

$$\begin{aligned} \operatorname{sgn} \left\{ \frac{d(\operatorname{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} &= \operatorname{sgn} \left\{ \operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega_0} \\ &= \operatorname{sgn} \left\{ -\frac{(p_1 - 3\omega_0^2)(\omega_0^2 - p_1) + 2p_2(p_0 - p_2\omega_0^2)}{(\omega_0^3 - p_1\omega_0)^2 + (p_0 - p_2\omega_0^2)^2} - \frac{q_1^2}{q_0^2 + q_1^2\omega_0^2} \right\}. \end{aligned} \quad (2.8)$$

We derive from (2.5) that $(\omega_0^3 - p_1\omega_0)^2 + (p_0 - p_2\omega_0^2)^2 = q_0^2 + q_1^2\omega_0^2$. Hence, it follows that

$$\operatorname{sgn} \left\{ \frac{d(\operatorname{Re}\lambda)}{d\tau} \right\}_{\lambda=i\omega_0} = \operatorname{sgn} \left\{ \frac{3\omega_0^4 + 2(p_2^2 - 2p_1)\omega_0^2 + p_1^2 - 2p_0p_2 - q_1^2}{q_0^2 + q_1^2\omega_0^2} \right\} > 0.$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0, \tau = \tau_0$.

We have the following theorem on stability and Hopf bifurcation.

Theorem 2.1 For system (1.2), we have the following:

- (i) The equilibrium $E_0(0, 0, 0)$ is always unstable;
- (ii) The predator-extinction equilibrium $E_1(\frac{r}{a}, 0, 0)$ is locally asymptotically stable if $a_2Dr^2 < r_2(D + r_1)(a^2 + mr^2)$; $E_1(\frac{r}{a}, 0, 0)$ is unstable if $a_2Dr^2 > r_2(D + r_1)(a^2 + mr^2)$;
- (iii) Let (H1) and $p_2(p_1 + q_1) > p_0 + q_0$ holds. If $p_0 > q_0$, then the coexistence equilibrium $E^*(x^*, y_1^*, y_2^*)$ is locally asymptotically stable for all $\tau \geq 0$; if $p_0 < q_0$, then there exist a positive number τ_0 such that $E^*(x^*, y_1^*, y_2^*)$ is locally asymptotically stable if $0 < \tau < \tau_0$ and is unstable if $\tau > \tau_0$. Further, system (1.2) undergoes a Hopf bifurcation at $E^*(x^*, y_1^*, y_2^*)$ when $\tau = \tau_0$.

3 Direction and Stability of Hopf Bifurcation

In the following part, we will investigate the direction of these Hopf bifurcations and stability of bifurcated periodic solutions arising through Hopf bifurcations based on the normal form theory and center manifold theorem introduced by Hassard et al. [24]. We have achieved the conditions under which a family of periodic solutions bifurcated from the positive equilibrium of system (1.2) when the delay crosses through the critical value τ_{0n} .

Which determine the direction of Hopf bifurcation and stability of bifurcated periodic solutions of system (1.2) at the critical value τ_{0n} :

$$g_{20} = 2\tau_0\bar{G}\hat{g}_{20}, \quad g_{11} = 2\tau_0\bar{G}\hat{g}_{11}, \quad g_{02} = 2\tau_0\bar{G}\hat{g}_{02}, \quad g_{21} = 2\tau_0\bar{G}\hat{g}_{21},$$

where

$$\begin{aligned}
 \hat{g}_{20} &= -(a + \frac{a_1 c_3}{c_2^3}) \rho_1^2 \bar{\rho}_1^* - \frac{2a_1 x^*}{c_2^2} \rho_1 \bar{\rho}_1^* + \frac{a_2 c_3}{c_2^3} e^{-2i\omega_0 \tau_0} \rho_1^2 \bar{\rho}_2^* + \frac{2a_2 x^*}{c_2^2} e^{-2i\omega_0 \tau_0} \rho_1 \bar{\rho}_2^*, \\
 \hat{g}_{11} &= -(a + \frac{a_1 c_3}{c_2^3}) \rho_1 \bar{\rho}_1 \bar{\rho}_1^* - \frac{a_1 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \bar{\rho}_1^* + \frac{a_2 c_3}{c_2^3} e^{-2i\omega_0 \tau_0} \rho_1^2 \bar{\rho}_2^* + \frac{a_2 x^*}{c_2^2} e^{-2i\omega_0 \tau_0} (\rho_1 + \bar{\rho}_1) \bar{\rho}_2^*, \\
 \hat{g}_{02} &= -a \bar{\rho}_1^2 \bar{\rho}_1^* - \frac{a_1 c_3}{c_2^3} \bar{\rho}_1^2 \bar{\rho}_1^* - \frac{2a_1 x^*}{c_2^2} \bar{\rho}_1 \bar{\rho}_1^* + \frac{a_2 c_3}{c_2^3} e^{-2i\omega_0 \tau_0} \bar{\rho}_1^2 \bar{\rho}_2^* + \frac{2a_2 x^*}{c_2^2} e^{-2i\omega_0 \tau_0} \bar{\rho}_1 \bar{\rho}_2^*, \\
 \hat{g}_{21} &= \frac{6mx^*(c_3 + c_2 y_2^*) \rho_1^2 \bar{\rho}_1}{c_2^4} (a_1 \bar{\rho}_1^* - a_2 \bar{\rho}_2^* e^{-i\omega_0 \tau_0}) + \frac{c_2 - 4mx^*}{c_2^3} (\rho_1^2 + 2\rho_1 \bar{\rho}_1) (a_2 \bar{\rho}_2^* e^{-i\omega_0 \tau_0} - a_1 \bar{\rho}_1^*) \\
 &\quad - \frac{a_1 x^* \bar{\rho}_1^*}{c_2^2} (W_{20}^{(3)}(0) \bar{\rho}_2 + W_{20}^{(3)}(0) + 2W_{11}^{(1)}(0) \rho_2 + 2W_{11}^{(3)}(0)) + \frac{a_2 x^* \bar{\rho}_2^*}{c_2^2} (W_{20}^{(3)}(-1) \bar{\rho}_1 e^{i\omega_0 \tau_0} \\
 &\quad + W_{20}^{(3)}(-1) e^{i\omega_0 \tau_0} + 2W_{11}^{(1)}(-1) \rho_1 e^{i\omega_0 \tau_0} + 2W_{11}^{(3)}(-1) e^{i\omega_0 \tau_0}) + \bar{\rho}_1 \bar{\rho}_1^* (W_{20}^{(1)}(0) \\
 &\quad + 2W_{11}^{(1)}(0)) (-a - \frac{a_1 c_3}{c_2^3}) + \bar{\rho}_1 \bar{\rho}_2^* (W_{20}^{(1)}(-1) + 2W_{11}^{(1)}(-1)) \frac{a_2 c_3}{c_2^3} e^{i\omega_0 \tau_0},
 \end{aligned}$$

where the terms \bar{G} , $W_{11}^1(0)$, $W_{11}^3(0)$, $W_{11}^1(-1)$, $W_{11}^3(-1)$, $W_{20}^1(0)$, $W_{20}^3(0)$, $W_{20}^1(-1)$, $W_{20}^3(-1)$ are calculated in Appendix. Now, we exhibit these coefficients we can evaluate the following values

$$\begin{aligned}
 C_1(0) &= \frac{i}{2\omega_0 \tau_0} (g_{11} g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\
 \mu_2 &= -\frac{\text{Re}C_1(0)}{\text{Re}\lambda'(\tau_0)}, \quad \beta_2 = 2\text{Re}\{C_1(0)\}, \quad T_2 = \frac{\text{Im}C_1(0) + \mu_2 \text{Im}\lambda'(\tau_0)}{\omega_0 \tau_0}.
 \end{aligned}$$

Now using the quantities above, the properties of the Hopf bifurcation is determined by the following results.

Theorem 3.1 For system (1.2) , we have the following:

(i) If $\mu_2 > 0$ ($\mu_2 < 0$) , then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ ($\tau < \tau_0$) .

(ii) β_2 determines the stability of the bifurcating periodic solutions: the bifurcating solutions on the center manifold are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

(iii) And T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0$ ($T_2 < 0$). From the discussion in Section 2, we know that $\text{Re}(\lambda'(\tau_0)) > 0$, therefore we have the following result.

Theorem 3.2 The direction of the Hopf bifurcation of system (1.2) at the origin when $\tau = \tau_j$ ($j = 0, 1, 2, \dots$) is supercritical (subcritical) and the bifurcating periodic solutions on the center manifold are stable (unstable) if $\text{Re}\{C_1(0)\} < 0$ (> 0); particularly, when $\tau = \tau_0$, the stability of the bifurcating periodic solutions is the same as that on the center manifold.

4 Numerical simulations

We now give an example to illustrate algorithm for determining the existence of Hopf bifurcation in Section 2 and the direction and stability of Hopf bifurcation in Section 3. Let

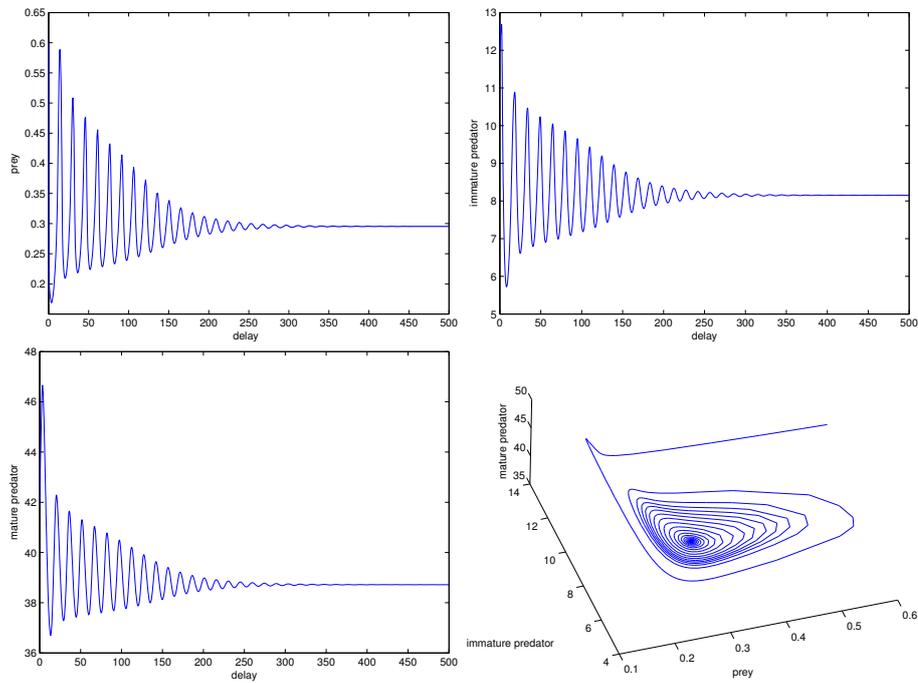


Figure 1: When $\tau = 2.2 < \tau_0$ and with the initial condition $x^0 = 0.6$, $y_1^0 = 12$, $y_2^0 = 42$, that show the positive equilibrium point E^* is locally asymptotically.

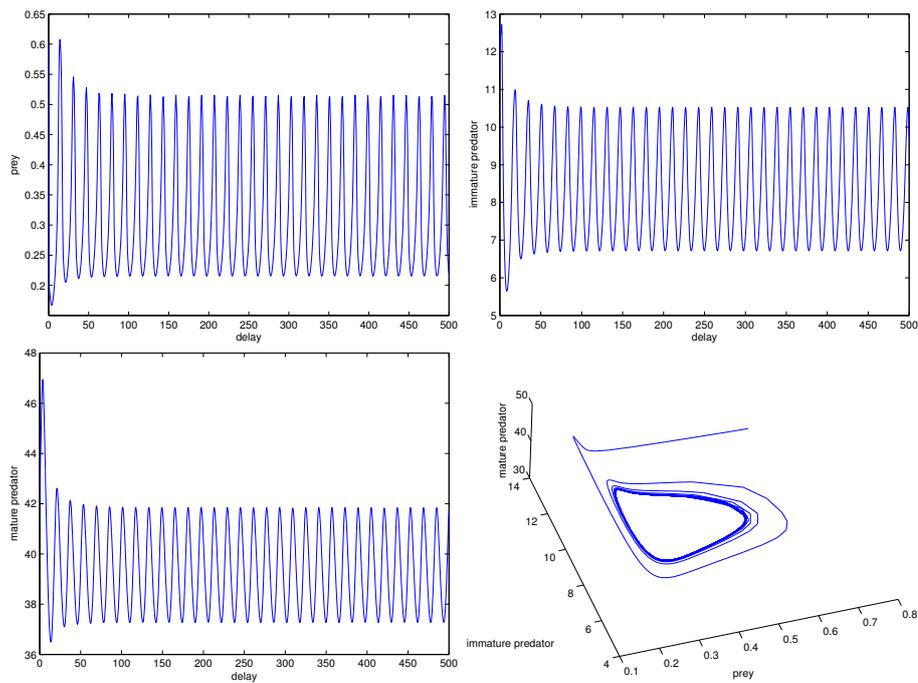


Figure 2: When $\tau = 2.4 > \tau_0$, with the same initial condition above that shows the bifurcating periodic solutions from the positive equilibrium point E^* .

$r = 25, r_1 = \frac{2}{19}, r_2 = \frac{2}{19}, a = 16, a_1 = 4, a_2 = \frac{7}{2}, m = 16, D = \frac{1}{2}$, it is easy to show that

$$a_2 D r^2 - r_2 (a^2 + m r^2) (D + r_1) = 440.3206 > 0.$$

Hence, system (1.2) has a unique coexistence equilibrium

$$E^* \left(\sqrt{\frac{92}{1055}}, \frac{3325}{92} \sqrt{\frac{92}{1055}} - \frac{2128}{1055}, \frac{63175}{368} \sqrt{\frac{92}{1055}} - \frac{10108}{1055} \right).$$

By calculation, we have

$$\begin{aligned} p_0 - q_0 &\approx -0.3258 < 0, & p_2(p_1 + q_1) - (p_0 + q_0) &\approx 1.6236 > 0, \\ \tau &\approx 2.3940, & C_1(0) &\approx -14.3078 + 2.4964i, \\ \mu_2 &\approx 1.6612, & \beta_2 &\approx -28.6155, & T_2 &\approx 24.9693. \end{aligned}$$

By Theorem 3.1, we obtain the result that the Hopf bifurcation of system (1.2) occurring at $\tau_0 = 2.3940$ is subcritical and the bifurcating periodic solution exist when τ cross τ_0 to the left and the bifurcating periodic solution is stable, and the equilibrium E^* is locally asymptotically stable if $0 < \tau < \tau_0$ as is illustrated by computer simulations in Fig.1 and is unstable if $\tau > \tau_0$ as is illustrated by computer simulations in Fig. 2. System (4.1) will show the complicated dynamical behaviors, here, we choose the initial condition $x^0 = 0.6, y_1^0 = 12, y_2^0 = 42$ in our simulations.

5 Conclusions

In this paper, we conclude the stability properties of this system based on the improvement of a stage structure predator-prey system proposed in [7]. By deriving the equation describing the flow on the center manifold, we determine the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions. It shows that under some conditions, the time delay due to the gestation of the mature predator may destabilize the coexistence equilibrium of the system and cause the population to fluctuate.

In addition, with the development of the society, the ordinary differential equations of the ecosystem is no longer satisfy mankind's significance. Correspondingly, the economic profit is becoming a important factor for governments, merchants and even every citizen. Therefore, it is necessary that we should study the differential-algebraic system centered on economic profit and commercial harvesting, and investigates the efforts of economic profit on the dynamics of the predator-prey system.

References

[1] Brauer F, Soudack A C. Stability regions and transition phenomena for harvested predator-prey system[J]. J. Math. Biol., 1979, 7: 319-337.
 [2] Brauer F, Soudack A C. Stability regions in predator-prey systems with constant rate prey harvesting[J]. J. Math. Biol., 1979, 8: 55-71.

- [3] Takeuchi Y. Global dynamical properties of lotka-volterra system[M]. Singapore: World Scientific, 1996.
- [4] Tian X H, Xu R. Global dynamics of a predator-prey model with Holling type II functional response[J]. *Nonlinear Anal. Modelling and Control*, 2011, 16(2): 242–253.
- [5] Liu B, Zhang Y, Chen L. Dynamics complexities in a Lotka-Volterra predator-prey model concerning impulsive control strategy[J]. *Dyn. Contin. Discrete Impuls. Syst: Ser. B Appl. Algorithms*, 2007, 15(2): 517–531.
- [6] Xu R. Global stability and Hopf bifurcation of a predator-prey model with stage structure and delayed predator response[J]. *Nonlinear Dynamics*, 2012, 67: 1683–1693.
- [7] Qu Y, Wei J J. Bifurcation analysis in a time-delay model for predator-prey growth with stage-structure[J]. *Nonlinear Dynamics*, 2007, 49(1): 285–294.
- [8] Liu S Q, Zhang J H. Coexistence and stability of predator-prey model with Beddington-DeAngelis functional response and stage structure[J]. *J. Math. Anal. Appl.*, 2008, 342(1): 446–460.
- [9] Cai L M, Song X Y. Permanence and stability of a predator-prey system with stage structure for predator[J]. *J. computational and Applied Math.*, 2007, 201(2): 356–366.
- [10] Wang W D, Mulone G, Salemi F, Salone V. Permanence and stability of a stage structured predator-prey model[J]. *J. Computational and Applied Math.*, 2001, 262(2): 499–528.
- [11] Wang W. Global dynamics of a population model with stage-structure for predator[J]. *J. Computational and Applied Math.*, 2001, 262(2): 499–528.
- [12] Agarwal M J, Devi S. A time-delay model for the effect of toxicant in a single species growth with stage-structure [J]. *Nonlinear Anal.: Real World Appl.*, 2010, 11(4): 2376–2389.
- [13] Bairagi N, Jana D. On the stability and Hopf bifurcation of a delay-induced predator-prey system with habitat complexity[J]. *Appl. Math. Model.*, 2011, 35(7): 3255–3267.
- [14] Chakraborty K, Chakraborty M, Kar T K. Bifurcation and control of a bioeconomic model of a predator-prey system with a time delay[J]. *Nonlinear Anal. Hybrid Syst.*, 2011, 5(4): 613–625.
- [15] Kar T K, Ghorai A. Dynamic behavior of a delayed predator-prey model with harvesting[J]. *Appl. Math. Comput.*, 2011, 217(22): 9085–9104.
- [16] Xu C J, Liao M X, He X F. Stability and Hopf bifurcation analysis for a Lokta-Volterra predator-prey model with two delays[J]. *Int. J. Appl. Math. Comput.*, 2011, 21(1): 97–107.
- [17] Xu C J, Tang X H, Liao M X. Stability and bifurcation analysis of a delayed predator-prey model of prey dispersal in two-patch environments[J]. *Appl. Math. Comput.*, 2010, 216(10): 2920–2936.
- [18] Xu C J, Tang X H, Liao M X, He X F. Bifurcation analysis in a delayed Lokta-Volterra predator-prey model with two delays[J]. *Nonlinear Dyn.*, 2011, 66(1-2): 169–183.
- [19] Yan X P, Zhang C H. Hopf bifurcation in a delayed Lokta-Volterra predator-prey system[J]. *Nonlinear Anal. Real World Appl.*, 2008, 9(1): 114–127.
- [20] Zhou X Y, Shi X Y, Song X Y. Analysis of non-autonomous predator-prey model with nonlinear diffusion and time delay[J]. *Appl. Math. Comput.*, 2008, 196(1): 129–136.
- [21] Kuang Y. Delay differential equations with application in population dynamical[M]. New York: Academic Press, 1993.
- [22] Hale J K. Theory of functional differential equations[M]. New York: Springer, 1976.
- [23] Hassard B, Kazarinoff N, Wan Y. Theory and applications of hopf bifurcation[M]. Cambridge: Cambridge University Press, 1981.
- [24] Carr J. Application of center manifold theory[M]. New York: Springer-Verlag, 1981.
- [25] Wei J J, Ruan S G. Stability and bifurcation in a neural network model with two delays[J]. *Physica D.*, 1999, 130(1): 255–272.

一类带有捕食者阶段结构的滞后型捕食食饵模型的Hopf分支

柯于胜, 李必文, 陈伯山

(湖北师范学院数学统计学院, 湖北 黄石 435002)

摘要: 本文研究了一类带有阶段结构和HollingIII型滞后函数响应的捕食食饵模型的稳定性和Hopf分支的问题. 利用微分动力系统的标准型和中心流形定理, 获得了内平衡点局部稳定和周期解的方向性, 推广了文献 [4] 所得出的结论.

关键词: 阶段结构; Hopf分支; 稳定性; Holling III型

MR(2010)主题分类号: 34C25; 34D05; 34F10; 34K18

中图分类号: O193

Appendix

We denote $\bar{x}(t) = x(t) - x^*$, $\bar{y}_1(t) = y_1(t) - y_1^*$, $\bar{y}_2(t) = y_2(t) - y_2^*$. By the transformation, the system (1.2) can be taken the form of

$$\begin{aligned} \dot{\bar{x}} &= c\bar{x} - \frac{r_2(D+r_1)}{D}\bar{y}_2 - a\bar{x}^2 - a_1h(\bar{x}, \bar{y}_2), \\ \dot{\bar{y}}_1 &= c_2\bar{x}(t-\tau) + \frac{a_1}{a_2}\frac{r_2(D+r_1)}{D}\bar{y}_2(t-\tau) - (D+r_1)\bar{y}_1 + a_2h(\bar{x}(t-\tau), \bar{y}_2(t-\tau)), \\ \dot{\bar{y}}_2 &= D\bar{y}_1 - r_2\bar{y}_2, \end{aligned} \quad (1)$$

where

$$\begin{aligned} h(x, y) &= \frac{c_1xy(x+2x^*) + x^2(c_3 - 2mx^*y_2^*)}{c_1^2(c_1 + m(x^2 + 2xx^*))}, \\ c_1 &= 1 + m(x^*)^2, \quad c_2 = \frac{2a_1x^*y_2^*}{c_1^2}, \quad c_3 = y_2^*(1 - 3m(x^*)^2), \quad c = r - 2ax^* - c_2. \end{aligned}$$

Then we imitate [25] to do the presentations as follow.

Let $t = s\tau$, $\bar{x}(s\tau) = \hat{x}(s)$, $\bar{y}_1(s\tau) = \hat{y}_1(s)$, $\bar{y}_2(s\tau) = \hat{y}_2(s)$, $\tau = \tau_0 + \mu$, $\mu \in R$, τ_0 is defined by (2.7), then system (1.2) can be transformed as an FDE in $C = C([-1, 0], R^3)$, we will denote $x = \hat{x}$, $y_1 = \hat{y}_1$, $y_2 = \hat{y}_2$,

$$\begin{aligned} \dot{x}(t) &= (\tau_0 + \mu) \left(cx(t) - \frac{r_2(D+r_1)}{D}y_2(t) - ax^2(t) - a_1h(x, y_2) \right), \\ \dot{y}_1(t) &= (\tau_0 + \mu) \left(c_2x(t-1) - \frac{a_1}{a_2}\frac{r_2(D+r_1)}{D}y_2(t-1) - (D+r_1)y_1(t) + a_2h(x(t-1), y_2(t-1)) \right), \\ \dot{y}_2(t) &= (\tau_0 + \mu) \left(Dy_1 - r_2y_2 \right). \end{aligned} \quad (2)$$

For $\phi = (\phi_1, \phi_2, \phi_3)^T \in C([-1, 0], R^3)$, system (1) is equivalent to the following operator equation

$$\dot{U}(t) = L(\mu)U_t + F(\mu)U_t, \quad (3)$$

where $U = (x, y_1, y_2)$, $U_t = U(t + \theta)$ for $\theta \in [-1, 0]$.

$$F(\mu, \phi) = \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix} = (\tau_0 + \mu) \begin{pmatrix} -a\phi_1^2(0) - a_1h(\phi_1(0), \phi_3(0)) \\ a_2h(\phi_1(-1), \phi_3(-1)) \\ 0 \end{pmatrix}.$$

We define

$$L_\mu \phi = (\tau_0 + \mu) \begin{pmatrix} c & 0 & -\frac{r_2(D+r_1)}{D} \\ 0 & -(D+r_1) & 0 \\ 0 & D & -r_2 \end{pmatrix} \phi(0) + (\tau_0 + \mu) \begin{pmatrix} 0 & 0 & 0 \\ c_2 & 0 & \frac{a_1 r_2(D+r_1)}{a_2 D} \\ 0 & 0 & 0 \end{pmatrix} \phi(-1).$$

By the Riesz representation theorem, there exists a 3×3 matrix functions $\eta(\theta, \mu) : [-1, 0] \rightarrow R^{3 \times 3}$ whose elements are of bounded variation such that

$$L_\mu \phi = \int_{-1}^0 [d\eta(\theta, \mu)] \phi(\theta) \text{ for } \phi \in C([-1, 0], R^3), \tag{4}$$

in fact, we chose $\eta(\theta, \mu) = B\delta(\theta) + K\delta(\theta + 1)$, then (3) is satisfied.

For $\phi \in C([-1, 0], R^3)$, define

$$A(\mu, \phi) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 [d\eta(s, \mu)] \phi(s), & \theta = 0, \end{cases}$$

$$R(\mu, \phi) = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu, \theta), & \theta = 0. \end{cases}$$

Then system (2) is equivalent to the following operator equation:

$$\dot{U}(t) = A(\mu)U_t + R(\mu)U_t.$$

For $\psi \in C([-1, 0], (R^3)^*)$, the adjoint operator A^* of the A is defined as

$$A^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & -1 \leq s < 0, \\ \int_{-1}^0 [d\eta^T(t, 0)] \psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1, 0], R^3)$ and $\psi \in C([-1, 0], (R^3)^*)$, define a bilinear form

$$\langle \phi, \psi \rangle = \bar{\psi}^T(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^\theta \bar{\psi}^T(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators.

From the discussion in Section 2, we know that $q(\theta)$ and $q^*(s)$ are eigenvectors of A and A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. We defined $q(\theta) = (\rho_1, \rho_2, 1)^T e^{i\omega_0\tau_0\theta}$, then $A(0)q(\theta) = i\omega_0\tau_0 q(\theta)$, it follows the above definition that

$$(\tau_0) \begin{pmatrix} c - i\omega_0\tau_0 & 0 & -\frac{r_2(D+r_1)}{D} \\ 0 & -(D+r_1) - i\omega_0\tau_0 & 0 \\ 0 & D & -r_2 - i\omega_0\tau_0 \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then, we can easily get $q(\theta) = (\rho_1, \rho_2, 1)^T e^{i\omega_0\tau_0\theta} = q(0)e^{i\omega_0\tau_0\theta}$, where $\rho_1 = \frac{r_2(D+r_1)}{D(c-i\omega_0)}$, $\rho_2 = \frac{r_2+i\omega_0}{D}$. Similarly, by definition of A^* ,

$$(\tau_0) \begin{pmatrix} c + i\omega_0\tau_0 & 0 & -\frac{r_2(D+r_1)}{D} \\ 0 & -(D + r_1) + i\omega_0\tau_0 & 0 \\ 0 & D & -r_2 + i\omega_0\tau_0 \end{pmatrix} q^*(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and $q^*(\theta) = \bar{G}(\rho_1^*, \rho_2^*, 1)^T e^{i\omega_0\tau_0\theta} = q^*(0)e^{i\omega_0\tau_0\theta}$, where $\rho_1^* = \frac{-c_2 D e^{-i\omega_0}}{(c+i\omega)(D+r_1-i\omega)}$, $\rho_2^* = \frac{D}{D+r_1-i\omega_0}$, in order to have $\langle q^*(s), q(\theta) \rangle = 1$, we evaluate the value \bar{G} , By the definition of the bilinear inner product, we have

$$\begin{aligned} & \langle q^*(0), q(0) \rangle \\ &= \bar{G}(\bar{\rho}_1^*, \bar{\rho}_2^*, 1)(\rho_1, \rho_2, 1)^T - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{G}(\bar{\rho}_1^*, \bar{\rho}_2^*, 1)e^{-i\omega_0(\xi-\theta)} d\eta(\theta, 0)(\rho_1, \rho_2, 1)e^{i\omega_0\xi} d\xi \\ &= \bar{G} \left(\rho_1 \bar{\rho}_1^* + \rho_2 \bar{\rho}_2^* + 1 - \int_{-1}^0 (\rho_1^*, \rho_2^*, 1)e^{-i\omega_0\theta} \theta \eta(\theta)(\rho_1, \rho_2, 1)^T \right) \\ &= \bar{G} \left(1 + \rho_1 \bar{\rho}_1^* + \rho_2 \bar{\rho}_2^* + (c_2 \rho_1 + \frac{a_1 r_2 (D+r_1)}{a_2 D}) \tau_0 e^{-i\omega_0 \tau_0} \bar{\rho}_2^* \right). \end{aligned}$$

Thus, we can choose \bar{G} as

$$\bar{G} = \frac{1}{1 + \rho_1 \bar{\rho}_1^* + \rho_2 \bar{\rho}_2^* + (c_2 \rho_1 + \frac{a_1 r_2 (D+r_1)}{a_2 D}) \tau_0 e^{-i\omega_0 \tau_0} \bar{\rho}_2^*},$$

such that $\langle q^*(s), q(\theta) \rangle = 1, \langle q^*(s), \bar{q}(\theta) \rangle = 0$.

According to the algorithms given in [22] and utilizing a computation process akin to that of Wei and Ruan [25], we compute the coordinates to describe the center manifold C_0 at $\mu = 0$. We define

$$z(t) = \langle q^*, x_t \rangle, \quad W(t, \theta) = x_t(\theta) - 2\text{Re}z(t)q(\theta), \tag{5}$$

thus, we have $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = \begin{pmatrix} W_{20}^{(1)}(\theta) \\ W_{20}^{(2)}(\theta) \\ W_{20}^{(3)}(\theta) \end{pmatrix} \frac{z^2}{2} + \begin{pmatrix} W_{11}^{(1)}(\theta) \\ W_{11}^{(2)}(\theta) \\ W_{11}^{(3)}(\theta) \end{pmatrix} z\bar{z} + \begin{pmatrix} W_{02}^{(1)}(\theta) \\ W_{02}^{(2)}(\theta) \\ W_{02}^{(3)}(\theta) \end{pmatrix} \frac{\bar{z}^2}{2} + \dots,$$

in fact, z and \bar{z} are local coordinates for center manifold C_0 in the direction of q^* and \bar{q}^* . Noting that W is also real if x_t is real, For solution $x_t \in C_0$ of (3.4),

$$\dot{z}(t) = i\omega_0\tau_0 z + \bar{q}^*(\theta)F(0, W(z(t), \bar{z}(t), \theta)) + 2\text{Re}zq(\theta) := i\omega_0\tau_0 z + \bar{q}^*(\theta)F_0 = i\omega_0\tau_0 z + g(z, \bar{z}).$$

By using (5), we have $x_t = (\phi_1, \phi_2, \phi_3) = W(t, \theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and $q(\theta) = (\rho_1, \rho_2, 1)^T e^{i\omega_0\tau_0\theta}$,

then we achieve

$$\begin{aligned}
 \phi_1(0) &= \rho_1 z + \bar{\rho}_1 \bar{z} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z \bar{z} + W_{02}^1(0) \frac{\bar{z}^2}{2}, \\
 \phi_2(0) &= \rho_2 z + \bar{\rho}_2 \bar{z} + W_{20}^2(0) \frac{z^2}{2} + W_{11}^2(0) z \bar{z} + W_{02}^2(0) \frac{\bar{z}^2}{2}, \\
 \phi_3(0) &= z + \bar{z} + W_{20}^3(0) \frac{z^2}{2} + W_{11}^3(0) z \bar{z} + W_{02}^3(0) \frac{\bar{z}^2}{2}, \\
 \phi_1(-1) &= \rho_1 z e^{-i\omega_0 \tau_0 \theta} + \bar{\rho}_1 \bar{z} e^{i\omega_0 \tau_0 \theta} + W_{20}^1(-1) \frac{z^2}{2} + W_{11}^1(-1) z \bar{z} + W_{02}^1(-1) \frac{\bar{z}^2}{2}, \\
 \phi_2(-1) &= \rho_1 z e^{-i\omega_0 \tau_0 \theta} + \bar{\rho}_1 \bar{z} e^{i\omega_0 \tau_0 \theta} + W_{20}^2(-1) \frac{z^2}{2} + W_{11}^2(-1) z \bar{z} + W_{02}^2(-1) \frac{\bar{z}^2}{2}, \\
 \phi_3(-1) &= z e^{-i\omega_0 \tau_0 \theta} + \bar{z} e^{i\omega_0 \tau_0 \theta} + W_{20}^3(-1) \frac{z^2}{2} + W_{11}^3(-1) z \bar{z} + W_{02}^3(-1) \frac{\bar{z}^2}{2}.
 \end{aligned} \tag{6}$$

From the above presentation, we have

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots = \bar{q}^* F_0(0, x_t) = \bar{G} \tau_0 (\bar{\rho}_1^*, \bar{\rho}_2^*, 1) (f_1, f_2, 0)^T, \tag{7}$$

where

$$\begin{aligned}
 f_1 &= -a \phi_1^2(0) - a_1 h(\phi_1(0), \phi_3(0)) \\
 &= -(a + \frac{a_1 c_3}{c_3^2}) \phi_1^2(0) - \frac{2a_1 x^*}{c_2^2} \phi_1(0) \phi_3(0) - \frac{a_1 (c_2 - 4mx^*)}{c_3^2} \phi_1^2(0) \phi_3(0) + O((|\phi_1(0)| + |\phi_3(0)|)^3), \\
 f_2 &= a_2 h(\phi_1(-1), \phi_3(-1)) \\
 &= \frac{a_2 c_3}{c_3^2} \phi_1^2(0) + \frac{2a_2 x^*}{c_2^2} \phi_1(0) \phi_3(0) + \frac{a_2 (c_2 - 4mx^*)}{c_3^2} \phi_1^2(0) \phi_3(0) + O((|\phi_1(-1)| + |\phi_3(-1)|)^3).
 \end{aligned} \tag{8}$$

Thus, By comparing the coefficients, we obtain $g_{20} = 2\tau_0 \bar{G} \hat{g}_{20}$, $g_{11} = 2\tau_0 \bar{G} \hat{g}_{11}$, $g_{02} = 2\tau_0 \bar{G} \hat{g}_{02}$, $g_{21} = 2\tau_0 \bar{G} \hat{g}_{21}$, where

$$\begin{aligned}
 \hat{g}_{20} &= -(a + \frac{a_1 c_3}{c_3^2}) \rho_1^2 \bar{\rho}_1^* - \frac{2a_1 x^*}{c_2^2} \rho_1 \bar{\rho}_1^* + \frac{a_2 c_3}{c_3^2} e^{-2i\omega_0 \tau_0} \rho_1^2 \bar{\rho}_2^* + \frac{2a_2 x^*}{c_2^2} e^{-2i\omega_0 \tau_0} \rho_1 \bar{\rho}_2^*, \\
 \hat{g}_{11} &= -(a + \frac{a_1 c_3}{c_3^2}) \rho_1 \bar{\rho}_1 \bar{\rho}_1^* - \frac{a_1 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \bar{\rho}_1^* + \frac{a_2 c_3}{c_3^2} e^{-2i\omega_0 \tau_0} \rho_1^2 \bar{\rho}_2^* + \frac{a_2 x^*}{c_2^2} e^{-2i\omega_0 \tau_0} (\rho_1 + \bar{\rho}_1) \bar{\rho}_2^*, \\
 \hat{g}_{02} &= -a \bar{\rho}_1^2 \bar{\rho}_1^* - \frac{a_1 c_3}{c_3^2} \bar{\rho}_1^2 \bar{\rho}_1^* - \frac{2a_1 x^*}{c_2^2} \bar{\rho}_1 \bar{\rho}_1^* + \frac{a_2 c_3}{c_3^2} e^{-2i\omega_0 \tau_0} \bar{\rho}_1^2 \bar{\rho}_2^* + \frac{2a_2 x^*}{c_2^2} e^{-2i\omega_0 \tau_0} \bar{\rho}_1 \bar{\rho}_2^*, \\
 \hat{g}_{21} &= \frac{6mx^* (c_3 + c_2 y_2^*) \rho_1^2 \bar{\rho}_1}{c_2^4} (a_1 \bar{\rho}_1^* - a_2 \bar{\rho}_2^* e^{-i\omega_0 \tau_0}) + \frac{c_2 - 4mx^*}{c_3^2} (\rho_1^2 + 2\rho_1 \bar{\rho}_1) (a_2 \bar{\rho}_2^* e^{-i\omega_0 \tau_0} - a_1 \bar{\rho}_1^*) \\
 &\quad - \frac{a_1 x^* \bar{\rho}_1^*}{c_2^2} (W_{20}^{(3)}(0) \bar{\rho}_2 + W_{20}^{(3)}(0) + 2W_{11}^{(1)}(0) \rho_2 + 2W_{11}^{(3)}(0)) + \frac{a_2 x^* \bar{\rho}_2^*}{c_2^2} (W_{20}^{(3)}(-1) \bar{\rho}_1 e^{i\omega_0 \tau_0} \\
 &\quad + W_{20}^{(3)}(-1) e^{i\omega_0 \tau_0} + 2W_{11}^{(1)}(-1) \rho_1 e^{i\omega_0 \tau_0} + 2W_{11}^{(3)}(-1) e^{i\omega_0 \tau_0}) \\
 &\quad + \bar{\rho}_1 \bar{\rho}_1^* (W_{20}^{(1)}(0) + 2W_{11}^1(0)) (-a - \frac{a_1 c_3}{c_3^2}) + \bar{\rho}_1 \bar{\rho}_2^* (W_{20}^{(1)}(-1) + 2W_{11}^1(-1)) \frac{a_2 c_3}{c_3^2} e^{i\omega_0 \tau_0}.
 \end{aligned}$$

Next, we calculate the $W_{20}^i(0)$, $W_{11}^i(0)$, $W_{20}^j(-1)$, $W_{11}^j(-1)$ ($i = 1, 2, 3; j = 1, 2$) in g_{21} . From (4) and (5), it is easily to derive:

$$W' = \begin{cases} AW - 2Re\bar{q}^*(0)\bar{F}q(\theta), & -1 \leq \theta < 0, \\ AW - 2Re\bar{q}^*(0)\bar{F}q(\theta), & \theta = 0, \end{cases} := AW + H(z, \bar{z}, \theta),$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z \bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \tag{9}$$

Substituting (6) to (5), we obtain

$$(A - 2i\omega_0\tau_0)W_{20} = -H_{20}(\theta), \tag{10}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{11}$$

According the previous definition, we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)F_0q(\theta) - q^*(0)\bar{F}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \tag{12}$$

Comparing the coefficients of (6) with (8) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{13}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{14}$$

It follow from (7) and (10) that

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

By the previous definition, we denote $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, and by solving W_{20} , we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{ig_{20}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + M_1e^{2i\omega_0\tau_0\theta}.$$

We can also obtain the follow result by utilizing the same method.

$$\dot{W}_{11}(\theta) = g_{11}(\theta)q(\theta) + \bar{g}_{11}(\theta)\bar{q}(\theta), W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{ig_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + M_2.$$

Now, we will try to find E_1 and E_2 . From the definition of A and (7), we obtain that

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2iw_0W_{20} - H_{20}(0)$$

and

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0),$$

where $d\eta(\theta) = \eta(\theta, 0)$.

Substituting (13) and (14) and noticing that

$$\left(iw_0\tau_0I - \int_{-1}^0 e^{iw_0\tau_0\theta}d\eta(\theta) \right) q(0) = 0, \quad \left(-iw_0\tau_0I - \int_{-1}^0 e^{-iw_0\tau_0\theta}d\eta(\theta) \right) \bar{q}(0) = 0, \tag{15}$$

we obtain

$$\left(-2iw_0\tau_0I + \int_{-1}^0 e^{2iw_0\tau_0\theta}d\eta(\theta) \right) M_1 = -2\tau_0 \begin{pmatrix} -(a + \frac{a_1c_3}{c_3^2})\rho_1^2 - \frac{2a_1x^*}{c_2^2}\rho_1 \\ a_2e^{-2i\omega_0\tau_0}(\frac{c_3}{c_3^2}\rho_1^2 + \frac{2x^*}{c_2^2}\rho_1) \\ 0 \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} 2i\omega_0 - c & 0 & -\frac{r_2(D+r_1)}{D} \\ -c_2 e^{-2i\omega} & 2i\omega_0 - (D+r_1) & -\frac{a_1}{a_2} \frac{r_2(D+r_1)}{D} e^{-2i\omega} \\ 0 & -D & 2i\omega_0 - r_2 \end{pmatrix} M_1 = \begin{pmatrix} -(a + \frac{a_1 c_3}{c_2^3}) \rho_1^2 - \frac{2a_1 x^*}{c_2^2} \rho_1 \\ a_2 e^{-2i\omega_0 \tau_0} (\frac{c_3}{c_2^3} \rho_1^2 + \frac{2x^*}{c_2^2} \rho_1) \\ 0 \end{pmatrix}.$$

Similarly, from (13) and (15), $-AW_{11}(\theta) = H_{11}(\theta)$ and for $\theta = 0$,

$$\int_{-1}^0 d\eta(\theta) W_{11}(\theta) = g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - \tau_0 \begin{pmatrix} -(a + \frac{a_1 c_3}{c_2^3}) \rho_1 \bar{\rho}_1 - \frac{a_1 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \\ \frac{a_2 c_3}{c_2^3} \rho_1 \bar{\rho}_1 + \frac{a_2 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \\ 0 \end{pmatrix},$$

from which we can get

$$\begin{aligned} & \frac{g_{11}}{i\omega_0} \int_{-1}^0 d\eta(\theta) q(0) e^{i\omega_0 \tau_0} - \frac{\bar{g}_{11}}{i\omega_0 \tau_0} \int_{-1}^0 d\eta(\theta) \bar{q}(0) e^{-i\omega_0 \tau_0} + E_2 \int_{-1}^0 d\eta(\theta) \\ &= g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - \tau_0 \begin{pmatrix} -2(a + \frac{a_1 c_3}{c_2^3}) \rho_1 \bar{\rho}_1 - \frac{2a_1 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \\ \frac{2a_2 c_3}{c_2^3} \rho_1 \bar{\rho}_1 + \frac{2a_2 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \\ 0 \end{pmatrix}, \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} -c & 0 & -\frac{r_2(D+r_1)}{D} \\ -c_2 & -(D+r_1) & -\frac{a_1}{a_2} \frac{r_2(D+r_1)}{D} \\ 0 & -D & -r_2 \end{pmatrix} M_2 = \begin{pmatrix} -(a + \frac{a_1 c_3}{c_2^3}) \rho_1 \bar{\rho}_1 - \frac{a_1 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \\ \frac{a_2 c_3}{c_2^3} \rho_1 \bar{\rho}_1 + \frac{a_2 x^*}{c_2^2} (\rho_1 + \bar{\rho}_1) \\ 0 \end{pmatrix}.$$

From the above calculation, we can get the vector M_1 and M_2 and put them in the equations (14) and (15), we can also obtain the terms

$$W_{11}^1(0), W_{11}^3(0), W_{11}^1(-1), W_{11}^3(-1), W_{20}^1(0), W_{20}^3(0), W_{20}^1(-1), W_{20}^3(-1), \quad (16)$$

and achieve the quantity g_{21} . For this propose, we express each g_{ij} in terms of the parameters and delay. and then, we can evaluate the following values:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0 \tau_0} (g_{11}g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\text{Re}C_1(0)}{\text{Re}\lambda'(\tau_0)}, \quad \beta_2 = 2\text{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\text{Im}C_1(0) + \mu_2 \text{Im}\lambda'(\tau_0)}{\omega_0 \tau_0}. \end{aligned}$$