Vol. 35 (2015)
No. 2

数 学 杂 志
J. of Math. (PRC)

COTORSION PAIRS OVER FINITE GROUP GRADED RINGS

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Abstract: In this paper, we study the relation of cotorsion pairs between the graded and ungraded cases. By using the graded theory and the relative homological algebra, we first consider the relationship of cotorsion pairs in R-mod and $S = R * G$-mod when R is any ring and G is a finite group. Then we study rigid cotorsion pairs in R-gr and consider the relationship of cotorsion pairs between R-gr and R-mod when R is a ring graded by a finite group G with $|G|^{-1} \in R$.

Keywords: cotorsion pair; finite group graded ring
2010 MR Subject Classification: 16W50; 18G25
Document code: A **Article ID:** 0255-7797(2015)02-0227-10

1 Introduction

The relative homological theory for graded rings was developed in [1, 2], where Gorenstein gr-injective, gr-projective, and gr-flat modules were introduced. Recent years, many authors have studied the existence of envelopes and covers in the category of graded modules and related them with the corresponding envelopes and covers in the category of modules (see [3, 10, 11, 20]).

The theory of cotorsion pairs is very important in relative homological theory. In this paper, we study the relation of cotorsion pairs in R-gr and R-mod when R is a graded ring by a finite group G. Since cotorsion pairs have a close relationship with the theory of envelopes and covers, we get a relationship of the existence of envelopes and covers between the graded and ungraded cases.

In the Second section, we first consider the case of cotorsion pairs over strongly graded rings of finite group. Then we give the relationship of cotorsion pairs in R-mod and $S = R*G$-mod when R is any ring and G is a finite group.

In the Third section, we give the definition of rigid cotorsion pairs in R-gr and give a equivalent characterization of it. Then we study the relationship of cotorsion pairs between R-gr and R-mod when R is a ring graded by a finite group G with $|G|^{-1} \in R$.

* **Received date:** 2014-05-12 **Accepted date:** 2014-09-16
Foundation item: Supported by by the School Foundation of Yangzhou University (2012CXJ004) and NSFC (11326065).

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In this paper, G is always a finite multiplicative group with identity 1. A G-graded ring R is a ring with identity 1, together with a direct sum $R = \oplus_{g\in G} R_g$ as additive subgroups, such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. It is well known that R_1 is a subring of R and $1 \in R_1$. If $R_g R_h = R_{gh}$ for all $g, h \in G$, then R is called a strongly graded ring.

A graded left R-module M is a left R-module endowed with an internal direct sum decomposition $M = \oplus_{\sigma\in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M satisfying $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For graded left R-modules M and N, $\mathrm{Hom}_{R-gr}(M, N) = \{f : M \to N \mid f \text{ is } R\text{−linear and } f(M_\sigma) \subseteq N_\sigma\}$ is the group of all morphisms from M to N in the category R-gr of all graded left R-modules. An R-linear map $f : M \to N$ is said to be a graded morphism of degree τ, $\tau \in G$, if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\mathrm{Hom}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{Hom}_R(M, N) = \oplus_{\sigma\in G}\mathrm{Hom}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote Ext^i_{R-gr} and Ext^i_R as the right derived functors of Hom_{R-gr} and Hom_R.

If $M = \oplus_{\sigma\in G} M_\sigma$ is a graded left R-module and $\sigma \in G$, then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\sigma\tau}$ for all $\tau \in G$, the graded module $M(\sigma)$ is called the σ-suspension of M. We can see the σ-suspension as an isomorphism of categories $T_\sigma : R - gr \to R - gr$, given on objects as $T_\sigma(M) = M(\sigma)$ for $M \in R - gr$.

There is a number of interesting functors relating the Grothendieck categories R-gr and R-mod (the category of all left R-modules). The forgetful functor $U : R - gr \to R - mod$ forgets the gradiation. This functor has a right adjoint F which associated to $M \in R - mod$ the graded R-module $F(M) = \oplus_{\sigma\in G}({}^\sigma M)$ (where each ${}^\sigma M$ is a copy of M, ${}^\sigma M = \{{}^\sigma m \mid m \in M\}$ with the structure of R-module given by $r *^\sigma m =^{\tau\sigma} m$ for $r \in R_\tau$). Following [13], when G is finite, (U, F) is an Frobenius pair (i.e., (U, F) and (F, U) are adjoint pairs). By [15], we know that the forgetful functor U is always separable, and if $n = |G|$ is invertible in R, then F is a separable functor.

We recall from [7] ([1]) that a (graded) left R-module M is called Gorenstein (gr-) injective if there exists an exact sequence

$$\mathscr{E} = \ \cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of (gr-)injective modules such that $M = \ker(E^0 \to E^1)$ and which remains exact whenever $\mathrm{Hom}_R(E, -)$ ($\mathrm{Hom}_{R-gr}(E, -)$) is applied for any (gr-)injective module E. Dually we have the definition of Gorenstein (gr-)projective modules.

A (graded) left R-module M is called Gorenstein (gr-)flat [8] ([2]) if there exists an exact sequence

$$\mathscr{F} = \ \cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of (gr-)flat modules such that $M = ker(F^0 \to F^1)$ and which remains exact whenever $E \otimes_R -$ is applied for any (gr-)injective R-module E.

A cotorsion pair (or cotorsion theory) in an abelian category $\mathcal{A}$ (see [9]) is a pair of classes $(\mathcal{F}, \mathcal{C})$ of objects of $\mathcal{A}$ if the following properties are satisfied:

(1) $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ for every $F \in \mathcal{F}, C \in \mathcal{C}$.

(2) $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ for every $F \in \mathcal{F}$, implies $C \in \mathcal{C}$.

(3) $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ for every $C \in \mathcal{C}$, implies $F \in \mathcal{F}$.

We use $\mathcal{C}^{\perp}$ (resp. ${}^{\perp}\mathcal{C}$) denote the the class of all objects such that $\mathrm{Ext}^1_{\mathcal{A}}(C, M) = 0$ (resp. $\mathrm{Ext}^1_{\mathcal{A}}(M, C) = 0$) for every $C \in \mathcal{C}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ of objects of $\mathcal{A}$ is called complete (see [9, Corollary 1.2.7]) if every object has a special $\mathcal{C}$-preenvelope (and a special $\mathcal{F}$-precover). Throughout this paper, we denote add($\mathcal{F}$)=$\{M \mid M$ is a direct summand of $\oplus_{i=1}^{i=n} F_i$, where n is any nonnegative integer and each $F_i \in \mathcal{F}\}$, where $\mathcal{F}$ is any class of modules closed under isomorphisms.

2 Cotorsion Pairs Over Strongly Graded Rings of Finite Group

We first consider the case of strongly graded ring. When R is a strongly graded ring, $R - gr$ and $R_1 - mod$ are equivalent categories. By [6, Theorem 2.8], we know that the equivalence is given by the functors $(-)_1 : R - gr \to R_1 - \mathrm{mod}$, $(M)_1 = M_1$ and $R \otimes_{R_1} - : R_1 - mod \to R - gr$. Using these functors, we can easily get the following proposition, but for the completeness, we give a proof here.

Proposition 2.1 Assume that R is a strongly graded ring.

(1) $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in R_1-mod if and only if $(R \otimes_{R_1} \mathcal{F}, R \otimes_{R_1} \mathcal{C})$ is a cotorsion pair in R-gr.

(2) $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in R-gr if and only if $((\mathcal{F})_1, (\mathcal{C})_1)$ is a cotorsion pair in R_1-mod.

Proof We only prove (1), the proof of (2) is similar to that of (1).

For any $M \in \mathcal{F}$ and $N \in \mathcal{C}$, we have that

$$Ext^1_{R-gr}(R \otimes_{R_1} M, R \otimes_{R_1} N) \cong Ext^1_{R_1}(M, (R \otimes_{R_1} N)_1) \cong Ext^1_{R_1}(M, N) = 0$$

since $(R\otimes_{R_1}, (-)_1)$ is an Frobenius pair. Then $R \otimes_{R_1} \mathcal{F} \subseteq^{\perp} (R \otimes_{R_1} \mathcal{C})$ and $R \otimes_{R_1} \mathcal{C} \subseteq (R \otimes_{R_1} \mathcal{F})^{\perp}$.

Next we prove that ${}^{\perp}(R \otimes_{R_1} \mathcal{C}) \subseteq R \otimes_{R_1} \mathcal{F}$. Let $X \in {}^{\perp}(R \otimes_{R_1} \mathcal{C})$, we get that

$$0 = Ext^1_{R-gr}(X, R \otimes_{R_1} N) \cong Ext^1_{R_1}((X)_1, N)$$

for any $N \in \mathcal{C}$. Hence $(X)_1 \in {}^{\perp}\mathcal{C} = \mathcal{F}$. Then $R \otimes_{R_1} (X)_1 \in R \otimes_{R_1} \mathcal{F}$. Since $X \cong R \otimes_{R_1} (X)_1$, $X \in R \otimes_{R_1} \mathcal{F}$.

Finally, we can prove that $(R \otimes_{R_1} \mathcal{F})^{\perp} \subseteq R \otimes_{R_1} \mathcal{C}$. Hence $(R \otimes_{R_1} \mathcal{F}, R \otimes_{R_1} \mathcal{C})$ is a cotorsion pair in R-gr.

Using a proof similar to the above, we can easily prove that if $(R \otimes_{R_1} \mathcal{F}, R \otimes_{R_1} \mathcal{C})$ is a cotorsion pair in R-gr, then $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in R_1-mod.

Let R be any ring and $S = R*G$ the skew group ring, as well known that S is a strongly graded ring. By [16, 8] and [12, Example 2.2], if G is a finite group and $|G|^{-1} \in R$, S is an excellent extension of R. Now we recall the definition of excellent extension ([16, 8]). Let R be a subring of the ring S and they have the same identity. The ring S is called an excellent extension of R if the following two conditions are satisfied:

1. S_R and $_RS$ are free modules with a basis $(1 = a_1, a_2, \cdots, a_n)$ such that $a_iR = Ra_i$ for $i = 1, \cdots, n$.

2. S is left R-projective, that is, if $_SM$ is a submodule of $_SN$ and $_RM$ is a direct summand of $_RN$, then $_SM$ is a direct summand of $_SN$.

Now, we study the relationship of cotorsion pairs between R-mod and S-mod. Let $i : R \to S$ be the inclusion map, then we have the induction functor $S \otimes_R - : R - \text{mod} \to S - \text{mod}$ and restriction of the scalars functor $_R(-) : S\text{-mod} \to R\text{-mod}$. If $M \in S$-mod, then $_RM$ will denote the image of M by the restriction of the scalars functor.

Proposition 2.2 Let R be any ring, $S = R * G$ be the skew group ring and $|G|^{-1} \in R$.

(1) If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in R-mod and $_R(S \otimes_R \mathcal{F}) \subseteq \mathcal{F}$, then $(\text{add}(S \otimes_R \mathcal{F}), \text{add}(S \otimes_R \mathcal{C}))$ is a cotorsion pair in S-mod.

(2) If $(\mathcal{F}', \mathcal{C}')$ is a cotorsion pair in S-mod and $S \otimes_R {}_R\mathcal{F}' \subseteq \mathcal{F}'$, then $(\text{add}(_R\mathcal{F}'), \text{add}(_R\mathcal{C}'))$ is a cotorsion pair in R-mod.

Proof (1) Assume that $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in R-mod and $_R(S \otimes_R \mathcal{F}) \subseteq \mathcal{F}$, we will prove that $(\text{add}(S \otimes_R \mathcal{F}), \text{add}(S \otimes_R \mathcal{C}))$ is a cotorsion pair in S-mod.

For any $M' \in \text{add}(S \otimes_R \mathcal{F})$ and $N' \in \text{add}(S \otimes_R \mathcal{C})$, if we want to prove that

$$\text{Ext}^1_S(M', N') = 0,$$

we only need to prove that $\text{Ext}^1_S(S \otimes_R M, S \otimes_R N) = 0$ for any $M \in \mathcal{F}$ and $N \in \mathcal{C}$. By [4, III,Proposition 4.14], we know $S \otimes_R - \cong \text{Hom}_R(S, -)$ (In fact, the condition R is artinian in [4, III, Proposition 4.14] is not necessary). By the definition of excellent extension, S is projective as right R-module. Then by [17, Excercise 9.21], we have that

$$\begin{aligned}&\text{Ext}^1_S(S \otimes_R M, S \otimes_R N) \cong \text{Ext}^1_S(S \otimes_R M, \text{Hom}_R(S, N))\\ &\cong Ext^1_R(\text{Hom}_S(S, S \otimes_R M), N) \cong \text{Ext}^1_R(S \otimes_R M, N) = 0,\end{aligned}$$

since $_R(S \otimes_R \mathcal{F}) \subseteq \mathcal{F}$. Thus

$$\text{add}(S \otimes_R \mathcal{F}) \subseteq^{\perp} (\text{add}(S \otimes_R \mathcal{C}))$$

and

$$\text{add}(S \otimes_R \mathcal{C}) \subseteq (\text{add}(S \otimes_R \mathcal{F}))^{\perp}.$$

Next we prove that $^{\perp}(\text{add}(S \otimes_R \mathcal{C})) \subseteq \text{add}(S \otimes_R \mathcal{F})$. Let $X \in {}^{\perp}(\text{add}(S \otimes_R \mathcal{C}))$, then $\text{Ext}^1_S(X, S \otimes_R N) = 0$ for any $N \in \mathcal{C}$. We get that

$$0 = \text{Ext}^1_S(X, S \otimes_R N) \cong \text{Ext}^1_S(X, \text{Hom}_R(S, N)) \cong \text{Ext}^1_R({}_RX, N)$$

for any $N \in \mathcal{C}$. Hence $_RX \in {}^{\perp}\mathcal{C} = \mathcal{F}$. Then $S \otimes_R {}_RX \in S \otimes_R \mathcal{F}$. Since S is left R-projective, by [19, Lemma 1.1], X is a direct summand of $S \otimes_R {}_RX \in S \otimes_R \mathcal{F}$. Thus $X \in \text{add}(S \otimes_R \mathcal{F})$.

Finally, let $Y \in (\text{add}(S \otimes_R \mathcal{F}))^{\perp}$, $\text{Ext}^1_S(S \otimes_R M, Y) = 0$ for any $M \in \mathcal{F}$. Then we know that $\text{Ext}^1_S(S \otimes_R M, Y) \cong \text{Ext}^1_R(M, \text{Hom}_S(S, Y)) \cong \text{Ext}^1_R(M, {}_RY)$. Thus $_RY \in \mathcal{F}^{\perp} = \mathcal{C}$, $S \otimes_R {}_RY \in S \otimes_R \mathcal{C}$. Therefore $Y \in \text{add}(S \otimes_R \mathcal{C})$.

Hence $(\mathrm{add}(S \otimes_R \mathcal{F}), \mathrm{add}(S \otimes_R \mathcal{C}))$ is a cotorsion pair in S-mod.

(2) Assume that $(\mathcal{F}', \mathcal{C}')$ is a cotorsion pair in S-mod and $S \otimes_R {}_R\mathcal{F}' \subseteq \mathcal{F}'$, we will prove that $(\mathrm{add}({}_R\mathcal{F}'), \mathrm{add}({}_R\mathcal{C}'))$ is a cotorsion pair in R-mod.

If we want to prove that $\mathrm{Ext}^1_R(M, N) = 0$ for any $M \in \mathrm{add}({}_R\mathcal{F}')$ and $N \in \mathrm{add}({}_R\mathcal{C}')$, we only need to prove that $\mathrm{Ext}^1_R({}_RM, {}_RN) = 0$ for any $M \in \mathcal{F}'$ and $N \in \mathcal{C}'$. By [17, Exercise 9.21] we have

$$\mathrm{Ext}^1_R({}_RM, {}_RN) \cong \mathrm{Ext}^1_R({}_RM, \mathrm{Hom}_S(S, N)) \cong \mathrm{Ext}^1_S(S \otimes_R {}_RM, N).$$

By hypothesis $S \otimes_R {}_R\mathcal{F}' \subseteq \mathcal{F}'$, we know that $\mathrm{Ext}^1_R({}_RM, {}_RN) \cong \mathrm{Ext}^1_S(S \otimes_R {}_RM, N) = 0$. Thus $\mathrm{add}({}_R\mathcal{F}') \subseteq {}^{\perp}(\mathrm{add}({}_R\mathcal{C}'))$, $\mathrm{add}({}_R\mathcal{C}') \subseteq (\mathrm{add}({}_R\mathcal{F}'))^{\perp}$.

Next we prove that

$${}^{\perp}(\mathrm{add}({}_R\mathcal{C}')) \subseteq \mathrm{add}({}_R\mathcal{F}').$$

Let $X \in {}^{\perp}(\mathrm{add}({}_R\mathcal{C}'))$, then

$$\mathrm{Ext}^1_R(X, {}_RN) = 0$$

for any $N \in \mathcal{C}'$. By [17, Excercise 9.21], we get that

$$\mathrm{Ext}^1_R(X, {}_RN) \cong \mathrm{Ext}^1_R(X, \mathrm{Hom}_S(S, N)) \cong \mathrm{Ext}^1_S(S \otimes_R X, N)$$

for any $N \in \mathcal{C}'$. Hence $S \otimes_R X \in {}^{\perp}\mathcal{C}' = \mathcal{F}'$. Then ${}_R(S \otimes_R X) \in {}_R\mathcal{F}'$. By the definition of excellent extension, X is a direct summand of ${}_R(S \otimes_R X)$, $X \in \mathrm{add}({}_R\mathcal{F}')$.

Finally, let $Y \in (\mathrm{add}({}_R\mathcal{F}'))^{\perp}$, then $\mathrm{Ext}^1_R({}_RM, Y) = 0$ for any $M \in \mathcal{F}'$. By [17, Exercise 9.21] we know that

$$0 = \mathrm{Ext}^1_R({}_RM, Y) \cong \mathrm{Ext}^1_R(S \otimes_S M, Y) \cong \mathrm{Ext}^1_S(M, \mathrm{Hom}_R(S, Y)) \cong \mathrm{Ext}^1_S(M, S \otimes_R Y)$$

for any $M \in \mathcal{F}'$. Thus $S \otimes_R Y \in \mathcal{F}'^{\perp} = \mathcal{C}'$, ${}_R(S \otimes_R Y) \in {}_R\mathcal{C}'$ and $Y \in \mathrm{add}({}_R\mathcal{C}')$.

Hence $(\mathrm{add}({}_R\mathcal{F}'), \mathrm{add}({}_R\mathcal{C}'))$ is a cotorsion pair in R-mod

3 Cotorsion Pairs Over Finite Group Graded Rings

Lemma 3.1 Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in R-gr. The following statements are equivalent:

(1) If $M \in \mathcal{F}$, then $M(\sigma) \in \mathcal{F}$ for every $\sigma \in G$.

(2) If $N \in \mathcal{C}$, then $N(\tau) \in \mathcal{C}$ for every $\tau \in G$.

Proof (1) $\Rightarrow$ (2) For any $M \in \mathcal{F}$, $N \in \mathcal{C}$ and any $\tau \in G$, by hypothesis we have

$$\mathrm{Ext}^1_{R-gr}(M, N(\tau)) \cong \mathrm{Ext}^1_{R-gr}(M(\tau^{-1}), N) = 0.$$

Then $N(\tau) \in \mathcal{F}^{\perp} = \mathcal{C}$ for any $\tau \in G$.

(2) $\Rightarrow$ (1) is similar to that of (1) $\Rightarrow$ (2).

Remark 3.2 (1) A cotorsion pair satisfying the equivalent conditions in Lemma 3.1 is said to be a rigid cotorsion pair.

(2) $(gr-\mathcal{P}(R), R-gr)$ and $(R-gr, gr-\mathcal{I}(R))$ are rigid cotorsion pairs in R-gr, where $gr-\mathcal{P}(R)$ (resp. $gr-\mathcal{I}(R)$) denotes the class of gr-projective (resp. gr-injective) left R-modules.

Proposition 3.3 Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in R-gr, then $(\mathcal{F}, \mathcal{C})$ is a rigid cotorsion pair if and only if $\mathrm{EXT}^1_R(M, N) = 0$ for all $M \in \mathcal{F}$ and $N \in \mathcal{C}$.

Proof Since $\mathrm{Hom}_R(-, N)_\sigma = \mathrm{Hom}_{R-gr}(-, N(\sigma))$ for every $\sigma \in G$, we get that

$$\mathrm{Ext}^1_R(M, N)_\sigma \cong \mathrm{Ext}^1_{R-gr}(M, N(\sigma))$$

for any $\sigma \in G$.

Let $M \in \mathcal{F}$ and $N \in \mathcal{C}$, if $(\mathcal{F}, \mathcal{C})$ is a rigid cotorsion pair, then $N(\sigma) \in \mathcal{C}$ for any $\sigma \in G$. Thus $\mathrm{Ext}^1_R(M, N)_\sigma = 0$ for any $\sigma \in G$, and hence $\mathrm{Ext}^1_R(M, N) = 0$.

Conversely, for any $M \in \mathcal{F}$ and $N \in \mathcal{C}$, we have $\mathrm{Ext}^1_{R-gr}(M, N(\sigma)) \cong \mathrm{Ext}^1_R(M, N)_\sigma = 0$ for any $\sigma \in G$. Then for any $N \in \mathcal{C}$ and any $\sigma \in G$, we know $N(\sigma) \in \mathcal{F}^\perp = \mathcal{C}$. Hence $(\mathcal{F}, \mathcal{C})$ is a rigid cotorsion pair.

By [5], we know when R is a graded ring by a finite group G, $R\#k[G]^*$ is the free right and left R-module with basis $\{p_g | g \in G\}$, and there is a category isomorphism between R-gr and $R\#k[G]^*$-mod given by the functors $(-)_{gr} : R\#k[G]^* - \mathrm{mod} \to R - gr$, $(-)^\# : R - gr \to R\#k[G]^* - \mathrm{mod}$. In [5], Cohen and Montgomery also proved the duality theorem for Coactions. The Duality Theorem for Coactions says that if R is graded by G, then there is an action of G on $R\#k[G]^*$, and $(R\#k[G]^*) * G \cong M_n(R)$ the $n \times n$ matrix ring over R, where $n = |G|$.

Theorem 3.4 Let R be a ring graded by a finite group G with $|G|^{-1} \in R$.

(1) If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in R-mod and $U(F(\mathcal{F})) \subseteq \mathcal{F}$, then $(\mathrm{add}(F(\mathcal{F})), \mathrm{add}(F(\mathcal{C})))$ is a cotorsion pair in R-gr.

(2) If $(\mathcal{F}', \mathcal{C}')$ is a cotorsion pair in R-gr and $F(U(\mathcal{F}')) \subseteq \mathcal{F}'$, then $(\mathrm{add}(U(\mathcal{F}')), \mathrm{add}(U(\mathcal{C}')))$ is a cotorsion pair in R-mod.

(3) If $(\mathcal{F}, \mathcal{C})$ is a rigid cotorsion pair in R-gr, then $(\mathrm{add}(U(\mathcal{F})), \mathrm{add}(U(\mathcal{C})))$ is a cotorsion pair in R-mod.

Proof (1) By [15], the functor $\mathrm{Col}_G(-) : R - \mathrm{mod} \to M_n(R) - \mathrm{mod}$ is an category equivalent functor, then $(\mathrm{Col}_G(\mathcal{F}), \mathrm{Col}_G(\mathcal{C}))$ is a cotorsion pair in $M_n(R) - \mathrm{mod}$. Let $i : R\#k[G]^* \to M_n(R) \cong (R\#k[G]^*) * G$ be inclusion map, we have the restriction functor $i_* : M_n(R) - \mathrm{mod} \to R\#k[G]^* - \mathrm{mod}$ and the induction functor

$$i^* : R\#k[G]^* - \mathrm{mod} \to M_n(R) - \mathrm{mod}.$$

Since $(R\#k[G]^*) * G \cong M_n(R)$ is a skew group ring of $R\#k[G]^*$, by Proposition 2.2, we know if $i^*(i_*(\mathrm{Col}_G(\mathcal{F}))) \subseteq \mathrm{Col}_G(\mathcal{F})$, then $(\mathrm{add}(i_*(\mathrm{Col}_G(\mathcal{F})), \mathrm{add}(i_*(\mathrm{Col}_G(\mathcal{C})))$ is a cotorsion pair in $R\#k[G]^*-$mod. Since $(-)_{gr} : R\#k[G]^*-\mathrm{mod} \to R-gr$ is an category equivalent functor, we get that if $i^*(i_*(\mathrm{Col}_G(\mathcal{F}))) \subseteq \mathrm{Col}_G(\mathcal{F})$, then $(\mathrm{add}((i_*(\mathrm{Col}_G(\mathcal{F}))_{gr}), \mathrm{add}((i_*(\mathrm{Col}_G(\mathcal{C}))_{gr}))$ is a cotorsion pair in R-gr.

Let $H = (-)_{gr} \circ i_* \circ Col_G(-)$, then $(\mathrm{add}(H(\mathcal{F})), \mathrm{add}(H(\mathcal{C})))$ is a cotorsion pair in R-gr. By [15, Lemma 3.2], we know that the functors F and H are isomorphic. Since $H = (-)_{gr} \circ i_* \circ Col_G(-)$, we know that $H' = {\mathrm{Col}_G}^{-1}(-) \circ i^* \circ (-)^{\#}$ is the left adjoint functor of H, where ${\mathrm{Col}_G}^{-1}(-) : M_n(R)-\mathrm{mod} \to R-\mathrm{mod}$ is the inverse of the functor $\mathrm{Col}_G(-)$. Hence the functor H' is isomorphic to U. Since $\mathrm{Col}_G(-) : R-\mathrm{mod} \to M_n(R)-\mathrm{mod}$ is an equivalent functor, we can prove that $U(F(\mathcal{F})) \subseteq \mathcal{F}$ if and only if $i^*(i_*(\mathrm{Col}_G(\mathcal{F}))) \subseteq \mathrm{Col}_G(\mathcal{F})$. Hence $(\mathrm{add}(F(\mathcal{F})), \mathrm{add}(F(\mathcal{C})))$ is a cotorsion pair in R-gr.

(2) By a proof similar to that of (1), we get the desired results.

(3) Since G is finite, by the remark following [14, I.2.12], we have

$$\mathrm{EXT}^1_R(M, N) \cong \mathrm{Ext}^1_R(M, N)$$

for any $M \in \mathcal{F}$ and $N \in \mathcal{C}$. By Proposition 3.3, we know that $Ext^1_R(M, N) = 0$. Hence

$$\mathrm{add}(U(\mathcal{F})) \subseteq {}^{\perp}\mathrm{add}(U(\mathcal{C}))$$

and

$$\mathrm{add}(U(\mathcal{C})) \subseteq (\mathrm{add}(U(\mathcal{F})))^{\perp}.$$

The rest of the proof is similar to that of Proposition 2.2, but for the completeness, we give a proof here. Next we prove that ${}^{\perp}(\mathrm{add}(U(\mathcal{C}))) \subseteq \mathrm{add}(U(\mathcal{F}))$. Let $X \in {}^{\perp}(\mathrm{add}(U(\mathcal{C})))$, then $\mathrm{Ext}^1_R(X, U(N)) = 0$ for any $N \in \mathcal{C}$. Since (U, F) is an Frobenius pair, we get that

$$\mathrm{Ext}^1_R(X, U(N)) \cong \mathrm{Ext}^1_{R-gr}(F(X), N)$$

for any $N \in \mathcal{C}$. Hence $F(X) \in {}^{\perp}\mathcal{C} = \mathcal{F}$, $U(F(X)) \in U(\mathcal{F})$. Since F is separable, by [11, Proposition 5], X is a direct summand of $U(F(X))$, $X \in \mathrm{add}(U(\mathcal{F}))$.

Finally, let $Y \in (\mathrm{add}(U(\mathcal{F})))^{\perp}$, then $\mathrm{Ext}^1_R(U(M), Y) = 0$ for any $M \in \mathcal{F}$. Since

$$\mathrm{Ext}^1_R(U(M), Y) \cong \mathrm{Ext}^1_{R-gr}(M, F(Y))$$

for any $M \in \mathcal{F}$, $F(Y) \in \mathcal{F}^{\perp} = \mathcal{C}$, $U(F(Y)) \in U(\mathcal{C})$. Since Y is a direct summand of $U(F(Y))$, $Y \in \mathrm{add}(U(\mathcal{C}))$.

Corollary 3.5 Let R be a ring graded by a finite group G with $|G|^{-1} \in R$.

(1) If $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair in R-mod and $U(F(\mathcal{F})) \subseteq \mathcal{F}$, then $(\mathrm{add}(F(\mathcal{F})), \mathrm{add}(F(\mathcal{C})))$ is a complete cotorsion pair in R-gr.

(2) If $(\mathcal{F}', \mathcal{C}')$ is a complete cotorsion pair in R-gr and $F(U(\mathcal{F}')) \subseteq \mathcal{F}'$, then $(\mathrm{add}(U(\mathcal{F}')), \mathrm{add}(U(\mathcal{C}')))$ is a complete cotorsion pair in R-mod.

Proof By a proof similar to that of [21, Theorem 3.7], we get the desired results.

From Corollary 3.5, we get a relationship of the existence of envelopes and covers between the graded and ungraded cases.

Lemma 3.6 Let R be a ring graded by a finite group G, M a graded left R-module and N a left R-module.

(1) If N is a Goreinstein injective left R-module, then $F(N)$ is Gorenstein gr-injective. If M is Gorenstein gr-injective, then $U(M)$ is Gorenstein injective.

(2) If N is a Goreinstein projective left R-module, then $F(N)$ is Gorenstein gr-projective. If M is Goreinstein gr-projective, then $U(M)$ is Gorenstein projective.

(3) Suppose that R is an n-FC ring. If M is Gorenstein gr-flat, then $U(M)$ is Gorenstein flat. If N is a Gorenstein flat left R-module, then $F(N)$ is Gorenstein gr-flat.

Proof (1) By [1, Proposition 3.6], we know that if N is a Gorenstein injective left R-module, then $F(N)$ is Gorenstein gr-injective. Since G is finite, (U, F) is an Frobenius pair. Using a proof similar to that of [1, Proposition 3.6], we can prove that if M is Gorenstein gr-injective, then $U(M)$ is Gorenstein injective.

(2) By [1, Proposition 4.5], we know that if N is a Gorenstein projective left R-module, then $F(N)$ is Gorenstein gr-projective. Since (U, F) is an Frobenius pair, we can similarly prove that if M is Gorenstein gr-projective, then $U(M)$ is Gorenstein projective.

(3) By [2, Proposition 3.1 and Theorem 3.3], we get the results.

Let $gr-\mathcal{GP}(R)$, $gr-\mathcal{GI}(R)$ and $gr-\mathcal{GF}(R)$ denote the class of Gorenstein gr-projective, Gorenstein gr-injective and Gorenstein gr-flat graded left R-modules respectively. As a application, we have the following proposition:

Proposition 3.7 If R is a ring graded by a finite group G with $|G|^{-1} \in R$, then we have

(1) $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a (complete) cotorsion pair in R-mod if and only if $(gr - \mathcal{GP}(R), gr - \mathcal{GP}(R)^{\perp})$ is a (complete) cotorsion pair in R-gr.

(2) $({}^{\perp}\mathcal{GI}(R), \mathcal{GI}(R))$ is a (complete) cotorsion pair in R-mod if and only if $({}^{\perp}gr - \mathcal{GI}(R), gr - \mathcal{GI}(R))$ is a (complete) cotorsion pair in R-gr.

(3) Suppose that R is an n-FC ring, then $(\mathcal{GF}(R), \mathcal{GF}(R)^{\perp})$ is a (complete) cotorsion pair in R-mod if and only if $(gr - \mathcal{GF}(R), gr - \mathcal{GF}(R)^{\perp})$ is a (complete) cotorsion pair in R-gr.

Proof (1) Let $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ be a cotorsion pair in R-mod, by Theorem 3.4 and Lemma 3.6, $(\mathrm{add}(F(\mathcal{GP}(R))), \mathrm{add}(F(\mathcal{GP}(R)^{\perp}))$ is a cotorsion pair. So we only need to prove that $\mathrm{add}(F(\mathcal{GP}(R))) = gr - \mathcal{GP}(R)$ and $\mathrm{add}(F(\mathcal{GP}(R)^{\perp})) = gr - \mathcal{GP}(R)^{\perp}$.

Since $F(\mathcal{GP}(R)) \subseteq gr - \mathcal{GP}(R)$ and $gr - \mathcal{GP}(R)$ is closed under finite direct sums and direct summands, $\mathrm{add}(F(\mathcal{GP}(R))) \subseteq gr - \mathcal{GP}(R)$. Let $M \in gr - \mathcal{GP}(R)$, since M is a direct summand of $FU(M)$, $M \in \mathrm{add}(F(\mathcal{GP}(R)))$ (since $U(M) \in \mathcal{GP}(R)$ by Lemma 3.6). Hence $\mathrm{add}(F(\mathcal{GP}(R))) = gr - \mathcal{GP}(R)$.

Then we prove that $\mathrm{add}(F(\mathcal{GP}(R)^{\perp})) = gr - \mathcal{GP}(R)^{\perp}$.

Let $N \in \mathcal{GP}(R)^{\perp}$ and $M \in gr - \mathcal{GP}(R)$, since $U(M) \in \mathcal{GP}(R)$, we have

$$\mathrm{Ext}^1_{R-gr}(M, F(N)) \cong \mathrm{Ext}^1_R(U(M), N) = 0.$$

Then $F(N) \in (gr - \mathcal{GP}(R))^{\perp}$, and $\mathrm{add}(F(\mathcal{GP}(R)^{\perp})) \subseteq gr - \mathcal{GP}(R)^{\perp}$ since $gr - \mathcal{GP}(R)^{\perp}$ is closed under finite direct sums and direct summands.

Let $N' \in gr - \mathcal{GP}(R)^{\perp}$ and $M' \in \mathcal{GP}(R)$, we have

$$\mathrm{Ext}^1_R(M', U(N')) \cong \mathrm{Ext}^1_{R-gr}(F(M'), N') = 0$$

and then $FU(N') \in F(\mathcal{GP}(R)^{\perp})$. Since N' is a direct summand of $FU(N')$, we know that $N' \in \mathrm{add}(F(\mathcal{GP}(R)^{\perp}))$. Thus $\mathrm{add}(F(\mathcal{GP}(R)^{\perp})) = gr - \mathcal{GP}(R)^{\perp}$, we get the desired results.

Conversely, let $(gr - \mathcal{GP}(R), gr - \mathcal{GP}(R)^{\perp})$ be a cotorsion pair in $R - gr$. Since $FU(gr - \mathcal{GP}(R)) \subseteq gr - \mathcal{GP}(R)$, $(\mathrm{add}(U(gr - \mathcal{GP}(R)), \mathrm{add}(U(gr - \mathcal{GP}(R)^{\perp}))$ is a cotorsion pair in R-mod. Using Lemma 3.6 and a proof similar to the above, we can prove that

$$\mathrm{add}(U(gr - \mathcal{GP}(R)) = \mathcal{GP}(R).$$

Next we prove that

$$\mathrm{add}(U(gr - \mathcal{GP}(R)^{\perp})) = \mathcal{GP}(R)^{\perp}.$$

For any $M \in \mathcal{GP}(R)$ and $N \in gr - \mathcal{GP}(R)^{\perp}$, we have the following isomorphisms

$$\mathrm{Ext}^1_R(M, U(N)) \cong \mathrm{Ext}^1_{R-gr}(F(M), N) = 0,$$

since $\mathrm{add}(F(\mathcal{GP}(R))) = gr - \mathcal{GP}(R)$. Hence $\mathrm{add}(U(gr - \mathcal{GP}(R)^{\perp})) \subseteq \mathcal{GP}(R)^{\perp}$ since $\mathcal{GP}(R)^{\perp}$ is closed under finite direct sums and direct summands. Similarly, we can prove that $\mathcal{GP}(R)^{\perp} \subseteq \mathrm{add}(U(gr - \mathcal{GP}(R)^{\perp}))$. Thus $\mathrm{add}(U(gr - \mathcal{GP}(R)^{\perp})) = \mathcal{GP}(R)^{\perp}$. Hence $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a cotorsion pair in R-mod.

By Corollary 3.5, we get the desired results.

Using Theorem 3.4, Corollary 3.5, Lemma 3.6 and a proof similar to that of (1), we get (2) and (3).

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有限群分次环上的余挠对研究

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摘要: 本文研究了余挠对在有限群分次和非分次情况下的联系. 利用分次理论以及相对同调, 我们首先研究了R是任意环G是有限群的情况下, 余挠对在R -模范畴以及斜群环$S = R * G$ -模范畴之间的关系; 然后我们研究了R-gr范畴中刚性余挠对的等价刻画, 同时给出了余挠对在R-gr范畴与R -模范畴之间的关系, 其中R是G分次环, 群G是有限群且$|G|^{-1} \in R$.

关键词: 余挠对; 有限群分次环

MR(2010)**主题分类号**: 16W50; 18G25 **中图分类号**: O154.2