Vol. 35 (2015)
No. 2

数 学 杂 志
J. of Math. (PRC)

THE OPTIMAL FUTURES HEDGING STRATEGIES WITH VAR

ZHANG Mao-jun [1], NAN Jiang-xia [2], YUAN Gong-lin [3]

(*1. School of Math. and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China*)

(*2. Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin 541004, China*)

(*3. School of Math. and Inform. Sci., Guangxi University, Nanning 530004, China*)

Abstract: This article studies a futures hedging optimization problem with the value-at-risk constraint. The existence of optimal hedging strategies, an augmented Lagrangian algorithm for solving this model and its convergence are obtained by the optimal methods. The studies about the single-variable hedging strategy with the return of futures following a normal distribution are extended via our results with random variables following elliptical distributions to describe some fat tail features of the market risk factors, value-at-risk to control risk of hedging strategies, the mean-VaR portfolio hedging model and an algorithm for solving this mode.

Keywords: futures contract; value-at-risk; hedging; augmented Lagrangian algorithm

2010 MR Subject Classification: 91G10; 90C90

Document code: A **Article ID:** 0255-7797(2015)02-0214-13

1 Introduction

The organized commodity futures as one of the most important kinds of derivative securities not only can be used to hedge a price risk of spot commodities but also may be selected as some assets included in an investment portfolio. Both academicians and practitioners have shown great interest in the issue of hedging spot commodities with futures because carrying costs of the spot commodities are difficult to predict, and a basis risk of futures is uncertain. One of the main theoretical issues in hedging involves the determination of the optimal hedging ratio. There are some mathematical models for the theoretical analysis of the futures hedging ratio in this literature. Minimum-variance hedging theory developed by Johnson (1960) was the first theory which attempted to minimize the price risk considered a reflection of a variance of the hedged portfolio to select the optimal number of futures contracts necessary to hedge a certain spot position. There are much of the debate

* **Received date:** 2013-06-11 **Accepted date:** 2013-09-06

Foundation item: Supported by National Natural Science Foundation of China (71461005; 71462004); Guangxi Natural Science Foundation (2014GXNSFFA118001; 2012GXNSFAA053013; 2014GXNSFAA118010); Postdoctoral Science Foundation of China (13R21414700; 2013M540372).

Biography: Zhang Maojun (1977–), male, born at Xinzhou, Shanxi, associate professor, Ph.D., major in mathematical finance.

whether minimum-variance criterion is appropriate, because it is based on two assumptions that investors have a quadratic utility function and returns of the spot and futures are normal.

In practice, neither of these assumptions is likely to hold. The variance as an objective function in the minimum-variance hedging theory is only one of many possible risk measures. Alternative hedging objectives may be applied. For instance, Howard and D'Antonio (1984) designed some hedging strategies to maximize the Sharpe ratio, Lien and Shaffer (1999) minimized the mean-Gini coefficient, Lien and Tse (2000) employed the objective functions including the generalized semivariance or higher lower partial moments. Furthermore, the returns are not always normally distributed, may be non-normal distributions, for instance, t-distribution which has a fat-tail feature. This has led to the emergence of alternative measures of risk. Of these, perhaps the most widely used risk measures are value-at-risk (VaR) and conditional value-at-risk (CVaR) evolved into risk measurement and portfolio optimization. Harris and Shen (2006) developed minimum-VaR futures hedging model, and showed that minimum-VaR hedging ratios are found to be significantly lower than minimum-variance hedging ratios. Hung et al. (2006) proposed zero-VaR hedging ratio which had an analytical solution and converged to the minimum-variance hedging ratio under a martingale process. Lee and Hung (2007) extended the one period zero-VaR hedging ratio proposed by Hung et al. (2006) to the multi-period case and incorporated the hedging horizon into the objective function under VaR framework. Cao et al. (2010) developed a semi-parametric approach for estimating VaR and CVaR under the assumption of the non-normality of futures returns.

The bulk of research on hedging mentioned above is concerned with a single commodity, and solely relies on minimizing the price risk as an objective function of the hedging theory about commodity futures. Nevertheless, as to a hedging portfolio of spot commodities with multiple commodities futures, the risk-minimizing and/or optimal positions to take in each contract must not only reflect its own covariance with the spot position, but also its degree of covariability wit other commodities futures. Anderson and Danthine (1980) made the first contributions to multivariate hedging theories. Eaker and Grant (1987) and Gagnon et al. (1998) used the variance of the portfolio including spot commodities and futures as a risk measure of price risk to study the multivariate hedging problem. Albrecht et al. (2011) proposed a multivariate futures hedging problem for minimizing VaR and CVaR based on an analysis approach.

However, no attempt has been made to develop the hedging strategies based on maximizing the expected value of the hedge portfolio with VaR as a price risk constraint. In order to fill this gap, we derive mean-VaR multivariate hedging rules that is applied to a very broad range of financial risk management, that is, to analyze a multivariate commodity cross hedging problem of a portfolio consisting of long position in spot commodities by selling short futures via numerical optimization techniques for the determination of optimal hedging strategies. In this study, we will provide a numerical approach to obtain the optimal hedging strategies for a mean-VaR hedging model under assumptions on the return rates of the spot commodities and the futures with elliptical distributions. Our contribution to the literature is threefold. First, we develop a multivariate futures hedging theory model with an expected return objective function and a price risk constraints to describe the hedging process of the hedger. We particularly make use of VaR under a weak assumption on the elliptical return distribution to control the price risk of the hedging portfolio, which allows us to capture asymmetries in the return distribution. Second, we analyze the first order conditions for the optimal hedging optimization problem to document the existence of the optimal hedging strategies, and derive the corresponding dual problem implying other financial implication. Third, we construct an augmented Lagrangian algorithm to solving the mean-VaR model in order to get the optimal hedging portfolio satisfying the objective of the hedger under the VaR constraint as a risk measure, and to prove the convergence of the algorithm proposed in our study. Moreover, some numerical examples are given to illustrate the effectiveness of both the theoretical model and the constructed algorithm.

This paper is organized as follows. A multivariate hedging model over one period is presented in the next section. In Section 3, an optimality condition and dual problem of the mean-VaR hedging problem are given. In Section 4, we provide an augmented Lagrangian algorithm for solving this problem and prove its convergence. Some numerical results are given in the last section.

2 Futures Hedging Portfolio Model

We consider a multivariate futures hedging problem in one-period world. At time $t-1$ an agent purchases the spot commodities which would sold at time t, and the amount of the ith commodity of a total of m commodities is S_i at the spot price P_{it-1} in the commodity markets. Let $S=(S_1,\cdots,S_m)^T$ be the long position vector of spot commodities and $P_{t-1}=(P_{1t-1},\cdots,P_{mt-1})^T$ be the spot price vector of these commodities at time $t-1$, respectively. These prices may fall prior to his reselling them because there are many uncertain factors in the spot market, which make him exposed to the price risk. According to the short selling regulation in the futures markets and hedging theories he would protect his long position of these commodities from the risk of such price fluctuation by selling a sufficient number of futures contracts to reduce the price risk. Suppose that there are a futures market for each of the commodities, the agent can take short positions in the given futures by selling a portfolio consisting of the n commodities futures at time $t-1$. Suppose that $C=(C_1,\cdots,C_n)^T$ is the short position vector of the futures, to be liquidated them at time t, a short position is represented by $C_i<0$ for $i=1,\cdots,n$, and $F_{t-1}=(F_{1t-1},\cdots,F_{nt-1})^T$ is the price vector of the futures at time $t-1$.

Furthermore, assume that the agent holds the long and short positions of the spot and futures commodities until the time t, and the spot prices and futures prices at time t are random. Let P_{it} and F_{it} be the ith spot price and futures price at time t, respectively. Then the return rate of the ith spot position is $R_{P_i}=\frac{P_{it}-P_{it-1}}{P_{it-1}}$ for $i=1,\cdots,m$, and the return rate of the ith futures position is $R_{F_i} = \frac{F_{it}-F_{it-1}}{F_{it-1}}$ for $i = 1, \cdots, n$.

Since futures contracts are used to reduce the fluctuations in spot positions, the resulting ratio between the amount of spot commodities and the amount of futures is known as a hedging ratio. Let $R_P = (R_{P_1}, \cdots, R_{P_m})^T$ and $R_F = (R_{F_1}, \cdots, R_{F_n})^T$ be the return rate vector of the spot positions and futures positions, respectively. Additionally, let h_i denote the ith hedging ratio, i.e.,

$$h_i = \frac{F_{it-1}C_i}{S^T P_{t-1}}$$

for $i = 1, \cdots, n$, and $h = (h_1, \cdots, h_n)^T$ be the vector of hedging ratios.

In general, one of the most fundamental issues in hedging is how to measure the risk and return of the hedge portfolio. In order to analyze the optimal hedge portfolio, some formulas for the return and risk of hedge portfolio are described in order. The amount of return gained on the hedging is expressed as the return of hedge portfolio consisting of the spot and futures positions,

$$R(x,h) = \frac{S^T(P_t - P_{t-1}) - C^T(F_t - F_{t-1})}{S^T P_{t-1}}, = x^T R_P - h^T R_F, \tag{2.1}$$

where

$$x_i = \frac{S_i P_{i,t-1}}{S^T P_{t-1}}$$

is the weight in the ith spot position, and $x = (x_1, \cdots, x_n)^T$ is the vector of the spot positions.

It is obvious that these vectors R_P and R_F are random because at time t the spot price P_{jt} (for $j = 1, \cdots, m$) and the futures price F_{it}(for $i = 1, \cdots, n$) are random. Therefore, the return of the hedging portfolio $R(x,h)$ is a function with respect to the two vectors x and h. With many uncertain factors in the spot and futures market, it is difficult for the hedger to predict the results of the $R(x,h)$ which is treated as a random variable. In view of the hedging practice, it seems be hopeful for the hedger to know the expectation of the return of the hedging. Hence, we will analyze the expected value of the return of the hedge portfolio. Furthermore, the expected value of the random vector R_P and R_F are denoted as $\mu_S = (E(R_{P_1}), \cdots, E(R_{P_m}))^T$ and $\mu_F = (E(R_{F_1}), \cdots, E(R_{F_m}))^T$, where the symbol $E(\cdot)$ represents the expectation of a random variable, respectively.

Given these notations and the equation(2.1), the expected return of the hedge portfolio can be written as

$$M(x,h) = E(R(x,h)) = x^T \mu_S - h^T \mu_F. \tag{2.2}$$

Therefore, the expected return function $M(x,h)$ is an objective function which the hedger will maximize by choosing the spot weight vector x and the hedging ratio vector h.

Hedging activities appear to be motivated by the desire to reduce risk, as described in risk management theory, the hedger should control the downside risk of the hedging. VaR is probably the most widely used risk measure in financial institutions. To develop our hedging model, we introduce VaR and use it to measure a downside risk of the spot and futures portfolio.

Definition 2.1 Given some confidence levels $\alpha \in (0,1)$, the VaR of the portfolio over the time period Δ at the confidence level α is given by the smallest number such that the probability that the loss L exceeds l is no larger than $1-\alpha$. Formally,

$$\mathrm{VaR}_\alpha = \inf\{l \in \mathbb{R} : \Pr(L > l) \leq 1-\alpha\} = \inf\{l \in \mathbb{R} : \mathrm{F}(l) \geq \alpha\}, \tag{2.3}$$

where $\Pr(\cdot)$ and $\mathrm{F}(\cdot)$ are a cumulative probability function and a loss distribution function with respect to the random variable L, respectively.

In probabilistic terms, VaR is thus simply a quantile of the loss distribution, and depends on the time period and the confidence level. Some typical values for Δ are usually 1 or 10 days in market risk management, and $\alpha = 0.95$ or $\alpha = 0.99$. If the probability density function of the loss distribution is given, then the value of VaR is obtained by computing quantile in certain case. To take an example about a normal loss distribution (see, McNeil et al., 2005), suppose that the loss distribution $F(\cdot)$ is normal with mean μ and variance σ^2. Fix $\alpha \in (0,1)$. Then

$$\mathrm{VaR}_\alpha = \mu + \sigma\Phi^{-1}(\alpha), \tag{2.4}$$

where Φ denotes the standard normal distribution function(df) and $\Phi^{-1}(\alpha)$ is the α-quantile of Φ. Of course, similar results may be obtained for the multivariate normal loss distribution.

As a generalization of the normal distribution, the family of elliptical distributions exhibits many favorable properties, which are important for financial modeling, especially able to capture fat-tails features of return series in financial market, such as the Student's t-loss distribution(see, McNeil et al., 2005). Suppose $(L-\mu)/\sigma$ has a standard t distribution with v degrees of freedom, and mean μ and variance $v\sigma^2/(v-2)$ when $v > 2$, so that σ is not the standard deviation of the distribution. We get

$$\mathrm{VaR}_\alpha = \mu + \sigma t_v^{-1}(\alpha), \tag{2.5}$$

where t_v denotes the distribution function of the standard t-distribution, which is available in most statistical computer package along with its inverse.

Since the returns of the spot commodities and futures contract are all random multivariates, we limit our analysis to multivariate elliptical distributions. Although there are several equivalent definitions of elliptical distributions, we use the following one for elliptical distributions to study our multivariate hedging problem. If the random vector $X = (X_1, \cdots, X_m)^{\mathrm{T}}$ is elliptically distributed with mean μ and correlation matrix Σ when its probability density is given by

$$J_X(x) = |\Sigma|^{-1/2} g((x-\mu)\Sigma^{-1}(x-\mu)^{\mathrm{T}}),$$

where $|\Sigma|$ stands for the determinant of Σ, and g is a scalar function referred as the density generator.

On the technical side, we thus assume that the return vector of the spot and futures position, R_P and R_F, are elliptically distributed, respectively. We will calculate the VaR of the return of hedge portfolio, $R(x,h)$, and the VaR at a confidence level of $1-\alpha$ is given by the solution of the following equation,

$$\Pr(R(x,h) < -\mathrm{VaR}_\alpha) = \alpha. \tag{2.6}$$

Here we follow the usual convention to record portfolio losses by negative numbers, and to start the VaR as a positive quantity of money.

Based on equation (2.6), the VaR depends on the two vectors x and h, that is, the VaR is a function with respect to the vectors x and h. In the hedging term, the agent can change the value of the VaR by choosing the linear portfolio of the spot wights x and the hedging radios vector h based on his expectation. In equation (2.6), it is important to derive a closed form of the VaR when the cumulative probability distribution of the two random vectors R_P and R_F are multivariate elliptically distributed. Kamdem (2005) got a formula used to compute VaR for linear portfolio with elliptically distributed risk factors, and the special attention is given to the particular case of a multivariate t-distribution. In the following, we limit our analysis to the elliptical distribution of the spot and future return, and take use of the computing results about VaR (see, Kamdem (2005)).

Lemma 2.2 [14] Suppose that the loss function of the portfolios over the time window of interest is, to good approximation, given by $\Delta X = \delta_1 X_1 + \cdots + \delta_n X_n$, with constant portfolio weights δ_j. Suppose moreover that the random vector $X = (X_1, \cdots, X_m)^T$ of underlying risk factors follows a continuous elliptic distribution, with probability density given by $J_X(x)$, where $g(s^2)$ is integrable over $\mathbb{R}$, continuous and nowhere 0. Then the portfolio's Delta-elliptic VaR at confidence level $1-\alpha$ is given by

$$\mathrm{VaR}_\alpha(\delta) = -\delta^T\mu + c^g_{\alpha,n}\sqrt{\delta\Sigma\delta^T}, \tag{2.7}$$

where $c_\alpha = c^g_{\alpha,n}$ is the unique positive solution of the transcendental equation $\alpha = G(c^g_{\alpha,n})$,

$$G(x) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_x^{-\infty}\int_{z^2}^{+\infty}(u-z^2)^{\frac{n-3}{2}}g(u)dudz$$

and $\Gamma(\cdot)$ is a Gamma function.

The value of $c^g_{\alpha,n}$ in the lemma is obtain by the Matlab (see, Kamdem (2005)) as follows:

Table 1: The value of $c^g_{\alpha,n}$

n	2	3	4	5	6	7	8	9
$c^g_{0.01,n}$	5.5722	5.9309	5.7879	5.4555	5.0799	4.7160	4.3819	4.0818
$c^g_{0.025,n}$	8.6113	7.6777	6.8216	6.0676	5.4326	4.9032	4.4601	4.0862
$c^g_{0.05,n}$	11.7123	9.0750	7.4966	6.3797	5.5457	4.9007	4.3880	3.9711

By the definition of VaR as maximum loss with a given confidence level, we can express the downside risk of the hedge portfolio over a given time period and confidence level $1-\alpha$ as a risk constraint function.

According to Lemma 2.2, we obtain a closed formula for the VaR of the hedge portfolios, that is,

$$\mathrm{VaR}_\alpha(x,h) = -x^T\mu_S + h^T\mu_F + c^g_{\alpha,n}(x\Sigma_{SS}x^T + h\Sigma^T_{FS} + x\Sigma_{SF}h^T + h\Sigma_{FF}h^T)^{1/2}, \tag{2.8}$$

where Σ_{SS} is the covariance matrix for R_P, Σ_{SF} is the covariance matrix between R_P and R_F, Σ_{FS} is the covariance matrix between R_F and R_P, and Σ_{FF} is the covariance matrix of R_F.

Let $y = (x^t\ \ h^T)^T$, $\mu = (\mu_S\ \ -\mu_F)^T$. Then equation (2.2) is changed into

$$x^T\mu_S - h^T\mu_F = y^T\tilde{\mu}.$$

Furthermore, denote the covariance matrix of the random vector $(R_P\ \ R_F)$ by

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma_{SS} & \Sigma_{SF} \\ \Sigma_{FS} & \Sigma_{FF} \end{pmatrix}. \tag{2.9}$$

Then $\mathrm{VaR}_\alpha(x,h)$ can be expressed in the matrix term,

$$\mathrm{VaR}_\alpha(x,h) = -y^T\tilde{\mu} + c^g_{\alpha,n}(y^T\tilde{\Sigma}y)^{1/2}. \tag{2.10}$$

Hence, we would consider the $\mathrm{VaR}_\alpha(x,h)$ as the constraint for controlling the hedging risk, which is given by

$$\mathrm{VaR}_\alpha(x,h) \le \gamma, \tag{2.11}$$

where $\gamma \in \mathbb{R}^+$.

Submitting the express of the VaR in equation (2.10) into equation (2.11), the risk constraint for VaR can be written as

$$-y^T\tilde{\mu} + c^g_{\alpha,n}(y^T\tilde{\Sigma}y)^{1/2} \le \gamma. \tag{2.12}$$

We thus can use equation (2.12) as a risk constraint to find the hedging policies of the hedger in the following section.

In view of the above analysis of the expected return treated as the objective function and the VaR denoted as a risk constraint function, suppose that the hedger can select x and h so as to maximize the expected return under the downside risk constraint. Our analysis of the futures hedging is called as mean-VaR hedging classified as the mean-risk hedging. Thus, according to equations (2.2) and (2.12), the mean-VaR hedging problem under assumption that the hedger is risk aversion, is given by

$$\begin{aligned} \max_y \quad & y^T\tilde{\mu}, \\ \text{s. t.} \quad & -y^T\tilde{\mu} + c^g_{\alpha,n}(y^T\tilde{\Sigma}y)^{1/2} \le \gamma, \\ & x_i \ge 0, i = 1, \cdots, m, \\ & h_j \ge 0, j = 1, \cdots, n. \end{aligned} \tag{2.13}$$

In order to solving problem (2.13), we will develop a duality theory and optimal condition of the primal problem in the next section.

3 Duality and Optimal Condition

In this section, we analyze the theory of problem (2.13) including its duality theory and optimal conditions.

To start with the duality theory of problem (2.13), let $f(y) = -y^{\mathrm{T}}\tilde{\mu}$, $g(y) = -y^T\tilde{\mu} + c^g_{\alpha,n}(y^T\tilde{\Sigma}y^{1/2} - \gamma$ and $X = \{y_i = (x_i, h_j)|x_i \geq 0, i = 1, \cdots, m; h_j \geq 0, i = 1, \cdots, n\}$. Then problem (2.13) can be written as

$$\begin{aligned} \min_{y\in X} \quad & f(y), \\ \text{s. t.} \quad & g(y) \leq 0. \end{aligned} \tag{3.1}$$

We will analyze the Lagrangian duality problem and corresponding properties of problem (3.1). The Lagrangian duality problem is denoted as

$$\begin{aligned} \max \quad & q(\lambda), \\ \text{s. t.} \quad & \lambda \geq 0, \end{aligned} \tag{3.2}$$

where $q(\lambda) = \inf\{L(y, \lambda)|y \in X\}$ and $L(y, \lambda) = f(y) + \lambda g(y)$.

It is possible that the value of the function $q(\lambda) = -\infty$ for some λ, so the domain of the duality problem is defined as

$$D = \{\lambda|q(\lambda) > -\infty\}.$$

Proposition 3.1 The domain D is convex, and the function $q(\lambda)$ is concave in D.

Proof For any y, $\lambda_1 > 0$, $\lambda_2 > 0$ and $a \in [0, 1]$, we have

$$L(y, a\lambda_1 + (1 - a)\lambda_2) = aL(y, \lambda_1) + (1 - a)L(y, \lambda_2).$$

Taking the infinimum of both sides of the above equation, we get

$$\inf_{y\in X} L(y, a\lambda_1 + (1 - a)\lambda_2) \geq a \inf_{y\in X} L(y, \lambda_1) + (1 - a) \inf_{y\in X} L(y, \lambda_2)$$

or

$$q(a\lambda_1 + (1 - a)\lambda_2) \geq aq(\lambda_1) + (1 - a)q(\lambda_2).$$

It is easy to obtain $a\lambda_1 + (1 - a)\lambda_2 \in D$ when $\lambda_1 \in D$ and $\lambda_2 \in D$. Thus, the set D is convex, and the function q is concave.

Theorem 3.2 $q^* \leq f^*$.

Proof For each $\lambda \geq 0$ and $y \in X$ satisfying $g(y) \leq 0$, we have

$$q(\lambda) = \inf_{y\in X} Ly, \lambda) \leq f(y) + \lambda g(y) \leq f(y).$$

So, we get

$$q^* = \sup_{\lambda\geq 0} q(\lambda) \leq \inf_{y\in X, g(y)\leq 0} f(y) = f^*.$$

Theorem 3.3 y^* is the optimal solution of the primal problem (3.1) and λ^* is the Lagrangian multiplier if and only if

$$y^* \in X, g(y^*) \leq 0, \tag{3.3}$$

$$\lambda^* \geq 0, \tag{3.4}$$

$$y^* = \arg\min_{\mathrm{y}\in \mathrm{X}} \mathrm{L}(\mathrm{y}, \lambda^*), \tag{3.5}$$

$$\lambda^* g(y^*) = 0. \tag{3.6}$$

Proof First, to analyze the sufficiency. If (y^*, λ^*) is the optimal solution of the primal problem and the Lagrangian multiplier, then equations (3.3) and (3.4) are satisfied. Furthermore, the following condition can be obtained

$$f^* = f(y^*) \geq f(y^*) + \lambda^* g(y^*) = L(y^*, \lambda^*) \geq \inf_{y\in X} L(y, \lambda^*).$$

According to the definition of the Lagrangian multiplier, $\lambda^* \geq 0$, and the feasibility of y^*, that is, $g(y^*) \leq 0$, the first inequality is satisfied. Moreover, one has $f^* = \inf\limits_{y\in X} L(y, \lambda^*)$ by the definition of the Lagrangian multiplier. So the equality in the above equation is true. That implies $\lambda^* g(y^*) = 0$ and $y^* = \arg\min\limits_{y\in X} L(y, \lambda^*)$.

Then we consider the necessity of the theorem. Using these equations from (3.3) to (3.6), one has

$$f^* \leq f(y) = L(y, \lambda^*) = \min_{y\in X} L(y, \lambda^*) = q(\lambda^*) \leq q^*. \tag{3.7}$$

The equality in the over equation (3.7) is obtained by the weak dual theorem. Thus, y^* and λ^* are the optimal solution of the primal problem and the one of the dual problem, respectively, and there does not exist the duality gap between the the primal problem and dual one.

4 An Algorithm

In this section an augmented Lagrangian algorithm is constructed to solve problem (2.13), and its convergent analysis is investigate. The constraint function $\tilde{g}(y)$ is defined as

$$\begin{aligned} \tilde{g}(y) &= (g_0, g_1, \cdots, g_n, g_{n+1}, \cdots, g_{n+m})^{\mathrm{T}}, \\ &= (-y^{\mathrm{T}}\tilde{\mu} + c^g_{\alpha,n}(y^{\mathrm{T}}\tilde{\Sigma}y)^{1/2} - \gamma, -x_1, \cdots, -x_m, -h_1, \cdots, -h_n)^{\mathrm{T}}. \end{aligned}$$

Then problem (2.13) is written as

$$\begin{aligned} \min_{y} \quad & f(y), \\ \text{s. t.} \quad & \tilde{g}(y) \leq 0. \end{aligned} \tag{4.1}$$

An augmented Lagrangian function of the problem (4.1) is denoted by

$$L_c(y, \lambda) = f(y) + \frac{1}{2c}\sum_{j=1}^{n+m}\{(\max\{0, \lambda_j + c_j g_j\})^2 - \lambda_j^2\}, \tag{4.2}$$

where $c_j > 0$ and $c > 0$ are called penalty parameters, and $\lambda_j \geq 0$ is the Lagrangian multipliers, $j = 1, \cdots, n+m$. So the augmented Lagrangian algorithm is given as follow.

Begin of Algorithm

Step 1 Initialization

Input parameters, $s > 0$, $\beta \in (0,1)$, $\sigma \in (0, 1/2)$, $\mu > 0, c_j^k > 0 (j = 1, \cdots, n+m)$, $\lambda_j^k = (\lambda_0, \cdots, \lambda_{n+m})$, and initialized points $y^0 = (x_1^0, \cdots, x_n^0, h_1^0, \cdots, h_m^0)$, the end parameter, ϵ_k. Let $k = 0$.

Step 2 Stopping criterion

If $\|\nabla_y L_c(y^k, \lambda^k)\| \leq \epsilon_k$, then the algorithm stop. Otherwise, to begin the next step.

Step 3 Parameter calibration

For $j = 0, \cdots, n+m$, to calibrate the augmented Lagrangian parameter $\lambda_j^{k+1} = \max\{0, \lambda_j^k + c_j^k g_j\}$ and c_j^{k+1} such that $c_j^{k+1} > c_j^k$.

Step 4 Iterative point calibration

Using the Armijo linear search to find the step length $\alpha_k = \beta^{m_k} s$ satisfying

$$L_c(y^k - \alpha_k \nabla_y L_c(y^k, \lambda^k), \lambda^k) - L_c(y^k, \lambda^k) \leq -\sigma \alpha_k \|\nabla_y L_c(y^k, \lambda^k)\|^2, \tag{4.3}$$

where m_k is the minimum positive integer number satisfying the above inequality. The next iterative point is $y^{k+1} = y^k - \alpha_k \nabla_y L_c(y^k, \lambda^k)$.

Step 5 Let $k = k+1$, and go back to Step 1.

End of Algorithm The above algorithm is called as the steepest descend augmented Lagrangian algorithm with the non-exact Armijio search method, and its convergence is discussed as follow.

Theorem 4.1 If the function f and g are continues, for each k the sequence $\{\epsilon_k\}$ and $\{c^k\}$ satisfying $0 < c^k < c^{k+1}$ and $c^k \to \infty$, and there exist the sequence $\{y^k\}$ satisfying

$$\|\nabla_y L_c(y^k, \lambda^k)\| \leq \epsilon_k,$$

and converging to y^*, where $0 \leq \epsilon_k$, and $\epsilon_k \to 0$, such that $\nabla g(y^*)$ has a rank n, and the sequence $\{\lambda_k\}$ is bound, then

$$\{\lambda^{k+1}\} \to \lambda^*,$$

where λ^* and h^* satisfying the following first order necessary condition and effective constraint

$$\nabla f(y^*) + \nabla g(y^*)\lambda^* = 0, \ g(y^*) = 0.$$

Proof Without loss of generality, if the sequence $\{y^k\}_k$ converges to the y^*, then for all k, we have

$$\nabla_y L_c(y^k, \lambda^k) = \nabla f(y^k) + \nabla g(y^k)\bar{\lambda^k}, \tag{4.4}$$

where $\bar{\lambda^k} = \max\{0, \lambda^k + c^k g(y^k)\}$.

Similarly, suppose that $\nabla g(y^k)$ has a rank n for each k, then to multiple

$$(\nabla g(y^k)^{\mathrm{T}} \nabla g(y^k))^{-1} \nabla g(y^k)^{\mathrm{T}}$$

on the both sides of the equation(4.4), we get

$$\bar{\lambda^k} = (\nabla g(y^k)^{\mathrm{T}} \nabla g(y^k))^{-1} \nabla g(y^k)^{\mathrm{T}} (\nabla_y L_c(y^k, \lambda^k) - \nabla f(y^k)). \tag{4.5}$$

According to $\nabla_y L_c(y^k, \lambda^k) \to 0$ when $k \to \infty$, the equation (4.5) implies $\bar{\lambda^k} \to \lambda^*$ when $k \to \infty$, where

$$\lambda^* = -(\nabla g(y^*)^{\mathrm{T}} \nabla g(y^*))^{-1} \nabla g(y^*)^{\mathrm{T}} \nabla f(y^*).$$

Using again $\nabla_y L_c(y^k, \lambda^k) \to 0$ when $k \to \infty$, and the equation (4.5), we get

$$\nabla f(y^*) + \nabla g(y^*) \lambda^* = 0.$$

Furthermore, since $\{\lambda^k\}$ are bound, and $\max\{0, \lambda^k + c^k g(y^k)\} \to \lambda^*$, $c^k g(y^k)$ are bound. Thus $g(y^k) \to 0$ because $c^k \to \infty$, that is, $g(y^*) = 0$. The proof is end.

5 Numerical Experiment

In the numerical analysis, the price risk of the China sugar index (i.e., CSI) is hedged by six sugar futures contracts with the DCE (i.e. Dalian Commodity Exchange) codes SR1109, SR1111, SR1201, SR1203, SR1205 and SR1207, respectively. The daily price data of each futures contracts was taken from Wind database over the period December 2009 through August 2011. Using the statistic software SAS, the covariance matrix $\tilde{\Sigma}$ in the equation (2.10) and the means $\tilde{\mu}$ of the CSI and six futures are given as follows

$$\begin{pmatrix} 0.00020041 & 0.00018769 & 0.00019247 & 0.00020451 & 0.00013485 & 0.00010641 & 0.00004071 \\ 0.00018769 & 0.00018578 & 0.00018679 & 0.00020049 & 0.00013164 & 0.00011003 & 0.00004211 \\ 0.00019247 & 0.00018679 & 0.00020382 & 0.00021674 & 0.00014315 & 0.00011626 & 0.00004027 \\ 0.00020451 & 0.00020049 & 0.00021674 & 0.00024732 & 0.00013472 & 0.00011453 & 0.00005772 \\ 0.00013485 & 0.00013165 & 0.00014315 & 0.00013472 & 0.00014316 & 0.00011557 & 0.00002047 \\ 0.00010641 & 0.00011004 & 0.00011626 & 0.00011454 & 0.00011557 & 0.00018791 & 0.00002995 \\ 0.00004072 & 0.00004212 & 0.00004028 & 0.00005773 & 0.00002048 & 0.00002995 & 0.00004535 \end{pmatrix}$$

and

$$\tilde{\mu} = (0.2230 \;\; 0.4658 \;\; 0.0489 \;\; 0.1024 \;\; 0.4473 \;\; 0.7821 \;\; 0.4321)^{\mathrm{T}},$$

respectively.

It is found that the return rates of the China sugar index and six futures are all not normal distributed, but approximately elliptical distribution with a fat-tail feature. However, it is difficult to determine what kind of of elliptical distribution the return of the seven futures follows. Therefore, the simulation method is used to get the value of $c^g_{\alpha,n}$ and γ, and for $c^g_{0.01,6} = 5.0799$ the algorithm proposed in the section of the article is executed by Matlab programming to solve e problem (2.13), and the corresponding optimal hedging ratio of each futures are given in the Table 2 with some different values of γ.

Table 2: Optimal hedging ratios

γ	SR1109	SR1111	SR1201	SR1203	SR1205	SR1207
0.5	0.1068	0.1068	0.5056	0.1728	0.7153	0.6550
0.7	0.1765	0.1765	0.3837	0.0224	0.0161	0.7531
1	0.3304	0.3304	0.4231	0.1183	0.4854	0.2416
1.5	0.2109	0.2109	0.4133	0.1129	0.0977	0.5757
2	0.2931	0.2931	0.5938	0.3045	0.1341	0.3313
2.5	0.1708	0.1708	0.0030	0.0115	0.1238	0.2061
3	0.1144	0.1144	0.1275	0.0308	0.0445	0.1818

The hedger can make use of the optimal hedging ratios in the Table 2 to decide the number of six futures contracts and China sugar index, and the maximal mean return of the portfolio and the VaR.

6 Conclusions

In this paper we consider a one-period mean-VaR multivariate futures hedging model at one period, in which the value-at-risk with elliptically distributed risk factors is used to measure the risk of portfolio consisting of some spot and futures positions. This model is a special kind of constrains optimization problems. An argument Lagrangian algorithm is proposed by analyzing the optimal condition of this problem in order to obtain the optimal hedging, and its convergence is proved. Furthermore, as a numerical experiment in which the hedger chooses the portfolio including six sugar futures contracts and a China sugar index, the solving method proposed in this paper is carried out to obtain the optimal hedging ratios.

References

[1] Alizadeh F, Goldfarb D. Second-order cone programming[J]. Math. Programming, 2003, 95(1): 3–51.

[2] Alexander G J, Baptista A M. Economic implications of using a mean-VaR model for portfolio selection: a comparison with mean-variance analysis[J]. J. of Economic Dynamics and Control, 2002, 26: 1159–1193.

[3] Alexander G J, Baptista A M. Portfolio performance evaluation using value at risk[J]. J. of Portfolio Management, 2003, 29: 93–102.

[4] Alexander G J, Baptista A M. A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model[J]. Management Science, 2004, 9: 1261–1273.

[5] Albrecht P, Huggenberger M, Pekelis A. VaR and CVaR minimal futures hedging strategies: an analytical approach[J]. working paper, 2011.

[6] Anderson R W, Danthine J P. Hedging and joint production: theory and illustrations[J]. J. of Finance, 1980, 35(2): 487–498.

[7] Cao Z, R. Harris D F, Shen J. Hedging and value at risk: a semi-parametric approach[J]. J. of Futures Markets, 2010, 30(8): 780–794.

[8] Eaker M R, Grant D M. Cross-hedging foreign currency risk[J]. J. of International Money and Finance, 1987, 6: 85–105.

[9] Gagnon L, Lypny G J, McCurdy T H. Hedging foreign currency portfolios[J]. J. of Empirical Finance, 1998, 5(3): 197–220.

[10] Harris R D F, Shen J. Hedging and value at risk[J]. J. of Futures Markets, 2006, 26(4): 369–390.

[11] Hung J C, Chiu C L and Lee M C. Hedging with zero-value at risk hedge ratio[J]. Applied Financial Economics, 2006, 16(3): 259–269.

[12] Howard C T, Dantonio L J. A risk-return measure of hedging effectiveness[J]. J. of Financial and Quantitative Analysis, 1984, 19: 101–112.

[13] Johnson L L. The theory of hedging and speculation in commodity futures[J]. Review of Economic Studies, 1960, 27(3): 139–151.

[14] Kamdem J S. Value-at-risk and expected shortfall for linear portfolios with elliptically distributed risk factors[J]. International J. of Theoretical and Applied Finance, 2005, 8(5): 537–551.

[15] Lien D, Luo X. Estimating multi-period hedge ratios in cointegrated markets[J]. J. of Futures Markets, 1993, 13: 909–920.

[16] Lien D, Shaffer D R. Note on estimating the minimum extended Gini hedge ratio[J]. J. of Futures Markets, 1999, 19: 101–113.

[17] Lien D, Tse Y K. Hedging time-varying downside risk[J]. J. of Futures Markets, 1998, 18: 705–722.

[18] Lien D, Tse Y K. Hedging downside risk with futures contracts[J]. Appl. Financial Economics, 2009, 10: 163–170

[19] Lee M C, Hung J C. Hedging for multi-period downside risk in the presence of jump dynamics and conditional heteroskasticity[J]. Applied Economics, 2007, 39: 2403–2412.

[20] McNeil A J, Frey R, Embrechts P. Quantitative risk management: concepts, techniques and tools[M]. Princeton: Princeton University Press, 2005.

带有风险价值的最优期货套期保值策略

张茂军[1], 南江霞[2], 袁功林[3]

(1.桂林电子科技的大学数学与计算科学学院, 广西 桂林 541004)

(2.桂林电子科技大学广西高校数据分析与计算重点实验室, 广西 桂林 541004)

(3.广西大学数学与信息科学学院, 广西 南宁 530004)

摘 要: 本文研究了带有风险价值约束的期货套期保值优化问题. 用最优化方法获得了套期保值策略的存在性、求解模型的增广拉格朗日算法及其收敛性. 文中的结果推广了期货收益率服从正态分布的单变量套期保值策略的研究, 表现为用服从椭圆分布的随机变量刻画市场风险因子的厚尾特征、用风险价值控制套期保值的风险、构建了均值-VaR组合套期保值理论模型并给出了求解算法.

关键词: 期货合约; 风险价值; 套期保值; 增广拉格朗日算法

MR(2010)**主题分类号**: 91G10; 90C90 **中图分类号**: F830.9