Vol. 35 (2015)
No. 1

数 学 杂 志
J. of Math. (PRC)

CONSTRUCTION OF AN ASSOCIATION SCHEME OVER DIHEDRAL GROUP

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Abstract: In this paper, we study the dihedral group of elements of the equivalence partitioning. By using the group acting on the set, we obtain the construction of association schemes on the dihedral group and all parameters of these schemes are computed. Moreover, we obtain a family of strongly regular graphs. The result enriches the theory of association scheme.

Keywords: dihedral group; association scheme; strongly regular graphs

2010 MR Subject Classification: 05E30

Document code: A **Article ID:** 0255-7797(2015)01-0103-07

1 Introduction

In the theory of (algebraic) combinatorics association schemes play an important role. Association schemes may be seen as colorings of the edges of the complete graph satisfying nice regularity conditions, and they are used in coding theory, design theory, graph theory and group theory. Many chapters of books or complete books are devoted to association schemes (see [1–3]).

We start with a brief introduction to association schemes. For (more) basic results on association schemes and their proofs we refer to [1].

Let V be a finite set of vertices. A d-class association scheme on V consists of a set of $d+1$ symmetric relations $R_0, R_1, \cdots, R_d$ on V, with identity relation $R_0 = \{(x,x)|x \in V\}$, such that any two vertices are in precisely one relation. Furthermore, there are intersection numbers p_{ij}^k such that for any $(x,y) \in R_k$, the number of vertices z such that $(x,z) \in R_i$ and $(z,y) \in R_j$ equals p_{ij}^k.

The nontrivial relations can be considered as graphs, which in our case are undirected. One immediately sees that the respective graphs are regular with degree $n_i = p_{ii}^0$. For the corresponding adjacency matrices A_i the axioms of the scheme are equivalent to

$$\sum_{i=0}^{d} A_i = J,\ A_0 = I,\ A_i = A_i^T,\ A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$

* **Received date:** 2013-10-09 **Accepted date:** 2014-03-03

Foundation item: Supported by Natural Science Foundation of Hebei Province (A2013408009); Natural Science Foundation of Hebei Education Department (ZH2012082); the Doctor Foundation of Langfang Teachers' College (LSBS201205).

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It follows that the adjacency matrices generate a $d+1$-dimensional commutative algebra $\mathfrak{A}$ of symmetric matrices. This algebra was first studied by Bose and Mesner (see [5]) and is called the Bose-Mesner algebra of the scheme. The corresponding algebra of a coherent configuration is called a coherent algebra, or by some authors a cellular algebra or cellular ring (with identity) (see [6]).

Kaishun Wang, Jun Guo, Fenggao Li (see [7]) constructed association schemes by attenuated spaces. Their construction stimulates us to consider the construction of association schemes by dihedral group.

In this paper, we provide a new family of symmetric association schemes, and obtain the following results:

Theorem 1.1 For $n=2m$, suppose that $D_{2n}=\{1,a,a^2,\cdots,a^{n-1},b,ba,ba^2,\cdots,ba^{n-1}\}$ be a dihedral group. Let us define

(1) $R_0=\{(x,x)|x\in D_{2n}\}$;

(2) $R_i=\{(a^l,a^{2m+l-i})|l=0,1,\cdots,2m-1\}\cup\{(a^l,a^{l+i})|l=0,1,\cdots,2m-1\}$ $\cup\{(ba^l,ba^{2m+l-i})|l=0,1,\cdots,2m-1\}\cup\{(ba^l,ba^{l+i})|l=0,1,\cdots,2m-1\}$, where $i=1,2,\cdots,m-1$;

(3) $R_m=\{(a^l,a^{m+l})|l=0,1,\cdots,2m-1\}\cup\{(ba^l,ba^{m+l})|l=0,1,\cdots,2m-1\}$;

(4) $R_{m+1}=\{(a^l,ba^{l+2j})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}\cup\{(ba^l,a^{2m+l-2j})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}$;

(5) $R_{m+2}=\{(a^l,ba^{l+2j+1})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}\cup\{(ba^l,a^{2m+l-2j-1})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}$.

Then we obtain a family of symmetric association scheme $\chi=(D_{2n},\{R_i\}_{0\le i\le m+2})$ with parameters

$$d=m+2;\quad v=2n;\quad n_i=\begin{cases}1, & i=0,\text{ or } i=m,\\ 2, & 1\le i\le m-1,\\ 0, & i=m+1\text{ or } i=m+2,\end{cases}$$

intersection numbers p_{ij}^k given by (1).

2 Some Lemmas

In this section we give some lemmas, which are needed in the proof of Theorem 1.1.

We assume that $D_{2n}=\{1,a,a^2,\cdots,a^{n-1},b,ba,ba^2,\cdots,ba^{n-1}\}$ is a dihedral group. It is know that $a^jb=ba^{-j}$ and $(ba^j)^{-1}=ba^j$. Here $j\in\{0,1,\cdots,n-1\}$.

Lemma 2.1 For $n=2m$, the conjugacy classes of the dihedral group D_{2n} are the following.

$$C_0=\{1\};\ C_i=\{a^i,a^{2m-i}\},i=1,2,\cdots,m-1;\ C_m=\{a^m\}$$

and

$$C_{m+1}=\{b,ba^2,\cdots,ba^{2m-2}\};\ C_{m+2}=\{ba,ba^3,\cdots,ba^{2m-1}\}.$$

Proof For any $a^j \in D_{2n}$ and $ba^j \in D_{2n}, j = 0, 1, \cdots, 2m-1$, pick $a^i \in D_{2n}, i = 1, 2, \cdots, m-1$. We have $a^j a^i (a^j)^{-1} = a^{j+i-j} = a^i$ and $(ba^j)a^i(ba^j)^{-1} = ba^{j+i}ba^j = a^{-i} = a^{2m-i}$. Therefore the conjugacy class containing a^i is

$$C_i = \{a^i, a^{2m-i}\}, i = 1, 2, \cdots, m-1.$$

Picking $a^m \in D_{2n}$, we have $a^j a^m (a^j)^{-1} = a^{j+m-j} = a^m$ and $(ba^j)a^m(ba^j)^{-1} = ba^{j+m}ba^j = a^{-m} = a^m$. Therefore the conjugacy class containing a^m is

$$C_m = \{a^m\}.$$

Picking $b \in D_{2n}$, we have $a^j b (a^j)^{-1} = ba^{-2j}$ and $(ba^j)b(ba^j)^{-1} = ba^{2j}$. Therefore the conjugacy class containing b is

$$C_{m+1} = \{b, ba^2, \cdots, ba^{2m-2}\}.$$

Picking $ba \in D_{2n}$, we have $a^j ba (a^j)^{-1} = ba^{-2j+1}$ and $(ba^j)ba(ba^j)^{-1} = ba^{2j-1}$. Therefore the conjugacy class containing ba is

$$C_{m+2} = \{ba, ba^3, \cdots, ba^{2m-1}\}.$$

Let $C_0 = \{1\}$. Since

$$\bigcup_{i=0}^{m+2} C_i = D_{2n},$$

the lemma holds.

Let $C_0, C_1, \cdots, C_{m+2}$ be the set of all the conjugacy classes of the group D_{2n} as Lemma 2.1. We define $X = D_{2n}$ and $(x, y) \in R_i \iff yx^{-1} \in C_i$. Then $(X, \{R_i\}_{0 \le i \le m+2})$ becomes an association scheme.

Lemma 2.2 For $n = 2m$, let D_{2n} be a dihedral group. Suppose that $(x, y) \in R_i \iff yx^{-1} \in C_i, i = 0, 1, \cdots, m+2$. Then

(1) $R_0 = \{(x, x) | x \in D_{2n}\}$.

(2) $R_i = \{(a^l, a^{2m+l-i}) | l = 0, 1, \cdots, 2m-1\} \cup \{(a^l, a^{l+i}) | l = 0, 1, \cdots, 2m-1\} \cup \{(ba^l, ba^{2m+l-i}) | l = 0, 1, \cdots, 2m-1\} \cup \{(ba^l, ba^{l+i}) | l = 0, 1, \cdots, 2m-1\}$, where $i = 1, 2, \cdots, m-1$.

(3) $R_m = \{(a^l, a^{m+l}) | l = 0, 1, \cdots, 2m-1\} \cup \{(ba^l, ba^{m+l}) | l = 0, 1, \cdots, 2m-1\}$.

(4) $R_{m+1} = \{(a^l, ba^{l+2j}) | l = 0, 1, \cdots, 2m-1, j = 0, 1, \cdots, m-1\} \cup \{(ba^l, a^{2m+l-2j}) | l = 0, 1, \cdots, 2m-1, j = 0, 1, \cdots, m-1\}$.

(5) $R_{m+2} = \{(a^l, ba^{l+2j+1}) | l = 0, 1, \cdots, 2m-1, j = 0, 1, \cdots, m-1\} \cup \{(ba^l, a^{2m+l-2j-1}) | l = 0, 1, \cdots, 2m-1, j = 0, 1, \cdots, m-1\}$.

Proof Let $b^k a^l \in D_{2n}$. If $b^k a^l x^{-1} \in C_i, i = 1, \cdots, m-1$, then $b^k a^l x^{-1} = a^i$ or $b^k a^l x^{-1} = a^{2m-i}$. Therefore

$$x = a^{2m-i} b^k a^l = b^k a^{(-1)^k (2m-i)+l} = \begin{cases} a^{2m+l-i}, & \text{if } k = 0, \\ ba^{l+i}, & \text{if } k = 1, \end{cases}$$

or

$$x=(a^{2m-i})^{-1}b^ka^l=b^ka^{(-1)^ki+l}=\begin{cases} a^{l+i}, & \text{if } k=0,\\ ba^{2m+l-i}, & \text{if } k=1.\end{cases}$$

Hence

$$\begin{aligned} R_i= \ & \{(a^l,a^{2m+l-i})|l=0,1,\cdots,2m-1\}\cup\{(a^l,a^{l+i})|l=0,1,\cdots,2m-1\}\\ &\cup\{(ba^l,ba^{2m+l-i})|l=0,1,\cdots,2m-1\}\cup\{(ba^l,ba^{l+i})|l=0,1,\cdots,2m-1\},\end{aligned}$$

where $i=1,2,\cdots,m-1$.
If $b^ka^lx^{-1}\in C_m$, then $b^ka^lx^{-1}=a^m$. Therefore

$$x=(a^m)^{-1}b^ka^l=a^mb^ka^l=\begin{cases} a^{m+l}, & \text{if } k=0\\ ba^{l-m}, & \text{if } k=1\end{cases}=\begin{cases} a^{m+l}, & \text{if } k=0,\\ ba^{m+l}, & \text{if } k=1.\end{cases}$$

Hence

$$R_m=\{(a^l,a^{m+l})|l=0,1,\cdots,2m-1\}\cup\{(ba^l,ba^{m+l})|l=0,1,\cdots,2m-1\}.$$

If $b^ka^lx^{-1}\in C_{m+1}$, then $b^ka^lx^{-1}=ba^{2j}, j=0,1,\cdots,m-1$. Therefore

$$x=(ba^{2j})^{-1}b^ka^l=b^{k+1}a^{(-1)^k2j+l}=\begin{cases} ba^{l+2j}, & \text{if } k=0,\\ a^{2m+l-2j}, & \text{if } k=1.\end{cases}$$

Hence

$$\begin{aligned} R_{m+1}= \ & \{(a^l,ba^{l+2j})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}\\ &\cup\{(ba^l,a^{2m+l-2j})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}.\end{aligned}$$

If $b^ka^lx^{-1}\in C_{m+2}$, then $b^ka^lx^{-1}=ba^{2j+1}, j=0,1,\cdots,m-1$. Therefore

$$x=(ba^{2j+1})^{-1}b^ka^l=b^{k+1}a^{(-1)^k(2j+1)+l}=\begin{cases} ba^{l+2j+1}, & \text{if } k=0,\\ a^{2m+l-2j-1}, & \text{if } k=1.\end{cases}$$

Hence

$$\begin{aligned} R_{m+2}= \ & \{(a^l,ba^{l+2j+1})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}\\ &\cup\{(ba^l,a^{2m+l-2j-1})|l=0,1,\cdots,2m-1,j=0,1,\cdots,m-1\}.\end{aligned}$$

3 Proof of Theorem 1.1

In this section we shall prove Theorem 1.1 and compute all the parameters of the corresponding association scheme.

By Lemma 2.2, for $n=2m$, we have that $\chi=(D_{2n},\{R_i\}_{0\le i\le m+2})$ is an association scheme, and obtain the following propositions.

Proposition 3.1 Suppose that $n=2m$. Then the set of the adjacency matrices of the association scheme χ is given by the following matrices.

(1) $A_0 = I$;

(2) $A_i = \begin{bmatrix} S^{n-i} + S^i & 0 \\ 0 & S^{n-i} + S^i \end{bmatrix}, i = 1, \cdots, m-1$;

(3) $A_m = \begin{bmatrix} S^m & 0 \\ 0 & S^m \end{bmatrix}$;

(4) $A_{m+1} = \begin{bmatrix} 0 & \sum\limits_{i=0}^{m-1} S^{2i} \\ \sum\limits_{i=0}^{m-1} S^{2i} & 0 \end{bmatrix}$;

(5) $A_{m+2} = \begin{bmatrix} 0 & \sum\limits_{i=0}^{m-1} S^{2i+1} \\ \sum\limits_{i=0}^{m-1} S^{2i+1} & 0 \end{bmatrix}$. Here $S = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix}$.

Proposition 3.2 Suppose that $n = 2m$. Then the intersection numbers of the association scheme χ are as following.

$$p_{ij}^k = \begin{cases} 1, \text{if } \{i,j\} = \{0,l\} \text{ and } k = l, l = 0, 1, \cdots, m+2, \\ 1, \text{if } 1 \le i+j \le m \text{ and } k = i+j, 1 \le i,j \le m-1, \\ 1, \text{if } m+1 \le i+j \le 2m-1 \text{ and } k = n-(i+j), 1 \le i,j \le m-1, \\ 1, \text{if } i \ne j \text{ and } k = |i-j|, 1 \le i,j,k \le m-1, \\ 2, \text{if } 1 \le i = j \le m-1 \text{ and } k = 0, \\ 1, \text{if } \{i,j\} = \{m,l\} \text{ and } k = m-l, l = 1, 2, \cdots, m-1, \\ 2, \text{if } \{i,j\} = \{m+1,l\} \text{ and } k = m+1, l = 2, 4, \cdots, 2m-2, \\ 2, \text{if } \{i,j\} = \{m+1,l\} \text{ and } k = m+2, l = 1, 3, \cdots, 2m-1, \\ 2, \text{if } \{i,j\} = \{m+2,l\} \text{ and } k = m+2, l = 2, 4, \cdots, 2m-2, \\ 2, \text{if } \{i,j\} = \{m+2,l\} \text{ and } k = m+1, l = 1, 3, \cdots, 2m-1, \\ 1, \text{if } \{i,j\} = \{m,m+1\} \text{ and } k = m+1, m \text{ is even}, \\ 1, \text{if } \{i,j\} = \{m,m+1\} \text{ and } k = m+2, m \text{ is odd}, \\ 1, \text{if } \{i,j\} = \{m,m+2\} \text{ and } k = m+2, m \text{ is even}, \\ 1, \text{if } \{i,j\} = \{m,m+2\} \text{ and } k = m+1, m \text{ is odd}, \\ m, \text{if } \{i,j\} = \{m+1,m+2\} \text{ and } k = m+2, \\ 1, \text{if } i = j = m \text{ and } k = 0, \\ m, \text{if } i = j = k = m+1, \\ m, \text{if } i = j = m+2, k = m+1, \\ 0, \text{otherwise}. \end{cases} \tag{1}$$

Proof By Proposition 3.1, we obtain the results as following. If $0 \le i \le m+2$, we have

$$A_0 A_i = A_i A_0 = A_i.$$

If $1 \le i, j \le m-1$, we have

$$\begin{aligned}
& A_iA_j = A_jA_i \\
= & \begin{bmatrix} S^{n-(i+j)} + S^{n+j-i} + S^{n+i-j} + S^{i+j} & 0 \\ 0 & S^{n-(i+j)} + S^{n+j-i} + S^{n+i-j} + S^{i+j} \end{bmatrix} \\
= & \begin{bmatrix} S^{n-(i+j)} + S^{i+j} & 0 \\ 0 & S^{n-(i+j)} + S^{i+j} \end{bmatrix} + \begin{bmatrix} S^{n+j-i} + S^{n+i-j} & 0 \\ 0 & S^{n+j-i} + S^{n+i-j} \end{bmatrix} \\
= & \begin{cases} A_{i+j} + A_{i-j}, & i > j, \\ A_{i+j} + A_{j-i}, & i < j, \\ A_{2i} + 2A_0, & i = j. \end{cases}
\end{aligned}$$

If $1 \le i \le m-1$, we have

$$A_iA_{m+1} = A_{m+1}A_i = \begin{cases} 2A_{m+1}, \text{if } i \text{ is even}, \\ 2A_{m+2}, \text{if } i \text{ is odd}, \end{cases}$$

and

$$A_iA_{m+2} = A_{m+2}A_i = \begin{cases} 2A_{m+2}, \text{if } i \text{ is even}, \\ 2A_{m+1}, \text{if } i \text{ is odd}. \end{cases}$$

Note that

$$A_mA_{m+1} = A_{m+1}A_m = \begin{cases} A_{m+1}, \text{if } m \text{ is even}, \\ A_{m+2}, \text{if } m \text{ is odd}, \end{cases}$$

$$A_mA_{m+2} = A_{m+2}A_m = \begin{cases} A_{m+2}, \text{if } m \text{ is even}, \\ A_{m+1}, \text{if } m \text{ is odd}. \end{cases}$$

$$A_m^2 = A_0,$$

$$A_{m+1}A_{m+2} = A_{m+2}A_{m+1} = \begin{bmatrix} 0 & m\sum\limits_{i=0}^{m-1} S^{2i+1} \\ m\sum\limits_{i=0}^{m-1} S^{2i+1} & 0 \end{bmatrix} = mA_{m+2},$$

and

$$A_{m+1}^2 = A_{m+2}^2 = \begin{bmatrix} 0 & m\sum\limits_{i=0}^{m-1} S^{2i} \\ m\sum\limits_{i=0}^{m-1} S^{2i} & 0 \end{bmatrix} = mA_{m+1}.$$

Therefore the desired result follows.

The proof of Theorem 1.1 is completed.

4 Strongly Regular Graphs

Bannai and Ito (see [1]) introduced strongly regular graphs. In this section we construct a family of strongly regular graphs from above association schemes $\chi = (D_{2n}, \{R_i\}_{0 \le i \le m+2})$.

A simple graph (X, R) is called a strongly regular graph with parameters (n, k, λ, μ) if the following conditions are satisfied:

(a) $|X| = n$;

(b) For each $x \in X$, we have $|\{y \in X|(x, y) \in R\}| = k$;

(c) For each pair of x, y with $(x, y) \in R$, we have $|\{z \in X|(x, z) \in R, (y, z) \in R\}| = \lambda$;

(d) For each pair of x, y with $(x, y) \notin R$, we have $|\{z \in X|(x, z) \in R, (y, z) \in R\}| = \mu$.

Let

$$
\begin{aligned}
&X = D_{2n}, \\
&R = R_1 = \{(a^l, a^{2m+l-1})|l = 0, 1, \cdots, 2m-1\} \cup \{(a^l, a^{l+1})|l = 0, 1, \cdots, 2m-1\} \\
&\cup\{(ba^l, ba^{2m+l-1})|l = 0, 1, \cdots, 2m-1\} \cup \{(ba^l, ba^{l+1})|l = 0, 1, \cdots, 2m-1\}.
\end{aligned}
$$

By Theorem 1.1, we obtain the following theorem.

Theorem 4.1 The graph $\Gamma = (X, R)$ is a strongly regular graph with parameters $(2n, 2, 0, 2)$.

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二面体群上构作的结合方案

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摘要: 本文研究了二面体群的元素的等价划分问题. 利用群在集合上的作用, 在二面体群上构造了一类新的结合方案, 并且计算了这类结合方案的所有参数. 进一步, 得到了一类强正则图. 所得到的结果丰富了结合方案理论.

关键词: 二面体群; 结合方案; 强正则图

MR(2010)**主题分类号:** 05E30 **中图分类号:** O157