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EXISTENCE AND CONCENTRATION BEHAVIOR OF NODE SOLUTIONS FOR A KIRCHHOFF EQUATION IN $\mathbb{R}^3$

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Abstract: In this paper, we investigate a nonlinear Kirchhoff type problem. By virtue of mini-max and truncation methods, we obtain the existence of nodal solutions and concentration behavior to Kirchhoff type problem, which extend the results in [4].

Keywords: truncation method; nodal solution; mini-max method; concentration behavior

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1 Introduction

In present paper, we concern with the following semi-linear Kirchhoff type equation

$$\begin{cases} -(\varepsilon^2 a+\varepsilon b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=f(u), \quad x\in\mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $\varepsilon>0$ is a parameter, $a,b>0$ are positive constants, $V(x)$ is a Hölder continuous function satisfying

$(\mathbf{V_1})$ $V(x)\ge\alpha>0,\ x\in\mathbb{R}^3$ for some constant $\alpha>0$.

$(\mathbf{V_2})$ There exists a bounded domain Λ compactly contained in $\mathbb{R}^3$ such that

$$V_0:=\inf_{x\in\Lambda}V(x)<\inf_{x\in\partial\Lambda}V(x)$$

and $f\in C^1(\mathbb{R})$ satisfying

$(\mathbf{f_1})$ $f(s)=o(s^3)$ as $s\to 0$.

$(\mathbf{f_2})$ $\displaystyle\lim_{|s|\to+\infty}\frac{|f(s)|}{|s|^p}=0$ for some $3<p<5$.

$(\mathbf{f_3})$ there exists some $\theta>4$ such that

$$0<\theta F(s)\le sf(s),\forall s\neq 0,$$

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where $F(s)=\int_0^s f(t)dt$.

$(\mathbf{f_4})$ $\dfrac{f(s)}{|s|^3}$ is increasing for any $s\neq 0$.

Equation (1.1) with $a=1, b=0$ and $\mathbb{R}^3$ replaced by $\mathbb{R}^N$, reduces to the well-known Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^N. \tag{1.2}$$

Equation (1.2) arises in different models. For instance, they are involved with the existence of standing waves of the nonlinear Schrödinger equations

$$i\varepsilon\partial_t z=-\varepsilon^2\Delta z+(V(x)+E)z-f(z),\quad \forall x\in\mathbb{R}^N, \tag{1.3}$$

where $f(s)=|s|^{p-2}s, 2<p<2^*:=2N/(N-4)$. A standing waves of (1.3) is a solution of the form $z(x,t)=\exp(-iEt/\varepsilon)u(x)$, where u is a solution of (1.2).

For (1.1), if $\varepsilon=1, V(x)=0$ and $\mathbb{R}^3$ replaced by Ω, it reduces to the following Kirchhoff type problem

$$\begin{cases} -(a+b\displaystyle\int_\Omega|\nabla u|^2dx)\Delta u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases} \tag{1.4}$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain.

Equation (1.4) is related to the stationary analogue of the equation

$$\begin{cases} u_{tt}-(a+b\displaystyle\int_\Omega|\nabla u|^2dx)\Delta u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases} \tag{1.5}$$

Equations of this type were first proposed by Kirchhoff in [19] to describe the transversal oscillations of a stretched string. Equation (1.5) began to attract much attention since the work of Lions [18] introduced an abstract framework to the problem. More results can refer to, for example [16–17]. Meanwhile, the presence of the term $(\int_\Omega|\nabla u|^2dx)\Delta u$ implies that the above two equation are no longer a point-wise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problem particularly interesting.

In recent years, a lot of work has been done by many authors related to equation (1.2) and (1.4). We can refer to [1–3, 5–7, 9–12, 14] and their references therein.

More recently, He and Zou in [15] considered the following general equation

$$\begin{cases} -(\varepsilon^2a+\varepsilon b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=f(u),\ u>0, & x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3). \end{cases} \tag{1.6}$$

By using Ljusternik-Schnirelmnn theory (see [13]) and mini-max methods, the multiplicity of positive solutions, which concentrate on the minima of $V(x)$ as $\varepsilon\to 0$, are obtained. But to the best of our knowledge, the existence and concentration behavior of the sign changing solutions to (1.1) have not ever been studied. Moreover, as far as we know, the existence and concentration behavior of node solutions are very interesting in both mathematicians and physicians. Fortunately, in [4], the author studied the following equation

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

Under some given conditions with $V(x), f(u)$, the existence of node solutions was obtained and such a solution has just one positive and negative peaks which are located around local minimal of $V(x)$.

Motivated by the above papers, we study the existence and concentration behavior of nodal solutions of problem (1.1). In our present paper, we mainly employ the method used in [4]. However, compared with [4], the term $(\int_\Omega |\nabla u|^2 dx)\Delta u$ and the lack of compactness of the embed-ding of $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3), 2 < p < 6$ cause us more difficulties. So we need to find some new arguments and our work is meaningful.

Our main result is as follows:

Theorem 1.1 Suppose that $V(x)$ satisfies $(\mathbf{V_1}) - (\mathbf{V_2})$ and f satisfies $(\mathbf{f_1}) - (\mathbf{f_4})$, then there exists $\varepsilon_0 > 0$ such that problem (1.1) possesses a nodal solutions $u_\varepsilon \in H^1(\mathbb{R}^3)$ for every $\varepsilon \in (0, \varepsilon_0)$. Moreover, u_ε possesses just one positive maximum point $P_\varepsilon^1 \in \Lambda$ and one negative minimum point $P_\varepsilon^2 \in \Lambda$. We also obtain $\lim\limits_{\varepsilon\to 0} V(P_\varepsilon^i) = V_0$ $(i = 1, 2)$ and

$$|u_\varepsilon(x)| \leq M\Big[\exp(-\beta|\frac{x - P_\varepsilon^1}{\varepsilon}|) + \exp(-\beta|\frac{x - P_\varepsilon^2}{\varepsilon}|)\Big],$$

where M, β are some positive constants.

To verify Theorem 1.1, we mainly employ the framework used in [4]. We first exploit the truncation method to modify the nonlinearity $f(u)$ in order to obtain the existence of a nodal solution. Furthermore, to show the phenomenon of concentration, we establish an upper estimate of the energy for the solution and make a careful study of its profile obtaining a relation between peak points, which imply that these points are concentrated around local minimal of V.

2 Preliminaries and Notations

In this section, we introduce some notations and prove the existence of a nodal solution to equation (1.1). Throughout this paper, we denote by H the Hilbert space given by

$$H = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < +\infty\}$$

endowed with the norm denote by $\|u\| = (\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx)^{1/2}$. It is clear that solutions of (1.1) are the critical points of the functional $I_\varepsilon : H \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2 a|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2 - \int_{\mathbb{R}^3}F(u)dx,$$

where $F(u) = \displaystyle\int_0^u f(s)ds$. By $(\mathbf{f_1}) - (\mathbf{f_2})$, I_ε is well defined and $I_\varepsilon \in C^1(H,\mathbb{R})$. Similar to the argument in [2], we choose $k > 0$ such that $k > \frac{\theta}{\theta-2} > 1$. Take $a_1 > 0, a_2 < 0$ such that $\frac{f(a_i)}{a_i} = \frac{\alpha}{k}, i = 1, 2$, where α is as in $(\mathbf{V_1})$. Meanwhile, we set

$$\tilde{f}(s) = \begin{cases} \dfrac{\alpha s}{k}, \ s > a_1 \text{ or } s < a_2, \\ f(s), \ a_2 \le s \le a_1 \end{cases}$$

and define the functional

$$g(x,s) = \chi_\Lambda(x)f(s) + (1-\chi_\Lambda(x))\tilde{f}(s),$$

where $\chi_\Lambda(x)$ denotes the characteristic function of Λ. Using the conditions $(\mathbf{f_1}) - (\mathbf{f_4})$, it is easy to verify $g(x,s)$ is a Carethéodory function and satisfies the following conditions:

$(\mathbf{g_1})$ $g(x,s) = o(|s|^3)$ as $s \to 0$, uniformly in $x \in \mathbb{R}^3$.

$(\mathbf{g_2})$ There exist some $3 < p < 5$ such that $\displaystyle\lim_{|s|\to+\infty}\frac{g(x,s)}{|s|^p} = 0$.

$(\mathbf{g_3})$ There exist some $\theta > 4$ such that

$$0 < \theta G(x,s) \le sg(x,s), \forall x \in \Lambda, s \neq 0,$$
$$0 < 2G(x,s) \le sg(x,s) \le \frac{1}{k}V(x)s^2, \forall x \in \mathbb{R}^3 \setminus \Lambda, s \neq 0,$$

where $G(x,s) = \displaystyle\int_0^s g(x,t)dt$.

$(\mathbf{g_4})$ The function $\dfrac{g(x,s)}{|s|^3}$ is increasing for any $x \in \mathbb{R}^3, s \neq 0$.

In the following discussion, we consider the following penalized problem

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^3, \\ u \in H. \end{cases} \tag{2.1}$$

Note that if u is a nodal solution of (2.1) with $a_2 \le u(x) \le a_1$, then $u(x)$ is indeed a nodal solution to equation (1.1).

For equation (2.1), the corresponding energy functional $J_\varepsilon : H \to \mathbb{R}$ is defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2 a|\nabla u|^2 + V(x)u^2)dx + \frac{\varepsilon b}{4}(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2 - \int_{\mathbb{R}^3}G(x,u)dx$$

and for any $\varphi \in H$,

$$\langle J'_\varepsilon(u), \varphi\rangle = (\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u|^2dx)\int_{\mathbb{R}^3}\nabla u\nabla\varphi dx + \int_{\mathbb{R}^3}V(x)u\varphi dx - \int_{\mathbb{R}^3}g(x,u)\varphi dx.$$

To prove the existence of a nodal solution, we define

$$M_\varepsilon = \{u \in H : u^\pm \not\equiv 0, J'_\varepsilon(u)u^\pm = 0\}$$

and

$$c_\varepsilon = \inf_{u \in M_\varepsilon} J_\varepsilon(u).$$

Lemma 2.1 c_ε is achieved by some $u_\varepsilon \in M_\varepsilon$. Moreover, u_ε is a nodal solution of equation (2.1).

Proof Since $(\mathbf{g_1}) - (\mathbf{g_3})$, there exist constants $C > 0, \mu > 0$ such that for any $u \in M_\varepsilon$, we have

$$J_\varepsilon(u) \geq C\|u\|^2, \qquad \int_\Lambda |u^\pm|^{p+1}dx \geq \mu > 0.$$

Take a sequence $\{u_n\} \subset M_\varepsilon$ such that $J_\varepsilon(u_n) \to c_\varepsilon$, then $\{u_n\}$ is bounded in H. Thus, there exist a subsequence, still denoted by $\{u_n\}$ and a function $u \in H$ such that $u_n \rightharpoonup u$ in H. Thus, we have

$$\int_\Lambda |u^\pm|^{p+1}dx = \lim_{n\to+\infty} \int_\Lambda |u_n^\pm|^{p+1}dx \geq \mu > 0.$$

Therefore, $u^\pm \not\equiv 0$.

Now, we claim that $J'_\varepsilon(u^\pm)u^\pm \leq 0$.

In fact, by the lower semi-continuous of the norm and Fatou Lemma, we derive that

$$\begin{aligned}
& \int_{\mathbb{R}^3} (\varepsilon^2 a|\nabla u^\pm|^2 + V(x)|u^\pm|^2)dx + b\varepsilon(\int_{\mathbb{R}^3} |\nabla u^\pm|^2dx)^2 - \int_{\mathbb{R}^3\setminus\Lambda} g(x,u^\pm)u^\pm dx \\
= & \int_{\mathbb{R}^3} \varepsilon^2 a|\nabla u^\pm|^2 + \int_\Lambda V(x)|u^\pm|^2dx + \int_{\mathbb{R}^3\setminus\Lambda} (1-\frac{1}{k})V(x)|u^\pm|^2dx + b\varepsilon(\int_{\mathbb{R}^3} |\nabla u^\pm|^2dx)^2 \\
& + \int_{\mathbb{R}^3\setminus\Lambda} (\frac{1}{k}V(x)|u^\pm|^2 - g(x,u^\pm)u^\pm)dx \\
\leq & \liminf_{n\to\infty} \Big[\int_{\mathbb{R}^3} \varepsilon^2 a|\nabla u_n^\pm|^2dx + \int_\Lambda V(x)|u_n^\pm|^2dx + \int_{\mathbb{R}^3\setminus\Lambda} (1-\frac{1}{k})V(x)|u_n^\pm|^2dx \\
& + b\varepsilon(\int_{\mathbb{R}^3} |\nabla u_n^\pm|^2dx)^2 \Big] + \liminf_{n\to\infty} \Big[\int_{\mathbb{R}^3\setminus\Lambda} (\frac{1}{k}V(x)|u_n^\pm|^2 - g(x,u_n^\pm)u_n^\pm)dx \Big] \\
\leq & \liminf_{n\to\infty} \Big[(\int_{\mathbb{R}^3} (\varepsilon^2 a|\nabla u_n^\pm|^2 + \int_\Lambda V(x)|u_n^\pm|^2)dx + \int_{\mathbb{R}^3\setminus\Lambda} (1-\frac{1}{k})V(x)|u_n^\pm|^2dx \\
& + b\varepsilon(\int_{\mathbb{R}^3} |\nabla u_n^\pm|^2dx)^2) + \int_{\mathbb{R}^3\setminus\Lambda} (\frac{1}{k}V(x)|u_n^\pm|^2 - g(x,u_n^\pm)u_n^\pm)dx \Big] \\
= & \liminf_{n\to\infty} \Big[\int_{\mathbb{R}^3} (\varepsilon^2 a|\nabla u_n^\pm|^2 + V(x)|u_n^\pm|^2)dx + b\varepsilon(\int_{\mathbb{R}^3} |\nabla u_n^\pm|^2dx)^2 - \int_{\mathbb{R}^3\setminus\Lambda} g(x,u_n^\pm)u_n^\pm dx \Big] \\
= & \liminf_{n\to\infty} \Big[\int_{\mathbb{R}^3} g(x,u_n^\pm)u_n^\pm dx - \int_{\mathbb{R}^3\setminus\Lambda} g(x,u^\pm)u^\pm dx \Big] \\
= & \liminf_{n\to\infty} \int_\Lambda g(x,u_n^\pm)u_n^\pm dx \\
= & \int_\Lambda g(x,u^\pm)u^\pm dx.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^3}(\varepsilon^2 a|\nabla u^{\pm}|^2+V(x)|u^{\pm}|^2)dx+b\varepsilon(\int_{\mathbb{R}^3}|\nabla u^{\pm}|^2dx)^2-\int_{\mathbb{R}^3}g(x,u^{\pm})u^{\pm}dx\le 0,$$

i.e., $\langle J'_\varepsilon(u^{\pm}),u^{\pm}\rangle\le 0$. By the above statement, there exists constant $t^{\pm}\in(0,1]$ such that $\langle J'_\varepsilon(t^{\pm}u^{\pm}),t^{\pm}u^{\pm}\rangle=0$, i.e., $u_\varepsilon=t^+u^++t^-u^-\in M_\varepsilon$. In addition, by $(\mathbf{g_4})$, $(\mathbf{V_1})$, $k>1$ and Fatou lemma, then we have

$$\begin{aligned}
&J_\varepsilon(t^{\pm}u^{\pm})=\frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2a|t^{\pm}\nabla u^{\pm}|^2+V(x)|t^{\pm}u^{\pm}|^2)dx+\frac{\varepsilon b}{4}(\int_{\mathbb{R}^3}|t^{\pm}\nabla u^{\pm}|^2dx)^2-\int_{\mathbb{R}^3}G(x,t^{\pm}u^{\pm})dx\\
&=\int_{\mathbb{R}^3}[\frac{1}{4}g(x,t^{\pm}u^{\pm})t^{\pm}u^{\pm}-G(x,t^{\pm}u^{\pm})]dx+\frac{1}{4}\int_{\mathbb{R}^3}(\varepsilon^2a|t^{\pm}\nabla u^{\pm}|^2+V(x)|t^{\pm}u^{\pm}|^2)dx\\
&\le\liminf_{n\to\infty}\Big[\int_{\mathbb{R}^3}(\frac{1}{4}g(x,t^{\pm}u_n^{\pm})t^{\pm}u_n^{\pm}-G(x,t^{\pm}u_n^{\pm}))dx+\frac{1}{4}\int_{\mathbb{R}^3}(\varepsilon^2a|t^{\pm}\nabla u_n^{\pm}|^2+V(x)|t^{\pm}u_n^{\pm}|^2)dx)\Big]\\
&\le\liminf_{n\to\infty}\Big[\int_{\mathbb{R}^3}(\frac{1}{4}g(x,u_n^{\pm})u_n^{\pm}-G(x,u_n^{\pm}))dx+\frac{1}{4}\int_{\mathbb{R}^3}(\varepsilon^2a|\nabla u_n^{\pm}|^2+V(x)|u_n^{\pm}|^2)dx\Big]\\
&=\liminf_{n\to\infty}J_\varepsilon(u_n^{\pm}).
\end{aligned}$$

Thus

$$\begin{aligned}
c_\varepsilon &\le J_\varepsilon(u_\varepsilon)=J_\varepsilon(t^+u^+)+J_\varepsilon(t^-u^-)\\
&\le \liminf_{n\to\infty}[J_\varepsilon(u_n^+)+J_\varepsilon(u_n^-)]\\
&= \liminf_{n\to\infty}J_\varepsilon(u_n)\\
&= c_\varepsilon,
\end{aligned}$$

i.e., c_ε is attained by the function u_ε. Furthermore, by the elliptic regularity arguments, u_ε is a classical nodal solution of problem (2.1).

3 Estimate of the Energy

In this section, we turn to estimate the energy of u_ε. Let

$$f_{\pm}(s)=\begin{cases} f(s), & \pm s\ge 0\\ 0, & \text{otherwise.}\end{cases}$$

Suppose $\omega_{\pm}\in H^1(\mathbb{R}^3)$ are respectively the least energy nodal solutions of the following limit equation:

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V_0u=f_{\pm}(u),\quad x\in\mathbb{R}^3,$$

that is $\omega_{\pm}$ satisfy $c_{V_0}^{\pm}:=J_{V_0}^{\pm}(\omega_{\pm})=\inf\limits_{u\in H\setminus\{0\}}\sup\limits_{\tau\ge0}J_{V_0}^{\pm}(\tau u)$, where

$$J_{V_0}^{\pm}(\omega_{\pm})=\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla\omega_{\pm}|^2+V_0^{\pm}\omega_{\pm}{}^2)dx+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla\omega_{\pm}|^2dx)^2-\int_{\mathbb{R}^3}F_{\pm}(u)dx$$

and $F_{\pm}(s)=\displaystyle\int_0^s f_{\pm}(t)dt$.

Without loss of generality, we assume $\omega_+(0)=\max\limits_{x\in\mathbb{R}^3}\omega_+(x)$, $\omega_-(0)=\min\limits_{x\in\mathbb{R}^3}\omega_-(x)$.

Lemma 3.1 Given $\varepsilon>0$, the function u_ε satisfies

$$\limsup_{\varepsilon\to 0}\varepsilon^{-3}J_\varepsilon(u_\varepsilon)\le c_{V_0}^{+}+c_{V_0}^{-}.$$

Proof Let $x_0\in\mathrm{int}(\Lambda)$ be such that $V(x_0)=V_0$. Choosing $r>0$ such that $B_r(x_0)\subset\mathrm{int}(\Lambda)$ and η is a smooth function, $0\le\eta\le 1$, $|\nabla\eta|\le C$ and

$$\eta(x)=\begin{cases}1, & |x|\le r/4,\\ 0, & |x|\ge r/2.\end{cases}$$

Denote $\omega_{\varepsilon,\pm}(x)=\eta(x-x_0)\omega_{\pm}(\frac{x-x_0}{\varepsilon})$. By condition $(\mathbf{g_1})-(\mathbf{g_3})$, there exists $t_{\varepsilon,\pm}>0$ such that $J_\varepsilon(t_{\varepsilon,\pm}\omega_{\varepsilon,\pm})=\max\limits_{t>0}J_\varepsilon(t\omega_{\varepsilon,\pm})$, then

$$J'_\varepsilon(t_{\varepsilon,\pm}\omega_{\varepsilon,\pm})t_{\varepsilon,\pm}\omega_{\varepsilon,\pm}=0.$$

Consider the function

$$\omega_\varepsilon=t_{\varepsilon,+}\omega_{\varepsilon,+}+t_{\varepsilon,-}\omega_{\varepsilon,-},$$

then $\omega^{\pm}=t_{\varepsilon,\pm}\omega_{\varepsilon,\pm}$, $J'_\varepsilon(\omega_\varepsilon^{\pm})\omega_\varepsilon^{\pm}=0$, i.e., $\omega_\varepsilon\in M_\varepsilon$, thus

$$c_\varepsilon\le J_\varepsilon(\omega_\varepsilon)=J_\varepsilon(\omega_\varepsilon^{+})+J_\varepsilon(\omega_\varepsilon^{-}).$$

On the other hand, by direct computation we conclude that

$$J_\varepsilon(\omega_\varepsilon^{\pm})=\varepsilon^3(c_{V_0}^{\pm}+o(1)),$$

where $o(1)\to 0$ as $\varepsilon\to 0$. Thus we obtain our conclusion.

4 Properties Analysis of u_ε

In this section, we make a careful analysis the profile of u_ε.

Lemma 4.1 The positive local maximum and negative local minimum points of u_ε are both in Λ.

Proof Let x_ε be a positive local maximum of u_ε. Suppose by contradiction that $x_\varepsilon\in\Lambda^c$. Since $\Delta u_\varepsilon(x_\varepsilon)\le 0$, using the definition of g, we have

$$\begin{aligned}\alpha u_\varepsilon(x_\varepsilon) &\le V(x_\varepsilon)u_\varepsilon(x_\varepsilon)\\ &\le -(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx)\Delta u_\varepsilon(x_\varepsilon)+V(x_\varepsilon)u_\varepsilon(x)\\ &= \tilde{f}(u_\varepsilon(x_\varepsilon))\le\frac{\alpha}{k}u_\varepsilon(x_\varepsilon).\end{aligned}$$

But the above estimate is impossible by the fact $k>\frac{\theta}{\theta-2}>1$. A similar argument implies that every negative local minimum of u_ε is also in Λ.

Lemma 4.2 Let P_ε^1 be a local maximum of u_ε^+ and P_ε^2 a local minimum of u_ε^-, then

(1) $u_\varepsilon(P_\varepsilon^1) \geq a_1$, $u_\varepsilon(P_\varepsilon^2) \leq a_2$,

(2) $|\frac{P_\varepsilon^1 - P_\varepsilon^2}{\varepsilon}| \to +\infty$ as $\varepsilon \to 0$.

Proof By Lemma 4.1, then P_ε^1, $P_\varepsilon^2 \in \Lambda$. Moreover, from the definition of g, then for $i = 1, 2$, we have

$$\begin{aligned}(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx)\Delta u_\varepsilon(P_\varepsilon^i) &= V(P_\varepsilon^i)u_\varepsilon(P_\varepsilon^i) - f(u_\varepsilon(P_\varepsilon^i)) \\ &= (V(P_\varepsilon^i) - \frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)})u_\varepsilon(P_\varepsilon^i).\end{aligned}$$

Meanwhile, as $\Delta u_\varepsilon(P_\varepsilon^1) \leq 0, \Delta u_\varepsilon(P_\varepsilon^2) \geq 0, u_\varepsilon(P_\varepsilon^1) > 0$ and $u_\varepsilon(P_\varepsilon^2) < 0$, it follows that

$$V(P_\varepsilon^i) - \frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)} \leq 0, \quad i = 1, 2,$$

which together with (V_1) imply

$$\frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)} \geq \alpha > 0, \quad i = 1, 2.$$

Consequence, if $a_2 < u_\varepsilon(P_\varepsilon^i) < a_1$, $i = 1, 2$, then

$$\frac{\alpha}{k} = \frac{\frac{\alpha}{k} u_\varepsilon(P_\varepsilon^i)}{u_\varepsilon(P_\varepsilon^i)} \geq \frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)} \geq \alpha > 0,$$

which is a contradiction, thus $u_\varepsilon(P_\varepsilon^1) \geq a_1, u_\varepsilon(P_\varepsilon^2) \leq a_2$. Item (1) is proved. The proof of item 2 is similar to Lemma 3.2 in [4], here we omit it.

Lemma 4.3 If $\varepsilon_n \downarrow 0$ and $x_n^i \in \bar{\Lambda}, i = 1, 2$ are such that

$$u_{\varepsilon_n}(x_n^1) \geq b > 0, \quad u_{\varepsilon_n}(x_n^2) \leq -b < 0,$$

then

$$\lim_{n\to\infty} V(x_n^i) = V_0 \ (i = 1, 2).$$

Proof The proof is similar to Proposition 4.1 in [4].

Lemma 4.4 If $m_\varepsilon^+ = \max\limits_{x\in\partial\Lambda} u_\varepsilon^+(x),\ m_\varepsilon^- = \min\limits_{x\in\partial\Lambda} u_\varepsilon^-(x)$, then

$$\lim_{\varepsilon\to 0} m_\varepsilon^\pm = 0.$$

Moreover, for every $\varepsilon > 0$ small enough, u_ε possesses at most one positive local maximum $P_\varepsilon^1 \in \Lambda$ and one negative local minimum $P_\varepsilon^2 \in \Lambda$. Meanwhile, we have

$$\lim_{\varepsilon\to 0} V(P_\varepsilon^i) = V_0 = \min_{x\in\bar{\Lambda}} V(x), i = 1, 2.$$

Proof The proof is similar to Corollary 4.1 in [4].

5 Proof of Theorem 1.1

To conclude the proof of Theorem 1.1, we only need to show that $a_2 < u_\varepsilon(x) < a_1,\ \forall x \in \Lambda^c$. By Lemma 4.4, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, we have

$$|u_\varepsilon(x)| \le A = \min\{a_1, -a_2\},\ \forall x \in \partial\Lambda.$$

Using the same arguments in [2], the above inequality follows for u_ε and $x \in \Lambda^c$. Moreover, the arguments explored by [14], we can prove the following estimate

$$|u_\varepsilon(x)| \le M\Big[\exp(-\beta|\frac{x-P_\varepsilon^1}{\varepsilon}|) + \exp(-\beta|\frac{x-P_\varepsilon^2}{\varepsilon}|)\Big]$$

for some constant $\beta > 0$. So we complete the proof of Theorem 1.1.

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$\mathbb{R}^3$ 中一类Kirchhoff 型方程变号解的存在性及集中性

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摘 要: 本文研究了一类Kirchhoff 型方程. 利用极大极小原理及惩罚函数方法, 证明了上述方程变号解的存在性及集中性, 我们的结果推广了文献[4]的结果.

关键词: 惩罚函数; 变号解; 极大极小方法; 集中性

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