Vol. 35 (2015)
No. 1

数 学 杂 志
J. of Math. (PRC)

CRITICAL EXPONENTS IN POROUS MEDIA EQUATIONS WITH WEIGHTED NONLOCAL BOUNDARY CONDITIONS

LIU Bing-chen, ZHANG Chang-cheng

(*College of Science, China University of Petroleum, Qingdao 266580, China*)

Abstract: This paper deals with a class of porous media parabolic equations coupled via nonlinear norm-type sources, subject to nonlocal boundary conditions. We show the influences of weighted functions and the coefficients on global existence and blow-up of solutions. Moreover, the critical blow-up exponents are obtained by using the comparison principle, which gives some new results for the previous published papers.

Keywords: porous media parabolic equations; nonlocal boundary condition; blow up; global existence

2010 MR Subject Classification: 35K65; 35K61; 35B33

Document code: A **Article ID:** 0255-7797(2015)01-0051-12

1 Introduction

In this paper, we consider the following porous media parabolic systems

$$\begin{cases} u_t = \Delta u^{m_1} + a\|u^{p_1}v^{q_1}\|_{\alpha}^{k_1},\ v_t = \Delta v^{m_2} + b\|v^{p_2}u^{q_2}\|_{\beta}^{k_2}, & x \in \Omega,\ t > 0, \\ u(x,t) = \displaystyle\int_\Omega f(x,y)u(y,t)dy,\ v(x,t) = \int_\Omega g(x,y)v(y,t)dy, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $m_i > 1,\ k_i, p_i, q_i > 0,\ i = 1, 2,\ \alpha, \beta \geq 1$; coefficients a, b are positive constants; Ω is a bounded connected domain of R^N with smooth boundary $\partial\Omega$; the nonlinear norm-type sources are taken the forms,

$$\|u^{p_1}v^{q_1}\|_{\alpha}^{k_1} = \left(\int_\Omega (u^{p_1}v^{q_1})^\alpha dx\right)^{k_1/\alpha}, \quad \|v^{p_2}u^{q_2}\|_{\beta}^{k_2} = \left(\int_\Omega (v^{p_2}u^{q_2})^\beta dx\right)^{k_2/\beta},$$

weighted functions $f(x,y)$ and $g(x,y)$, for the sake of the meaning of nonlocal boundary, are nonnegative and continuous defined in $\partial\Omega \times \Omega$ and satisfying $\int_\Omega f(x,y)dy \leq 1$ and

* **Received date:** 2013-11-10 **Accepted date:** 2014-04-18

Foundation item: Supported by National Natural Science Foundation of China (11201483); Shandong Provincial Natural Science Foundation; the Fundamental Research Funds for the Central Universities.

Biography: Liu Bingchen(1976–), male, born at Weifang, Shandong, semiprofessor, major in nonlinear partial differential equations and their applications.

$\int_\Omega g(x,y)dy \le 1$; the initial data v_0, u_0 are positive and continuous, satisfying the compatibility conditions $u_0(x) = \int_\Omega f(x,y)u_0(y)dy$ and $v_0(x) = \int_\Omega g(x,y)v_0(y)dy$ on $\partial\Omega$, respectively.

System (1.1) can be found in the study of the flows of fluids through porous media with integral sources, and the absorption and download infiltration of fluids into porous media with nonlocal sources, and also in the population dynamics (see, for example, [1–4] and the papers cited therein).

The homogeneous Dirichlet problem

$$\begin{cases} u_t = \Delta u^{m_1} + \|u^{p_1}v^{q_1}\|_\alpha^{k_1},\ v_t = \Delta v^{m_2} + \|v^{p_2}u^{q_2}\|_\beta^{k_2}, & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

with $m_1, m_2 > 1$ has been studied by Ling and Wang (see [5]). Suppose that one of the following conditions holds:

(a) $m_1 > p_1k_1$, $m_2 > p_2k_2$, $q_1k_1q_2k_2 < (m_1 - p_1k_1)(m_2 - p_2k_2)$;

(b) $m_1 > p_1k_1$, $m_2 > p_2k_2$, $q_1k_1q_2k_2 > (m_1 - p_1k_1)(m_2 - p_2k_2)$, and the initial data are sufficiently small;

(c) $m_1 > p_1k_1$, $m_2 > p_2k_2$, $q_1k_1q_2k_2 = (m_1 - p_1k_1)(m_2 - p_2k_2)$, and the domain $|\Omega|$ is sufficiently small, then every nonnegative solution exists globally.

On the contrary, if one of the following conditions holds:

(a) $m_1 > p_1k_1$, $m_2 > p_2k_2$, $q_1k_1q_2k_2 > (m_1 - p_1k_1)(m_2 - p_2k_2)$, and the initial data are sufficiently large;

(b) $m_1 > p_1k_1$, $m_2 > p_2k_2$, $q_1k_1q_2k_2 = (m_1 - p_1k_1)(m_2 - p_2k_2)$, the domain contains a sufficiently large ball, and the initial data are sufficiently large, then the nonnegative solutions blow up in finite time.

Ye and Xu in [6] studied the following system

$$\begin{cases} u_t = \Delta u^{m_1} + au^{p_1}\displaystyle\int_\Omega v^{q_1}dy,\ v_t = \Delta v^{m_2} + bv^{p_2}\int_\Omega u^{q_2}dy, & x \in \Omega,\ t > 0, \\ u(x,t) = \displaystyle\int_\Omega f(x,y)u(y,t)dy,\ v(x,t) = \int_\Omega g(x,y)v(y,t)dy, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where $m_1, m_2 > 1$ and a, b are positive constants. They obtained that:

(i) For any $\delta > 0$ such that $\delta \le \displaystyle\int_\Omega f(x,y)dy, \int_\Omega g(x,y)dy \le 1$ on $\partial\Omega$, and if $m_1 > p_1$, $m_2 > p_2$ and $q_1q_2 < (m_1 - p_1)(m_2 - p_2)$, then an arbitrary nonnegative solution (u,v) exists globally.

(ii) If $\displaystyle\int_\Omega f(x,y)dy, \int_\Omega g(x,y)dy < 1$ on $\partial\Omega$, and if one of the following conditions holds:

(A) $m_1 < p_1$,

(B) $m_2 < p_2$,

(C) $q_1q_2 > (m_1 - p_1)(m_2 - p_2)$, then an arbitrary nonnegative solution (u,v) exists globally for sufficiently small initial data.

(iii) If $m_1 < p_1$, or $m_2 < p_2$, or $q_1q_2 > (m_1-p_1)(m_2-p_2)$, then an arbitrary nonnegative solution of the system blows up in finite time for sufficiently large initial data.

(iv) For any $\delta > 0$ such that $\delta \le \int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy \le 1$ on $\partial\Omega$, and if $m_1 > p_1$, $m_2 > p_2$, and $q_1q_2 = (m_1-p_1)(m_2-p_2)$, then an arbitrary nonnegative solution (u,v) exists globally for small a and b. For $m_1 = m_2 = 1$, $p_1, p_2 < 1$, and $q_1q_2 > (1-p_2)(1-p_1)$, the blow-up rates are obtained provided that $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy \le c < 1$.

Chen, Mi and Mu in [7] studied the following system

$$\begin{cases} u_t = \Delta u^{m_1} + u^{p_1}v^{q_1},\ v_t = \Delta v^{m_2} + v^{p_2}u^{q_2}, & x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega f(x,y)u(y,t)dy,\ v(x,t) = \int_\Omega g(x,y)v(y,t)dy, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where $m_i, p_i, q_i > 1$, $i = 1, 2$. The following results have been obtained:

(i) Suppose that $\int_\Omega f(x,y)dy \ge 1$, $\int_\Omega g(x,y)dy \ge 1$ for any $x \in \partial\Omega$. If $q_2 > p_1 - 1$ and $q_1 > p_2 - 1$, then any solution with positive initial data blows up in finite time.

(ii) Suppose that $\int_\Omega f(x,y)dy < 1$, $\int_\Omega g(x,y)dy < 1$ for any $x \in \partial\Omega$. If $m_1 > p_1$, $m_2 > p_2$, and $q_1q_2 > (m_1-p_1)(m_2-p_2)$, then every nonnegative solution is global; while if $m_1 < p_1$, or $m_2 < p_2$, or $q_1q_2 > (m_1-p_1)(m_2-p_2)$, then the nonnegative solution exists globally for sufficiently small initial data and blows up in finite time for sufficiently large initial data.

(iii) If $\int_\Omega f(x,y)dy \ge 1$, $\int_\Omega g(x,y)dy \ge 1$ for any $x \in \partial\Omega$, $q_1 > m_2$, $q_2 > m_1$ and satisfy $q_2 > p_1 - 1$ and $q_1 > p_2 - 1$, and some assumptions on $u_0(x)$, $v_0(x)$, blow-up rates are given.

There are also some good works on the nonlocal parabolic equations with nonlocal boundary conditions (see [8–12], and the papers cited therein).

This paper is arranged as follows: In the next section, we show the main results and some remarks of the paper. The global existence and blow-up of solutions will be proved in Sections 3 and 4, respectively.

2 Main Results and Remarks

It is well-known that the porous media parabolic equations need not posses local classical solutions. We give a precise definition of a weak solution of (1.1).

Definition 2.1 A function $(u(x,t), v(x,t))$, $(x,t) \in \bar\Omega \times [0,T]$ is called a sub- (or super) solution of (1.1), if the following conditions hold:

$u(x,t), v(x,t) \in L^\infty(\bar\Omega \times [0,T])$;

$(u(x,t), v(x,t)) \le\ (\ge)(\int_\Omega f(x,y)u(y,t)dy, \int_\Omega g(x,y)v(y,t)dy)$ on $\partial\Omega \times [0,T]$;

$(u(x,t), v(x,t)) \le\ (\ge)(u(x,0), v(x,0))$ on $\bar\Omega \times [0,T]$;

For any $t \in [0,T]$ and test function $\phi(x,t) \in C(\bar\Omega \times [0,T], R^+)$, satisfying $\phi_t, \Delta\phi \in$

$C(\Omega\times[0,T])\cap L^2(\Omega\times[0,T])$ and $\phi(x,t)\geq 0$ on $\partial\Omega\times[0,T]$,

$$\begin{aligned}\int_\Omega u(x,t)\phi(x,t)dx \leq (\geq) &\int_\Omega u(x,0)\phi(x,0)dx\\ &+\int_0^t\int_{\bar\Omega}(u\phi_\tau+u^{m_1}\Delta\phi+\phi a\|u^{p_1}v^{q_1}\|_\alpha^{k_1})dxd\tau\\ &-\int_0^t\int_{\partial\Omega}\frac{\partial\phi}{\partial\eta}\left(\int_\Omega f(x,y)u(y,\tau)dy\right)^{m_1}dSd\tau,\\ \int_\Omega v(x,t)\phi(x,t)dx \leq (\geq) &\int_\Omega v(x,0)\phi(x,0)dx\\ &+\int_0^t\int_{\bar\Omega}(v\phi_\tau+v^{m_2}\Delta\phi+\phi b\|v^{p_2}u^{q_2}\|_\beta^{k_2})dxd\tau\\ &-\int_0^t\int_{\partial\Omega}\frac{\partial\phi}{\partial\eta}\left(\int_\Omega g(x,y)v(y,\tau)dy\right)^{m_2}dSd\tau.\end{aligned}$$

A weak solution is both a sub-solution and a super solution of (1.1).

The local existence of weak solution and the comparison principle of (1.1) can be obtained by [14, 15]. We omit the detail here. Our main results are stated as follows.

Theorem 2.1 If one of the following conditions holds, then the nonnegative solution of system (1.1) is global.

(i) $m_1 > p_1k_1$, $m_2 > p_2k_2$ and $q_1k_1q_2k_2 < (m_1-p_1k_1)(m_2-p_2k_2)$, and $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy < 1$, $x\in\partial\Omega$;

(ii) $m_1 < p_1k_1$ or $m_2 < p_2k_2$ or $q_1k_1q_2k_2 > (m_1-p_1k_1)(m_2-p_2k_2)$, the initial data are small enough, and $\int_\Omega f(x,y)dy$, $\int_\Omega g(x,y)dy < 1$, $x\in\partial\Omega$;

(iii) $m_1 > p_1k_1$, $n_2 > p_2k_2$, $q_1k_1q_2k_2 = (m_1-p_1k_1)(m_2-p_2k_2)$ and, for any constant $\delta > 0$ such that $\delta \leq \int_\Omega f(x,y)dy, \int_\Omega g(x,y)dy \leq 1$ on $\partial\Omega$, moreover, a, b are sufficiently small.

Theorem 2.2 If one of the following conditions holds, then the nonnegative solution of system (1.1) blows up in finite time.

(i) $m_1 < p_1k_1$ or $m_2 < p_2k_2$ or $q_1k_1q_2k_2 > (m_1-p_1k_1)(m_2-p_2k_2)$, and the initial data are sufficiently large;

(ii) $m_1 > p_1k_1$, $m_2 > p_2k_2$ and $q_1k_1q_2k_2 = (m_1-p_1k_1)(m_2-p_2k_2)$, the initial data are large enough, and the domain Ω contains a sufficiently large ball.

Remark 2.1 It can be checked from Theorems 2.1 and 2.2 that all of the classifications of the ten exponents in the equations of (1.1) are complete. All of the solutions remain global if and only if $\max\left\{p_1k_1-m_1,\ p_2k_2-m_2,\ q_1k_1q_2k_2 > (m_1-p_1k_1)(m_2-p_2k_2)\right\} < 0$. For $\max\left\{p_1k_1-m_1,\ p_2k_2-m_2,\ q_1k_1q_2k_2 > (m_1-p_1k_1)(m_2-p_2k_2)\right\} \geq 0$, both blow-up solutions and global solutions may exist under different assumptions.

Remark 2.2 One can find out from the proofs of the main results that some of the results can be extended to [5–7], which are not obtained there.

The criteria of Theorem 2.1 (ii) and Theorem 2.2 (i) can be used directly to [5].

By using the methods in the proof of Theorem 2.2 (ii), one can obtain the same results for [6].

By using the methods in Theorem 2.1 (iii) and Theorem 2.2 (ii), the same results hold for the main systems in [7].

3 Global Solutions

Compared with the traditional null Dirichlet boundary, the weight functions $f(x,y)$ and $g(x,y)$ play an important role in the global existence results for system (1.1).

Proof of Theorem 2.1 (i) Let $\Psi_1(x)$ be the positive solution of the linear elliptic problem

$$-\Delta\Psi_1(x)=\varepsilon_1,\quad x\in\Omega;\quad \Psi_1(x)=\int_\Omega f(x,y)dy,\quad x\in\partial\Omega; \tag{3.1}$$

Let $\Psi_2(x)$ be the positive solution of the linear elliptic problem

$$-\Delta\Psi_2(x)=\varepsilon_2,\quad x\in\Omega;\quad \Psi_2(x)=\int_\Omega g(x,y)dy,\quad x\in\partial\Omega, \tag{3.2}$$

where ε_1, ε_2 are positive constants such that $0\le\Psi_1(x)\le 1$, $0\le\Psi_2(x)\le 1$. We remark that $\int_\Omega f(x,y)dy<1$, $\int_\Omega g(x,y)dy<1$ ensure the existence of such ε_1, ε_2.

Denote that

$$\max_{\bar\Omega}\Psi_1=\bar K_1,\ \min_{\bar\Omega}\Psi_1=\underline{K}_1;\quad \max_{\bar\Omega}\Psi_2=\bar K_2,\ \min_{\bar\Omega}\Psi_2=\underline{K}_2. \tag{3.3}$$

We define the functions

$$\bar u(x,t)=\bar u(x)=M^{l_1}\Psi_1^{1/m_1},\quad \bar v(x,t)=\bar v(x)=M^{l_2}\Psi_1^{1/m_2}, \tag{3.4}$$

where M is a constant to be determined later. Then, we have

$$\begin{aligned}\bar u(x,t)|_{x\in\partial\Omega}&=M^{l_1}\Psi^{1/m_1}=M^{l_1}\Big(\int_\Omega f(x,y)dy\Big)^{1/m_1}\\&>M^{l_1}\int_\Omega f(x,y)dy\ge M^{l_1}\int_\Omega f(x,y)\Psi_1^{1/m_1}(y)dy=\int_\Omega f(x,y)\bar u(y)dy.\end{aligned} \tag{3.5}$$

In a similar way, we can obtain that

$$\bar v(x,t)|_{x\in\partial\Omega}>\int_\Omega g(x,y)\bar v(y)dy. \tag{3.6}$$

Here, we used $0\le\Psi_1(x)\le 1$, $0\le\Psi_2(x)\le 1$, $\int_\Omega f(x,y)dy<1$ and $\int_\Omega g(x,y)dy<1$.

On the other hand, we have

$$\begin{aligned}\bar{u}_t-\Delta\bar{u}^{m_1}-a\|\bar{u}^{p_1}\bar{v}^{q_1}\|_{\alpha}^{k_1} &= M^{m_1l_1}\varepsilon_1-a\Big[\int_\Omega\Big(M^{p_1l_1+q_1l_2}\Psi_1^{p_1/m_1}\Psi_2^{q_1/m_2}\Big)^{\alpha}dx\Big]^{k_1/\alpha}\\ &\geq M^{m_1l_1}\varepsilon_1-aM^{k_1(p_1l_1+q_1l_2)}\bar{K}_1^{p_1k_1/m_1}\bar{K}_2^{q_1k_1/m_2}|\Omega|^{k_1/\alpha}, \quad (3.7)\\ \bar{v}_t-\Delta\bar{v}^{m_2}-b\|\bar{v}^{p_2}\bar{u}^{q_2}\|_{\beta}^{k_2} &\geq M^{m_2l_2}\varepsilon_2-bM^{k_2(q_2l_2+p_2l_1)}\bar{K}_1^{q_1k_2/m_1}\bar{K}_2^{q_1k_2/m_2}|\Omega|^{k_2/\beta}. \quad (3.8)\end{aligned}$$

Let

$$\begin{cases} M_1=\left(\frac{a\bar{K}_1^{p_1k_1/m_1}\bar{K}_2^{q_1k_1/m_2}|\Omega|^{k_1/\alpha}}{\varepsilon_1}\right)^{\frac{1}{m_1l_1-k_1(p_1l_1+q_1l_2)}},\\ M_2=\left(\frac{b\bar{K}_1^{q_1k_2/m_1}\bar{K}_2^{q_1k_2/m_2}|\Omega|^{k_2/\beta}}{\varepsilon_2}\right)^{\frac{1}{m_2l_2-k_2(q_2l_2+p_2l_1)}}. \end{cases} \tag{3.9}$$

If $m_1>p_1k_1$, $m_2>p_2k_2$ and $q_1k_1q_2k_2>(m_1-p_1k_1)(m_2-p_2k_2)$, then there exist positive constants l_1, l_2 such that

$$p_1k_1l_1+q_1k_1l_2<m_1l_1, \quad q_2k_2l_2+p_2k_2l_1>m_2l_2. \tag{3.10}$$

Therefore, we can choose M sufficiently large such that

$$\begin{cases} M>\max\{M_1,M_2\},\\ M^{l_1}\Psi_1^{1/m_1}\geq u_0(x), \quad M^{l_2}\Psi_2^{1/m_2}\geq v_0(x). \end{cases} \tag{3.11}$$

Now, it follows from (3.5)–(3.11) that $(\bar{u},\bar{v})$ defined by (3.4) is a positive super-solution of system (1.1). By the comparison principle, we conclude that $(u,v)\leq(\bar{u},\bar{v})$, which implies (u,v) exists globally.

(ii) If $m_1<p_1k_1$, or $m_2<p_2k_2$, or $q_1k_1q_2k_2>(m_1-p_1k_1)(m_2-p_2k_2)$, there exist positive constants l_1, l_2 such that

$$p_1k_1l_1+q_1k_1l_2>m_1l_1, \quad q_2k_2l_1+p_2k_2l_2>m_2l_2. \tag{3.12}$$

So we can choose $M=\min\{M_1,M_2\}$. Furthermore, assume that $u_0(x)$, $v_0(x)$ are small enough to satisfy $M^{l_1}\Psi_1^{1/m_1}\geq u_0(x)$, $M^{l_2}\Psi_2^{1/m_2}\geq v_0(x)$. It follows that $(\bar{u},\bar{v})$ defined by (3.4) is a positive super-solution of system (1.1). Hence, (u,v) exists globally.

(iii) If $m_1>p_1k_1$, $m_2>p_2k_2$, $(m_1-p_1k_1)(m_2-p_2k_2)=p_1k_1p_2k_2$, then there exist two positive numbers $l_1,l_2<1$ satisfying $\frac{q_1k_1}{m_1-p_1k_1}=\frac{l_1}{l_2}=\frac{m_2-p_2k_2}{q_2k_2}$, $m_1l_1,m_2l_2<1$.

We define the following elliptic boundary value problems:

$$\begin{cases} -\Delta\phi(x)=\eta_1, & x\in\Omega,\\ \phi(x)=\displaystyle\int_\Omega\varphi_1(x,y)dy, & x\in\partial\Omega, \end{cases} \tag{3.13}$$

$$\begin{cases} -\Delta\psi(x)=\eta_2, & x\in\Omega,\\ \psi(x)=\displaystyle\int_\Omega\varphi_2(x,y)dy, & x\in\partial\Omega, \end{cases} \tag{3.14}$$

where η_1, η_2 are both positive constants such that $\delta \le \phi(x), \psi(x) \le 1$. We have

$$\delta \le \int_\Omega \varphi_1(x,y)dy, \int_\Omega \varphi_2(x,y)dy \le 1,$$

which guarantee the existence of $\phi(x)$ and $\psi(x)$. We define

$$K_1 = \max_{x\in\bar\Omega} \phi(x), \quad K_2 = \min_{x\in\bar\Omega} \phi(x), \quad L_1 = \max_{x\in\bar\Omega} \psi(x), \quad L_2 = \min_{x\in\bar\Omega} \psi(x).$$

Applying the classical elliptic theorems to problems (3.13) and (3.14), it is easy to see that

$$K_2 \ge \delta, \quad L_2 \ge \delta.$$

Define $\bar u = (K\phi(x))^{l_1}$, $\bar v = (K\psi(x))^{l_2}$, where K is to be determined later.

A simple computation shows

$$\bar u_t - \Delta \bar u^{m_1} = -\Delta (K\phi(x))^{m_1 l_1} \ge \eta_1 m_1 l_1 K^{m_1 l_1} K_1^{m_1 l_1 - 1},$$
$$a\|\bar u^{p_1}\bar v^{q_1}\|_\alpha^{k_1} = a\Big[\int_\Omega (K\phi(x))^{p_1 l_1 \alpha}(K\psi(x))^{q_1 l_2 \alpha} dx\Big]^{\frac{k_1}{\alpha}} \le aK^{p_1k_1l_1+q_1k_1l_2} K_1^{p_1k_1l_1} L^{q_1k_1l_2} |\Omega|^{\frac{k_1}{\alpha}}.$$

Therefore, we have

$$\begin{aligned}\bar u_t - \Delta \bar u^{m_1} - a\|\bar u^{p_1}\bar v^{q_1}\|_\alpha^{k_1} \ge & \eta_1 m_1 l_1 K^{m_1 l_1} K_1^{m_1 l_1 - 1} \\ & - aK^{p_1k_1l_1+q_1k_1l_2} K_1^{p_1k_1l_1} L_1^{q_1k_1l_2} |\Omega|^{\frac{k_1}{\alpha}}. \end{aligned} \tag{3.15}$$

By the similar way, it is easy to verify that

$$\begin{aligned}\bar v_t - \Delta \bar v^{m_2} - b\|\bar v^{p_2}\bar v^{q_2}\|_\beta^{k_2} \ge & \eta_2 m_2 l_1 K^{m_2 l_2} K_1^{m_2 l_2 - 1} \\ & - aK^{p_2k_2l_2+q_2k_2l_1} K_1^{q_2k_2l_1} L_1^{p_2k_2l_2} |\Omega|^{\frac{k_2}{\beta}}. \end{aligned} \tag{3.16}$$

The numbers on the right-hand side of (3.15) and (3.16) are both nonnegative provided that

$$a \le \frac{\eta_1 m_1 l_1 K_1^{(m_1-p_1k_1)l_1 - 1}}{L_1^{q_1k_1l_2}|\Omega|^{\frac{k_1}{\alpha}}}, \quad b \le \frac{\eta_2 m_2 l_2 L_1^{(m_2-p_2k_2)l_2 - 1}}{K_1^{q_2k_2l_1}|\Omega|^{\frac{k_2}{\beta}}}.$$

Take

$$K = \max\left\{ \frac{u_0^{1/l_1}(x)}{K_2}, \quad \frac{v_0^{1/l_2}(x)}{L_2} \right\}.$$

Now we turn our attention to the boundary conditions, that is for every $x \in \partial\Omega$,

$$\begin{aligned}\bar u(x,t) = (K(\phi(x)))^{l_1} = K^{l_1}\left(\int_\Omega \varphi_1(x,y)dy\right)^{l_1} & \ge K^{l_1}\int_\Omega \varphi_1(x,y)dy \\ & \ge K^{l_1}\int_\Omega \varphi_1(x,y)\phi^{l_1}(y)dy = \int_\Omega \varphi_1(x,y)\bar u(x,t)dy.\end{aligned}$$

Similarly, we get

$$\bar{v}(x,t) = (K\psi(x))^{l_2} \geq \int_\Omega \varphi_2(x,y)\bar{v}(x,t)dy,$$

where we have used $l_1, l_2 \in (0,1)$; $\int_\Omega \varphi_1(x,y), \int_\Omega \varphi_2(x,y) \leq 1$; $\phi(x), \psi(x) \in (0,1)$. By means of the comparison principle, we obtain $(u,v) \leq (\bar{u},\bar{v})$. Hence, it yields that (u,v) exists globally.

4 Blow-up Solutions

Proof of Theorem 2.2 (i) Due to the requirement of the comparison principle, we will construct blow-up sub-solution in some sub-domain of Ω in which $u, v > 0$. Let $\varphi(x)$ be a nontrivial nonnegative continuous function and vanished on $\partial\Omega$. Without loss of generality, we may assume that $O \in \Omega$ and $\varphi(0) > 0$. We will construct a blow-up positive sub-solution to complete proof. Set

$$\underline{u}(x,t) = \frac{1}{(T-t)^{l_1}} w^{1/m_1}\Big(\frac{|x|}{(T-t)^\delta}\Big), \underline{v}(x,t) = \frac{1}{(T-t)^{l_2}} w^{1/m_2}\Big(\frac{|x|}{(T-t)^\delta}\Big) \tag{4.1}$$

with

$$w(r) = \frac{R^3}{12} - \frac{R}{4}r^2 + \frac{1}{6}r^3, \quad r = \frac{|x|}{(T-t)^\delta}, \quad 0 \leq r \leq R, \tag{4.2}$$

where $l_1, l_2, \delta > 0$ and $0 < T < 1$ are to be determined later. Clearly, $0 \leq w(r) \leq R^3/12$ and $w(r)$ is non-increasing since $w'(r) = r(r-R)/2 \leq 0$. Note that

$$\operatorname{supp}\underline{u}(\cdot,t) = \operatorname{supp}\underline{v}(\cdot,t) = \overline{B(0,R(T-t)^\delta)} \subset \overline{B(0,RT^\delta)} \subset \Omega \tag{4.3}$$

for sufficiently small $T > 0$. Obviously, $(\underline{u},\underline{v})$ becomes unbounded as $t \to T^-$, at the point $x = 0$. Calculating directly, we obtain that

$$\begin{aligned}\underline{u}_t - \Delta\underline{u}^{m_1}(x,t) &= \frac{m_1 l_1 w^{1/m_1}(r) + \delta r w'(r) w^{\frac{1-m_1}{m_1}}(r)}{m_1(T-t)^{l_1+1}} \\ &\quad + \tfrac{R-2r}{2(T-t)^{m_1+2\delta}} + \frac{(N-1)(R-r)}{2(T-t)^{m_1 l_1+2\delta}} \\ &\leq \frac{l_1(\frac{R^3}{12})^{1/m_1}}{(T-t)^{l_1+1}} + \frac{NR-(N+1)r}{2(T-t)^{m_1 l_1+2\delta}},\end{aligned} \tag{4.4}$$

noticing that $T < 1$ is sufficiently small.

Similarly, we have

$$\underline{v}_t - \Delta\underline{v}^{m_2}(x,t) \leq \frac{l_2(\frac{R^3}{12})^{1/m_2}}{(T-t)^{l_2+1}} + \frac{NR-(N+1)r}{2(T-t)^{m_2 l_2+2\delta}}. \tag{4.5}$$

Case 1 If $0 \le r \le NR/(N+1)$, we have $w(r) \ge (3N+1)R^3/12(N+1)^3$, then

$$\begin{cases} a\|\underline{u}^{p_1}\underline{v}^{q_1}\|_\alpha^{k_1} & = a\Big[\displaystyle\int_\Omega \Big(\frac{w^{\frac{m_1q_1+m_2p_1}{m_1m_2}}}{(T-t)^{p_1l_1+q_1l_2}}\Big)^\alpha dx\Big]^{\frac{k_1}{\alpha}} \\ & \ge \dfrac{a}{(T-t)^{k_1(p_1l_1+q_1l_2)}}\Big[\dfrac{(3N+1)R^3}{12(N+1)^3}\Big]^{\frac{p_1k_1}{m_1}+\frac{q_1k_1}{m_2}}|\Omega|^{\frac{k_1}{\alpha}}, \\ b\|\underline{v}^{p_2}\underline{u}^{q_2}\|_\alpha^{k_2} & \ge \dfrac{b}{(T-t)^{k_2(p_2l_2+q_2l_1)}}\Big[\dfrac{(3N+1)R^3}{12(N+1)^3}\Big]^{\frac{q_2k_2}{m_1}+\frac{p_2k_2}{m_2}}|\Omega|^{\frac{k_2}{\beta}}. \end{cases} \tag{4.6}$$

Hence,

$$\begin{cases} \underline{u}_t - \Delta\underline{u}^{m_1} - a\|\underline{u}^{p_1}\underline{v}^{q_1}\|_\alpha^{k_1} \le & \dfrac{l_1(\frac{R^3}{12})^{1/m_1}}{(T-t)^{l_1+1}} + \dfrac{NR-(N+1)r}{2(T-t)^{m_1l_1+2\delta}} \\ & -\dfrac{a}{(T-t)^{k_1(p_1l_1+q_1l_2)}}\Big[\dfrac{(3N+1)R^3}{12(N+1)^3}\Big]^{\frac{p_1k_1}{m_1}+\frac{q_1k_1}{m_2}}|\Omega|^{\frac{k_1}{\alpha}}, \\ \underline{v}_t - \Delta\underline{v}^{m_2} - b\|\underline{v}^{p_2}\underline{u}^{q_2}\|_\alpha^{k_2} \le & \dfrac{l_2(\frac{R^3}{12})^{1/m_2}}{(T-t)^{l_2+1}} + \dfrac{NR-(N+1)r}{2(T-t)^{m_2l_2+2\delta}} \\ & -\dfrac{b}{(T-t)^{k_2(p_2l_2+q_2l_1)}}\Big[\dfrac{(3N+1)R^3}{12(N+1)^3}\Big]^{\frac{q_2k_2}{m_1}+\frac{p_2k_2}{m_2}}|\Omega|^{\frac{k_2}{\beta}}. \end{cases} \tag{4.7}$$

Case 2 If $NR/(N+1) < r \le R$, then

$$\begin{cases} \underline{u}_t - \Delta\underline{u}^{m_1} - a\|\underline{u}^{p_1}\underline{v}^{q_1}\|_\alpha^{k_1} \le & \dfrac{l_1(\frac{R^3}{12})^{1/m_1}}{(T-t)^{l_1+1}} \\ & -\dfrac{a}{(T-t)^{k_1(p_1l_1+q_1l_2)}}\Big[\dfrac{(3N+1)R^3}{12(N+1)^3}\Big]^{\frac{p_1k_1}{m_1}+\frac{q_1k_1}{m_2}}|\Omega|^{\frac{k_1}{\alpha}}, \\ \underline{v}_t - \Delta\underline{v}^{m_2} - b\|\underline{v}^{p_2}\underline{u}^{q_2}\|_\alpha^{k_2} \le & \dfrac{l_2(\frac{R^3}{12})^{1/m_2}}{(T-t)^{l_2+1}} \\ & -\dfrac{b}{(T-t)^{k_2(p_2l_2+q_2l_1)}}\Big[\dfrac{(3N+1)R^3}{12(N+1)^3}\Big]^{\frac{q_2k_2}{m_1}+\frac{p_2k_2}{m_2}}|\Omega|^{\frac{k_2}{\beta}}. \end{cases} \tag{4.8}$$

There exist positive constants l_1, l_2 large enough to satisfy

$$\begin{cases} p_1k_1l_1 + q_1k_1l_2 > m_1l_1 + 1, & q_2k_2l_1 + p_2k_2l_2 > m_2l_2 + 1, \\ (m_1-1)l_1 > 1, & (m_2-1)l_2 > 1, \end{cases} \tag{4.9}$$

and we can choose $\delta > 0$ be sufficiently small that

$$\delta < \min\Big\{\frac{p_1k_1l_1 + q_1k_1l_2 - m_1l_1}{2}, \ \frac{p_2k_2l_2 + q_2k_2l_1 - m_2l_2}{2}\Big\}. \tag{4.10}$$

Thus, we have

$$p_1k_1l_1 + q_1k_1l_2 > m_1l_1 + 2\delta > l_1 + 1, \quad p_2k_2l_2 + q_2k_2l_1 > m_2l_2 + 2\delta > l_2 + 1. \tag{4.11}$$

Hence, for sufficiently small $T > 0$, (4.7) and (4.8) imply that

$$\begin{cases} \underline{u}_t - \Delta\underline{u}^{m_1} - a\|\underline{u}^{p_1}\underline{v}^{q_1}\|_\alpha^{k_1} \le 0, & (x,t) \in \Omega \times (0,T), \\ \underline{v}_t - \Delta\underline{v}^{m_2} - b\|\underline{v}^{p_2}\underline{u}^{q_2}\|_\alpha^{k_2} \le 0, & (x,t) \in \Omega \times (0,T). \end{cases} \tag{4.12}$$

Since $\varphi(0) > 0$ and $\varphi(x)$ is continuous, there exist two positive constants ρ and ε such that $\varphi(x) \geq \varepsilon$, for all $x \in B(0,\rho) \subset \Omega$. Choose T small enough to insure $B(0, RT^{\delta}) \subset B(0,\rho)$, hence $\underline{u} \leq 0$, $\underline{v} \leq 0$ on $\partial\Omega \times (0,T)$. Under the assumption that $\int_{\Omega} f(x,y)dy < 1$ and $\int_{\Omega} g(x,y)dy < 1$ on $\partial\Omega$, we have

$$\underline{u}(x,t) \leq \int_{\Omega} f(x,y)\underline{u}(y,t)dy, \underline{v}(x,t) \leq \int_{\Omega} g(x,y)\underline{v}(y,t)dy$$

on $\partial\Omega \times (0,T)$. Furthermore, choose $u_0(x)$, $v_0(x)$ so large that $u_0(x) > \underline{u}(x,0)$ and

$$v_0(x) > \underline{v}(x,0).$$

By the comparison principle, we have $(\underline{u}, \underline{v}) < (u,v)$. It shows that solution (u,v) to system (1.1) blows up in finite time.

(ii) In this section, we consider the case $m_1 > p_1k_1$, $m_2 > p_2k_2$ and

$$q_1k_1q_2k_2 = (m_1 - p_1k_1)(m_2 - p_2k_2).$$

Clearly, there exist two positive constants l_1, l_2 such that

$$m_1l_1 = p_1k_1l_1 + q_1k_1l_2, m_2l_2 = p_2k_2l_2 + q_2k_2l_1, (m_1-1)l_1 > 1, (m_2-1)l_2 > 1. \tag{4.13}$$

Denote by $\lambda_{B_R} > 0$ and $\phi_R(r)$ the first eigenvalue and the corresponding eigenfunction of the following eigenfunction problem

$$-\phi''(r) - \frac{N-1}{r}\phi'(r) = \lambda\phi(r),\ r \in (0,R);\quad \phi'(0) = 0,\ \phi(R) = 0. \tag{4.14}$$

It is well known that $\phi_R(r)$ can be normalized as $\phi_R(r) > 0$ in B and

$$\phi_R(0) = \max_{x\in B} \phi_R(r) = 1.$$

By the property (Let $\tau = \frac{r}{R}$) of eigenvalues and eigenfunctions, we see that

$$\lambda_{B_R} = R^{-2}\lambda_{B_1}$$

and

$$\phi_R(r) = \phi_1(r/R) = \phi_1(\tau),$$

where λ_{B_1} and ϕ_1 are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball $B_1(0)$. Moreover,

$$\max_{B_1} \phi_1(\tau) = \phi_1(0) = \phi_R(0) = \max_{B} \phi_R(r) = 1. \tag{4.15}$$

We define the functions $\underline{u}(x,t)$, $\underline{v}(x,t)$ in the forms

$$\underline{u}(x,t) = \frac{1}{(T-t)^{l_1}}\phi_R^{l_1}(|x|),\ \underline{v}(x,t) = \frac{1}{(T-t)^{l_2}}\phi_R^{l_2}(|x|). \tag{4.16}$$

In the following, we will prove that $(\underline{u}, \underline{v})$ blows up in finite time in the ball $B = B(0, R)$. So $(\underline{u}, \underline{v})$ does blow up in the larger domain Ω. Calculating directly, we have

$$\begin{cases} \underline{u}_t - \Delta \underline{u}^{m_1} - a\|\underline{u}^{p_1}\underline{v}^{q_1}\|_\alpha^{k_1} & \le \frac{\phi_R^{l_1}}{(T-t)^{l_1+1}}\left[l_1 - \frac{a}{(T-t)^{m_1 l_1 - l_1 - 1}}(\|\phi_R^{p_1 l_1 + q_1 l_2}\|_\alpha^{k_1} - \lambda_{B_R} m_1 l_1)\right], \\ \underline{v}_t - \Delta \underline{v}^{m_2} - b\|\underline{u}^{p_2}\underline{v}^{q_2}\|_\beta^{k_2} & \le \frac{\phi_R^{l_2}}{(T-t)^{l_2+1}}\left[l_2 - \frac{b}{(T-t)^{m_2 l_2 - l_2 - 1}}(\|\phi_R^{p_2 l_1 + q_2 l_2}\|_\beta^{k_2} - \lambda_{B_R} m_2 l_2)\right], \end{cases} \tag{4.17}$$

where

$$\|\phi_R^{p_1 l_1 + q_1 l_2}\|_\alpha^{k_1} \le K_1 R^{\frac{N k_1}{\alpha}}, \|\phi_R^{p_2 l_1 + q_2 l_2}\|_\beta^{k_2} \le K_2 R^{\frac{N k_2}{\beta}}$$

and K_1, K_2 are independent of R. Then, in view of $\lambda_{B_R} = R^{-2}\lambda_{B_1}$, we may assume that R, that is the ball B, is sufficiently large that

$$\lambda_{B_R} < \min\left\{\frac{\|\phi_R^{p_1 l_1 + q_1 l_2}\|_\alpha^{k_1}}{m_1 l_1}, \frac{\|\phi_R^{p_2 l_1 + q_2 l_2}\|_\beta^{k_2}}{m_2 l_2}\right\}, \tag{4.18}$$

so for small $T > 0$, we get

$$\underline{u}_t - \Delta \underline{u}^{m_1} - a\|\underline{u}^{p_1}\underline{v}^{q_1}\|_\alpha^{k_1} \le 0, \quad \underline{v}_t - \Delta \underline{v}^{m_2} - b\|\underline{u}^{p_2}\underline{v}^{q_2}\|_\beta^{k_2} \le 0. \tag{4.19}$$

Therefore, $(\underline{u}, \underline{v})$ is a positive sub-solution in the ball B, which blows up in finite time provided the initial data is sufficiently large that

$$\underline{u}(x, 0) = T^{-l_1}\phi_R^{l_1}(|x|) \le u_0(x), \underline{v}(x, 0) = T^{-l_2}\phi_R^{l_2}(|x|) \le v_0(x)$$

in the ball B. Thanks to the comparison principle, the arbitrary nonnegative solution (u, v) of (1.1) must blow up in finite time now.

References

[1] Diaz J I, Kerser R. On a non-linear degenerate parabolic equation in infiltration or evaporation through a porous medium [J]. J. Differential Equations, 1987, 69(3): 368–403.

[2] Cantrell R S, Conser C. Diffusive logistic equations with indefinite weightes: population models in disrupted environments II [J]. SIAM J. Math. Anal., 1989, 22(4): 1043–1064.

[3] Furter J, Grinfield M. Local vs. nonlocal interactions in populations dynamics [J]. J. Math. Biol., 1989, 27(1): 65–80.

[4] Galaktionov A. On asymptotic self-similar behavior for a quasilinear heat equation: single point blow-up [J]. SIAM J. Math. Anal., 1995, 26(3): 695–693.

[5] Ling Z Q, Wang Z J. Global existence and blow-up for a degenerate reaction-diffusion system with nonlocal sources [J]. Appl. Math. Lett., 2012, 25(12): 2198–2202.

[6] Ye Z, Xu X J. Global existence and blow-up for a porous medium system with nonlocal boundary conditions and nonlocal sources [J]. Nonlinear Anal., 2013, 82(4): 115–126.

[7] Chen B T, Mi Y S, Mu C L. A quasilinear parabolic system with nonlocal boundary condition [J]. Boundary Value Problems, 2011, Article ID 750769, 1–18.

[8] Cui Z J, Yang Z D. Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition [J]. J. Math. Anal. Appl., 2008, 324(1): 559–570.

[9] Lin Z G, Liu Y R. Uniform blow-up profiles for diffusion equations with nonlocal source and nonlocal boundary [J]. Acta Math. Sci., 2004, 24B(3): 443–450.

[10] Wang M X, Wei Y F. Blow-up properties for a degenerate parabolic system with nonlinear localized sources [J]. J. Math. Anal. Appl., 2008, 343(2): 621–635.

[11] Du L L. Blow-up for a degenerate reaction-diffusion system with nonlinear nonlocal sources [J]. J. Comput. Anal. Math., 2007, 202(2): 237–247.

[12] Zheng S N, Kong L H. Roles of weight functions in a nonlinear nonlocal parabolic system [J]. Nonlinear Anal., 2008, 68(8): 2406–2416.

[13] Souplet P. Blow-up in nonlocal reaction-diffusion equations [J]. SIAM J. Math. Anal., 1998, 29(6): 1301–1334.

[14] Deng W B, Li Y X, Xie C H. Blow-up and global existence for a nonlocal degenerate parabolic system [J]. J. Math. Anal. Appl., 2003, 277(1): 199–217.

[15] Deng W B. Global existence and finite time blow-up for a degenerate reaction-diffusion system [J]. Nonlinear Anal., 2005, 60(5): 977–991.

具有加权非局部边界条件的多孔介质方程组的临界指标问题

刘丙辰, 张长城

(中国石油大学(华东)理学院, 山东 青岛 266580)

摘要: 本文研究了具有非局部边界条件和非线性内部源的多孔介质抛物型方程组问题. 利用比较原理, 获得了权函数和系数对整体解和爆破解的影响, 并得到了解的爆破临界指标, 推广了先前的研究结果.

关键词: 多孔介质方程组; 非局部边界条件; 爆破; 整体存在

MR(2010)**主题分类号:** 35K65; 35K61; 35B33 **中图分类号:** O175.26