A NOTE ON QUASI-∗-A(K) OPERATORS

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Abstract: In this note, we introduce quasi-∗-A(k) operators and obtain their spectral properties as follows: (i) If $T$ is quasi-∗-A(k) for $0 < k \leq 1$, then the spectral mapping theorem holds for the essential approximate point spectrum. (ii) If $T$ is quasi-∗-A(k) for $0 < k \leq 1$, then $\sigma_{ja}(T)\{0\} = \sigma_a(T)\{0\}$. Besides, we consider tensor product of ∗-A(k) operators.

Keywords: quasi-∗-A(k) operators; SVEP; joint approximate point spectrum; tensor product

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1 Introduction

Let $H$ be an arbitrary complex Hilbert space and $T$ be a bounded linear operator on $H$. We denote the ∗-algebra of all bounded linear operators on $H$ by $B(H)$.

An operator $T$ is ∗-paranormal if $||T^2x|| \geq ||T^*x||^2$ for unit vector $x$. ∗-paranormal operators have been studied by many researchers, see [5, 9, 11], etc. An operator $T$ is said to be class ∗-A if $|T^2| \geq |T^*|^2$, where $|T| = (T^*T)^{1/2}$. As an easy extension of class ∗-A operators, an operator $T$ is said to be quasi-∗-A if $T^*|T^2|T \geq T^*|T^*|^2T$ in [17]. Moreover, for $k > 0$, an operator $T$ belongs to quasi-∗-A(k) if $(T^*|T^2T|)^{1/k} \geq |T^*|^2$. An operator $T$ is absolute-∗-k-paranormal if $||T^kx||^{k+1} \leq |||T^kT^2x||| |x|^k$ for every $x \in H$. Particularly an operator $T$ is a class ∗-A(resp. ∗-paranormal) operator if and only if $T$ is a ∗-A(1)(resp. absolute-∗-1-paranormal) operator.

In this note we extend ∗-A(k)(resp. ∗-paranormal) operators and quasi-∗-A operators to a new class of operators called quasi-∗-A(k)(resp.quasi-absolute-∗-k-paranormal) operators, and study their spectral properties.

Definition 1.1 Let $T \in B(H)$.

(i) For each $k > 0$, $T$ belongs to quasi-∗-A(k) if

$$T^* (T^* |T^2|^k T)^{1/k} T \geq |T|^4.$$ 

(ii) For each $k > 0$, $T$ belongs to quasi-absolute-∗-k-paranormal if

$$||T^kT^2x||^{k+1} \leq |||T^kT^2x||| T^k x ||^k$$ for every $x \in H$. 

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It’s known that quasi-$\ast A(1)$ operator is quasi-$\ast A$, and a quasi-$\ast A(k)$ operator is quasi-
absolute-$\ast k$-paranormal (see Lemma 2.2), then we have the following implications:

$\ast A(k) \Rightarrow \text{quasi-$\ast A(k)$} \Rightarrow \text{quasi-absolute-$\ast k$-paranormal.}$

By simple calculation, we have the following lemma as appeared in [22].

**Lemma 1.2** Let $K = \oplus_{n=1}^{\infty} H_n$, where $H_n \cong H$. For given positive operators $A$ and $B$
on $H$, define the operator $T_{A,B}$ on $K$ as follows:

\[
T_{A,B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
A & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & B & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & B & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & B & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then the following assertions hold:

(i) $T_{A,B}$ belongs to $\ast A(k)$ if and only if $B^2 \geq A^2$.

(ii) $T_{A,B}$ belongs to quasi-$\ast A(k)$ if and only if $AB^2A \geq A^4$.

The following example provides an operator which is quasi-$\ast A(k)$ but not $\ast A(k)$.

**Example 1.3** A non-$\ast A(k)$ and quasi-$\ast A(k)$ operator.

Take $A$ and $B$ as

\[
A = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \quad B = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

Then

\[
B^2 - A^2 = \begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix} \not\geq 0.
\]

Hence $T_{A,B}$ is not a $\ast A(k)$ operator.

On the other hand,

\[
A(B^2 - A^2)A = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \geq 0.
\]

Thus $T_{A,B}$ is a quasi-$\ast A(k)$ operator.

Consider unilateral weighted shift operator as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \cdots$ (called weights), the unilateral weighted shift $W_{\alpha}$ associated with $\alpha$ is the operator on $H = l_2$ defined by $W_{\alpha}e_n := \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthogonal basis.
for $l_2$. Straightforward calculations show that $W_\alpha$ is quasi-*$A(k)$ if and only if

$$W_\alpha = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_1 & 0 & 0 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & 0 & 0 & \cdots \\
0 & 0 & \alpha_3 & 0 & 0 & \cdots \\
0 & 0 & 0 & \alpha_4 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where

$$(\alpha_{i+1}^{k} \alpha_{i+2}^{k})^{1/k} \geq \alpha_i \ (i = 1, 2, 3, \cdots).$$

The following examples show that quasi-*$A$ operator and quasi-*$A(2)$ operator are independent.

**Example 1.4** A non-quasi-*$A$ and quasi-*$A(2)$ operator.

Let $T$ be a unilateral weighted shift operator with weighted sequence $(\alpha_i)$, given $\alpha_1 = 3, \alpha_2 = 1, \alpha_3 = 8, \alpha_4 = 3, \alpha_5 = \cdots$. Simple calculations show that $T$ is quasi-*$A(2)$ and a non-quasi-*$A$ operator.

**Example 1.5** A non-quasi-*$A(2)$ and quasi-*$A$ operator.

Let $T$ be a unilateral weighted shift operator with weighted sequence $(\alpha_i)$, given $\alpha_1 = 1, \alpha_2 = \frac{1}{3}, \alpha_3 = 2, \alpha_4 = \frac{1}{5}, \alpha_5 = 64, \alpha_6 = \cdots$. Simple calculations show that $T$ is quasi-*$A$ but not a quasi-*$A(2)$ operator.

### 2 Spectrum of Quasi-*$A(k)$ Operators

In the sequel, let $\sigma(T), \sigma_a(T), \sigma_p(T), \sigma_{ea}(T), \sigma_{jp}(T), \sigma_{ja}(T)$ for the spectrum of $T$, the approximate point spectrum of $T$, the point spectrum of $T$, the essential approximate point spectrum of $T$, the joint point spectrum of $T$, the joint approximate point spectrum of $T$, respectively. $\lambda \in \sigma_p(T)$ if there is a nonzero $x \in H$ such that $(T - \lambda)x = 0$. If in addition, $(T^* - \lambda)x = 0$, then $\lambda \in \sigma_{jp}(T)$. Analogously, $\lambda \in \sigma_a(T)$ if there is a sequence $\{x_n\}$ of unit vectors in $H$ such that $(T - \lambda)x_n \rightarrow 0$. If in addition, $(T^* - \lambda)x_n \rightarrow 0$, then $\lambda \in \sigma_{ja}(T)$.

Clearly, $\sigma_{jp}(T) \subseteq \sigma_p(T), \sigma_{ja}(T) \subseteq \sigma_a(T)$. In general, $\sigma_{jp}(T) \neq \sigma_p(T), \sigma_{ja}(T) \neq \sigma_a(T)$.

Recently, it was shown that, for some nonnormal operator $T$, the nonzero points of its point spectrum and joint point spectrum are identical, the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical[3, 7, 19–21]. In this section, we will extend that result to quasi-*$A(k)$ for $0 < k \leq 1$.

To prove the inclusion relation between quasi-*$A(k)$ operator and quasi-absolute-*-$k$-paranormal operator, we need the following lemma.

**Lemma 2.1** [10] Let $A$ be a positive linear operator on a Hilbert space $H$. Then

(i) $(A^\lambda x, x) \geq (Ax, x)^\lambda$ for any $\lambda > 1$ and $||x|| = 1$.

(ii) $(A^\lambda x, x) \leq (Ax, x)^\lambda$ for any $0 \leq \lambda \leq 1$ and $||x|| = 1$.

**Lemma 2.2** For each $k > 0$, every quasi-*$A(k)$ operator is a quasi-absolute-*-$k$-paranormal operator.
Hence, for all unit vector $T$ implies $\sigma$ such that $K$ that is, Therefore, Example 1.3 shows that $ja$ $T$ $\lambda x$ $\lambda x$ $\lambda Tx$ $\lambda Tx$ $\lambda x$. 

**Proof** Suppose that $T$ belongs to quasi-$*$-$A(k)$ for $k > 0$, i.e., $T^*|T|^2 T \leq T^*(T^*|T|^{2k} T)^{\frac{1}{2k+1}} T$. Then, for every $x \in H$, 

$$
||T^*T x||^{2(k+1)} = (T^*|T|^2 T x, x)^{(k+1)}

\leq (T^*(T^*|T|^{2k} T)^{1/2k+1} T x, x)^{(k+1)}

= ((T^*|T|^{2k} T)^{1/2k+1} T x, T x)^{(k+1)}

\leq (T^*|T|^{2k-2} T x) ||T x||^{2k}

= ||T^* T x||^2 ||T x||^k.
$$

Therefore, $||T^*T x||^{k+1} \leq ||T^* T^2 x|| ||T x||^k$ for every $x \in H$, that is, $T$ is quasi-absolute-$*$-$k$-paranormal for $k > 0$.

**Lemma 2.3** [6] Let $H$ be a complex Hilbert space. Then there exists a Hilbert space $K$ such that $H \subset K$ and a map $\varphi : B(H) \rightarrow B(K)$ such that 

(i) $\varphi$ is a faithful $*$-representation of the algebra $B(H)$ on $K$; 
(ii) $\varphi(A) \geq 0$ for any $A \geq 0$ in $B(H)$; 
(iii) $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.

**Lemma 2.4** [21] Let $\varphi : B(H) \rightarrow B(K)$ be Berberian’s faithful $*$-representation. Then $\sigma_ja(T) = \sigma_jp(\varphi(T))$.

**Lemma 2.5** Let $T$ be a quasi-$*$-$A(k)$ operator for $0 < k \leq 1$ and $\lambda \neq 0$. Then $T x = \lambda x$ implies $T^* x = \overline{\lambda} x$.

**Proof** Let $\lambda \neq 0$ and suppose $x \in N(T - \lambda)$, we get $T x = \lambda x$. Since $T$ is quasi-$*$-$A(k)$, for every unit vector $x \in H$, $||T^*T x||^{k+1} \leq ||T^*T^2 x|| ||T x||^k$, then 

$$
||T^* x||^{k+1} |\lambda|^{k+1} \leq |\lambda|^{k+2} ||T x||^k = |\lambda|^{k+2} (||T^{2k} x, x||)^{\frac{1}{2}}

\leq |\lambda|^{k+2} (||T^{2k} x||)^{\frac{1}{2}} = |\lambda|^{k+2} ||T x||^k

= |\lambda|^{2k+2}.
$$

Hence, for all $||x|| = 1$, $||T^* x||^2 \leq |\lambda|^2$, and then 

$$
||T^* x - \overline{\lambda} x|| = ((T - \lambda)^* x, (T - \lambda)^* x)

= ||T^* x||^2 - (x, \overline{T x}) - (\overline{T x}, x) + |\lambda|^2 ||x||^2

\leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2

= 0.
$$

Therefore, we have $||T^* x - \overline{\lambda} x|| = 0$, that is, $T^* x = \overline{\lambda} x$.

**Remark 2.6** The condition “$\lambda \neq 0$” cannot be omitted in Lemma 2.5. In fact, Example 1.3 shows that $T_{A,B}$ is a quasi-$*$-$A(k)$ operator, however for the vector $x =$
Thus we have

Hence

We have

\[ \lambda \]

\[ \mu \]

\[ T_{A,B}(x) = 0 \]

Therefore, the relation \( N(T_{A,B}) \subseteq N(T_{A,B}^*) \) does not always hold.

**Theorem 2.7** Let \( T \in B(H) \) be quasi-*A(k) for \( 0 < k \leq 1 \). Then

(i) \( \sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} \);

(ii) If \( (T - \lambda)x = 0,(T - \mu)y = 0 \), and \( \lambda \neq \mu \), then we have \( < x,y > = 0 \);

(iii) \( \sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\} \).

**Proof**: (i) Clearly by Lemma 2.5.

(ii) Without loss of generality, we assume \( \mu \neq 0 \). Then \( (T - \mu)^*y = 0 \) by Lemma 2.5. Thus we have \( \mu < x,y > = < x,T^*y > = \lambda < x,y > \). Since \( \lambda \neq \mu \), \( < x,y > = 0 \).

(iii) Let \( \varphi : B(H) \to B(K) \) be Berberian’s faithful *-representation of Lemma 2.3. In the following, we shall show that \( \varphi(T) \) is also a quasi-*A(k) operator.

In fact, since \( T \) is a quasi-*A(k) operator, by Lemma 2.3, we have

\[
(\varphi(T))^*[(\varphi(T))^*|\varphi(T)|_{2^k}^2 \varphi(T)]^{1/2} - |(\varphi(T))^*|^2 \varphi(T) \\
= \varphi(T^*[(T^*|T|_{2^k}^2)^{1/2} - |T^*|^2]T) \geq 0.
\]

Then

\[
\sigma_a(T) \setminus \{0\} = \sigma_a(\varphi(T)) \setminus \{0\} \quad \text{by Lemma 2.3}
\]

\[
= \sigma_p(\varphi(T)) \setminus \{0\} \quad \text{by Lemma 2.3}
\]

\[
= \sigma_{jp}(\varphi(T)) \setminus \{0\} \quad \text{by (i)}
\]

\[
= \sigma_{ja}(T) \setminus \{0\} \quad \text{by Lemma 2.4.}
\]

Recall that \( T \in B(H) \) is said to have finite ascent if \( N(T^n) = N(T^{n+1}) \) for some positive integer \( n \), where \( N(T) \) is the null space of \( T \).

**Theorem 2.8** If \( T \) is quasi-*A(k) for \( 0 < k \leq 1 \), then \( T - \lambda \) has finite ascent for each \( \lambda \in \mathbb{C} \).

**Proof**: If \( \lambda \neq 0 \), then \( N(T - \lambda) \subseteq N(T^* - \overline{\lambda}) \) by Lemma 2.5, thus \( N(T - \lambda) = N(T - \lambda)^2 \). If \( \lambda = 0 \), let \( x \in N(T^2) \), since \( T \) is quasi-*A(k), then

\[
||T^*Tx||^{k+1} \leq ||T|^kT^2x||||Tx||^k \quad \text{for every} \ x \in H.
\]

We have

\[
||T^2x||^{k+1} = ||T^*Tx||^{k+1} \leq ||T|^kT^2x||||Tx||^k
\]

\[
= (||T|^2T^2x,T^2x||^{1/2})||Tx||^k
\]

\[
\leq (||T^2T^2x,T^2x||^{1/2})||Tx||^k||T^2x||^{(1-k)}
\]

\[
= ||T^3x||^k||Tx||^k||T^2x||^{(1-k)} \quad \text{for every} \ x \in H.
\]

Hence \( ||T^2x|| = 0 \) implies \( Tx = 0 \). This shows that \( T - \lambda \) has finite ascent for each \( \lambda \in \mathbb{C} \).
Recall that $T \in B(H)$ has the single valued extension property (abbrev. SVEP), if for every open set $U$ of $\mathbb{C}$, the only analytic solution $f: U \to H$ of the equation

$$(T - \lambda)f(\lambda) = 0$$

for all $\lambda \in U$ is the zero function on $U$.

**Theorem 2.9** If $T$ is quasi-$\ast$-$A(k)$ for $0 < k \leq 1$, then $T$ has SVEP.

**Proof** Clearly by Theorem 2.8 and [14, Proposition 1.8].

As a simple consequence of the preceding result, we obtain

**Corollary 2.10** If $T$ is quasi-$\ast$-$A(k)$ for $0 < k \leq 1$, then

(i) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ is the space of functions analytic on an open neighborhood of $\sigma(T)$;

(ii) $T$ obeys $a$-Browder’s theorem, that is $\sigma_{ea}(T) = \sigma_{ab}(T)$, where $\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \text{ is a compact operator}\}$;

(iii) $a$-Browder’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

**Proof** Note that $T$ has SVEP, Corollary 2.10 follows by [1].

### 3 Tensor Products of $\ast$-$A(k)$ Operators

Given non-zero $T, S \in B(H)$, let $T \otimes S$ denote the tensor product on the product space $H \otimes H$. The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in B(H)$, but by no means all of them. The normaloid property is invariant under tensor products [16, p.623], $T \otimes S$ is normal if and only if $T$ and $S$ are normal [12, 18], however, there exist paranormal operators $T$ and $S$ such that $T \otimes S$ is not paranormal [4]. Duggal [8] showed that for non-zero $T, S \in B(H)$, $T \otimes S \in H(p)$ if and only if $T, S \in H(p)$. Recently, this result was extended to class $\ast$-$A$ operators and class $A$ operators in [9, 13], respectively.

In this section we consider the tensor products of $\ast$-$A(k)$ operators. The following key lemma is due to J. Stochel.

**Lemma 3.1** [18, Proposition 2.2] Let $A_1, A_2 \in B(H), B_1, B_2 \in B(K)$ be non-negative operators. If $A_1$ and $B_1$ are non-zero, then the following assertions are equivalent:

(i) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.

(ii) There exists $c > 0$ for which $A_1 \leq cA_2$ and $B_1 \leq c^{-1}B_2$.

**Theorem 3.2** Let $T, S \in B(H)$ be non-zero operators. Then $T \otimes S$ is a $\ast$-$A(k)$ operator if and only if $T$ and $S$ are $\ast$-$A(k)$ operators.

**Proof** By simple calculation we have

$T \otimes S$ is a $\ast$-$A(k)$ operator

$\Leftrightarrow [(T \otimes S)^*] [T \otimes S]^{2k} (T \otimes S)]^{1/2} \geq |(T \otimes S)^*|^2$

$\Leftrightarrow [(T \otimes S)^*] [T \otimes S]^{2k} (T \otimes S)]^{1/2} - |T^*|^2 \otimes |S^*|^2 \geq 0$

$\Leftrightarrow [(T^* \otimes S^*) ((T)^{2k} \otimes |S|^{2k})(T \otimes S)]^{1/2} - |T^*|^2 \otimes |S^*|^2 \geq 0$

$\Leftrightarrow (T^* [T^{2k}T])^{1/2} \otimes (S^* |S|^{2k}S)^{1/2} - |T^*|^2 \otimes |S^*|^2 \geq 0$

$\Leftrightarrow |T^*|^2 \otimes (S^* |S|^{2k}S)^{1/2} - |S^*|^2 + |(T^* |T^{2k}T)|^{1/2} - |T^*|^2 \otimes (S^* |S|^{2k}S)^{1/2} \geq 0$. 

Thus the sufficiency is easily proved. Conversely, suppose that \( T \otimes S \) belongs to \( *-A(k) \). Without loss of generality, it is enough to show that \( T \) belongs to \( *-A(k) \). Since \( T \otimes S \) is \( *-A(k) \), we obtain
\[
(T^* | T|^{2k} T)^{1\over 1+k} \otimes (S^* | S|^{2k} S)^{1\over 1+k} \geq |(T \otimes S)^*|^2.
\]
Therefore, by Lemma 3.1, there exists a positive real number \( l \) for which
\[
l(T^* | T|^{2k} T)^{1\over 1+k} \geq |T^*|^2
\]
and
\[
l^{-1}(S^* | S|^{2k} S)^{1\over 1+k} \geq |S^*|^2.
\]
Consequently, for arbitrary \( x, y \in H \), using Lemma 2.1 we have
\[
||T||^2 = ||T^*||^2 = \sup\{(|T^*|^2 x, x) : ||x|| = 1\}
\leq \sup\{l(T^* | T|^{2k} T)^{1\over 1+k} x, x) : ||x|| = 1\}
\leq l \sup\{((T^* | T|^{2k} T)x, x)^{1\over 1+k} : ||x|| = 1\}
\leq l(||T^*||^{2k} T||^{1\over 1+k} \leq l||T||^2
\]
and
\[
||S||^2 = ||S^*||^2 = \sup\{(|S^*|^2 y, y) : ||y|| = 1\}
\leq \sup\{l^{-1}(S^* | S|^{2k} S)^{1\over 1+k} y, y) : ||y|| = 1\}
\leq l^{-1} \sup\{((S^* | S|^{2k} S)y, y)^{1\over 1+k} : ||y|| = 1\}
\leq l^{-1}||S^*||^{2k} S||^{1\over 1+k} \leq l^{-1}||S||^2.
\]
Clearly, we must have \( l = 1 \), and hence \( T \) is a \( *-A(k) \) operator.

References


关于拟-\(\ast\)-A\(\langle k \rangle\)算子的注记

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摘要：本文引入了拟-\(\ast\)-A\(\langle k \rangle\)算子并研究其谱性质如下：(i) 如果\(T\) 是拟-\(\ast\)-A\(\langle k \rangle\) 算子，其中\(0 < k \leq 1\)，则谱映射定理对\(T\) 的本质近似谱成立。 (ii) 如果\(T\) 是拟-\(\ast\)-A\(\langle k \rangle\) 算子，其中\(0 < k \leq 1\)，则\(\sigma_m(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}\)。最后对拟-\(\ast\)-A\(\langle k \rangle\) 算子的张量积性质也进行了讨论。

关键词：拟-\(\ast\)-A\(\langle k \rangle\) 算子；本质近似谱；张量积

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