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CONSTRUCTION OF BIORTHOGONAL 3-BAND WAVELETS WITH SYMMETRY AND HIGH VANISHING MOMENTS

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Abstract: In this paper, we investigate the biorthogonal matrix extension problem with symmetry and its application to construction of 3-band biorthogonal wavelets with compact supported, symmetry and high vanishing moments. By using the theory of matrix extension, we obtain the method of three-dimensional biorthogonal matrix extension with symmetry and a step-by-step algorithm for construction of 3-band biorthogonal wavelets, which is easily implemented by computer program. Several examples are provided to illustrate the proposed algorithm and results in this paper.

Keywords: biorthogonality; symmetry; polyphase matrices; matrix extension; vanishing moments

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1 Introduction

As a generalization of orthogonal wavelets, the biorthogonal wavelets have become a fundamental tool in many areas of applied mathematics, from signal processing to numerical analysis [2, 8, 12, 13]. It is well known that 2-band orthogonal wavelet suffers from severe constraints, such as nontrivial symmetric (linear-phase) 2-band orthogonal wavelet does not exist [1]. Fortunately, multiwavelets and multiband wavelets with linear phase are designed as alternatives for more freedom and flexibility [2–5, 9, 10, 12]. (Bi)Orthogonal real-valued wavelets with symmetry and dilations being greater than two have been reported in [3–7, 9–14]. For example, symmetric compactly supported orthogonal real-valued wavelets with dilation factor 3 have been obtained by the method of undetermined coefficient and some special treatments in Chui and Lian [3]. By complicated calculation, only some examples of compactly supported C^1 symmetric orthogonal real-valued wavelets with dilation factor 4 have been obtained in [4].

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Biography: Zou Qingyun (1977–), male, born at Changde, Hunan, associate professor, major in wavelet analysis theory and image processing.

Under the framework of MRA, the construction of multiwavelet systems having some desirable properties say, (bi)orthogonality, symmetry, regularity, and so on can be reduced to two main parts: One firstly constructs dual scaling filters and functions and then the wavelet filters and wavelets, which should be able to inherit certain properties similar to those of their scaling filters. Therefore, in order to obtain a wavelet system, the linchpin is to design the scaling functions with good properties. It is well known that the second part can be formulated as a matrix extension problem, see [7, 17]. Goh and Yap in [15] studied the biorthogonal matrix extension problem and presented a step-by-step algorithm for deriving the extension matrices. Yet neither did they concern about the lengths of the coefficient supports of the extension matrices, nor did they considered any symmetry constrain on the extension matrices. In [16], Chui, Han and Zhuang proposed a dual chain approach for this problem, which first constructs a top-down dual-chain that essentially reduces the lengths of the coefficient supports of the given pair of vectors to zero and then derives a bottom-up dual-chain that produces the desired pair of extension matrices. However, the multiwavelet systems have unsolved questions in many applied areas such as image compression coding. In view of this, further study for multi-band wavelet systems is still valuable.

In this paper, we construct the 3-band wavelets, which is perfect reconstruction, biorthogonal, symmetric/antisymmetric, the high vanishing moments and compactly supported. First, by designating the length of the filters and the vanishing moments, we construct the scaling filters with high vanishing moments and symmetry by solving nonlinear equations with less parameters. Then, we provide a step-by-step algorithm to construct a class of biorthogonal 3-band wavelets with symmetry.

2 3-Band Scaling Function and Wavelets

In this section, we shall construct the scaling function with desired properties and propose an algorithm for obtaining symbols of the corresponding 3-band wavelets. First, we discuss several results that are useful in the following.

As the case of dyadic wavelet, a pair $(\phi(x), \widetilde{\phi}(x))$ of dual scaling functions can be expressed as the following dilation equations

$$\phi(x) = \sum_{k\in Z} h_0(k)\phi(3x-k) \tag{2.1}$$

and

$$\widetilde{\phi}(x) = \sum_{k\in Z} \widetilde{h}_0(k)\widetilde{\phi}(3x-k), \tag{2.2}$$

respectively, with

$$\langle \phi(. - k), \widetilde{\phi}(. - j)\rangle = \delta(k-j), \tag{2.3}$$

or equivalently,

$$\sum_k h_0(k)\widetilde{h}_0(k+3j) = 3\delta(j), \tag{2.4}$$

where h, g are real number, $\delta(j)$ denotes the Dirac sequence such that $\delta(j) = 1$ for $j = 0$ otherwise $\delta(j) = 0$.

The polyphase decomposition of $H_0(z)$ and $\widetilde{H}_0(z)$ are defined by $H_{0,i}(z) = \frac{1}{\sqrt{3}} \sum_k h_0(i+3k)z^k, i = 0, 1, 2$ and $\widetilde{H}_{0,i}(z) = \frac{1}{\sqrt{3}} \sum_k \widetilde{h}_0(i + 3k)z^k, i = 0, 1, 2$, respectively. It is well known that (2.4) is equivalent to the following equation

$$H_{0,0}(z)\widetilde{H}_{0,0}(z^{-1}) + H_{0,1}(z)\widetilde{H}_{0,1}(z^{-1}) + H_{0,2}(z)\widetilde{H}_{0,2}(z^{-1}) = 1. \tag{2.5}$$

In addition, we assume

$$\sum_k h_0(k) = \sum_k \widetilde{h}_0(k) = 3. \tag{2.6}$$

Definition 2.1 The filter h_i is symmetric/antisymmetric if and only if

$$h_i(k) = \pm h_i(N - 1 - k)(0 \leq k < N), \tag{2.7}$$

where N is the length of the filter. Equivalently,

$$H_i(z) = \pm z^{(N-1)} H_i(z^{-1}), \tag{2.8}$$

where $H_i(z) = \sum_{k \in Z} h_i(k)z^k$.

In [18], Zhang provided an effective algorithm to construct compactly supported symmetric biorthogonal multiwavelets by given a pair of biorthogonal matrices. Inspired by this idea, we shall provide a step-by-step algorithm to construct a class of compactly supported biorthogonal 3-band wavelets with symmetry. First, we need the following theorem.

Theorem 2.2 Suppose that $(A(z), \widetilde{A}(z))$ be a pair of 1×3 Laurent polynomial vectors such that

$$A(z)\widetilde{A}^T(z^{-1}) = 1, \tag{2.9}$$

and

$$\begin{aligned} &A_1(z) = A_1(z^{-1}), A_2(z) = A_2(z^{-1}), A_3(z) = -A_3(z^{-1}), \\ &\widetilde{A}_1(z) = \widetilde{A}_1(z^{-1}), \widetilde{A}_2(z) = \widetilde{A}_2(z^{-1}), \widetilde{A}_3(z) = -\widetilde{A}_3(z^{-1}), \end{aligned} \tag{2.10}$$

where $A(z) = (A_1(z), A_2(z), A_3(z))$, $\widetilde{A}(z) = (\widetilde{A}_1(z), \widetilde{A}_2(z), \widetilde{A}_3(z))$.

Let their coefficient support be $\mathrm{csupp}(A) = [-k, k]$, $\mathrm{csupp}(\widetilde{A}) = [-\widetilde{k}, \widetilde{k}]$, respectively. Then $A(z)$, and $\widetilde{A}(z)$ can be written as

$$\begin{aligned} A(z) &= (a_0, b_0, c_0)z^{-k} + (a_1, b_1, c_1)z^{1-k} + \cdots + (a_1, b_1, -c_1)z^{k-1} + (a_0, b_0, -c_0)z^k, \\ \widetilde{A}(z) &= (\widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0)z^{-\widetilde{k}} + (\widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1)z^{1-\widetilde{k}} + \cdots + (\widetilde{a}_1, \widetilde{b}_1, -\widetilde{c}_1)z^{\widetilde{k}-1} + (\widetilde{a}_0, \widetilde{b}_0, -\widetilde{c}_0)z^{\widetilde{k}}. \end{aligned}$$

Assume that $a_0, b_0, c_0, \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0$ are nonzero. Then there exists a pair $(u(z), \widetilde{u}(z))$ of 3×3 Laurent polynomial matrices with symmetry such that

(a) $(u(z), \widetilde{u}(z))$ are biorthogonal: $u(z)\widetilde{u}^T(1/z)) = I_3$ and the coefficient supports of $u(z), \widetilde{u}(z)$ satisfy $\mathrm{csupp}(u) \subseteq [-1, 1]$ and $\mathrm{csupp}(\widetilde{u}) \subseteq [-1, 1]$, respectively.

(b) $u(z), \widetilde{u}(z)$ reduce the lengths of the coefficient support of $A(z)$ and $\widetilde{A}(z)$, respectively.

(c) $B(z) = A(z)u_(z)$, $\widetilde{B}(z) = \widetilde{A}(z)\widetilde{u}_(z)$ satisfies (2.9) and (2.10).

Proof Let us recall the theory of symmetric polynomial operations.

Assume that Laurent polynomials $f_1(z)$, $f_2(z)$ are symmetric about the points l_1, l_2, respectively, and $f_3(z)$, $f_4(z)$ are antisymmetric about the points l_3, l_4, respectively. Then

(i) $f_1(z)f_2(z)$, $f_3(z)f_4(z)$ are symmetric about the points $l_1 + l_2$, $l_3 + l_4$, respectively, and $f_1(z)f_3(z)$, $f_2(z)f_4(z)$ are antisymmetric about the points $l_1 + l_3$, $l_2 + l_4$, respectively.

(ii) If $l_1 = l_2$, then $f_1(z) + f_2(z)$ is symmetric about the point l_1; If $l_3 = l_4$, then $f_3(z) + f_4(z)$ is antisymmetric about the points l_3.

Form (2.9), we have

$$a_0\widetilde{a}_0 + b_0\widetilde{b}_0 - c_0\widetilde{c}_0 = 0.$$

Define

$$q_1(z) = \frac{1}{2\sqrt{|c_0\widetilde{c}_0|}} \begin{bmatrix} \widetilde{a}_0(z^{-1}+1) & 2b_0 & \widetilde{a}_0(z^{-1}-1) \\ \widetilde{b}_0(z^{-1}+1) & -2a_0 & \widetilde{b}_0(z^{-1}-1) \\ -\widetilde{c}_0(z^{-1}-1) & 0 & -\widetilde{c}_0(z^{-1}+1) \end{bmatrix},$$

$$\widetilde{q}_1(z) = \mathrm{sgn}(c_0\widetilde{c}_0)\frac{1}{2\sqrt{|c_0\widetilde{c}_0|}} \begin{bmatrix} a_0(z^{-1}+1) & 2\widetilde{b}_0 & a_0(z^{-1}-1) \\ b_0(z^{-1}+1) & -2\widetilde{a}_0 & b_0(z^{-1}-1) \\ -c_0(z^{-1}-1) & 0 & -c_0(z^{-1}+1) \end{bmatrix}, \tag{2.11}$$

where $\mathrm{sgn}(x)$ is a sign function such that $\mathrm{sgn}(x) = 1$ for $x > 0$, $\mathrm{sgn}(x) = 0$ for $x = 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$.

By direct calculation, we can see that $q_1(z)$ and $\widetilde{q}_1(z)$ are biorthogonal. It is obvious that the columns of $q_1(z)$ are symmetric/antisymmetric about $-1/2, 0, -1/2$, respectively. Since

$$(a_0, b_0, -c_0)z^k q_1(z) = \frac{1}{2\sqrt{|c_0\widetilde{c}_0|}}(a_0\widetilde{a}_0 + b_0\widetilde{b}_0 + c_0\widetilde{c}_0)z^{k-1}(1, 0, 1)$$

and

$$(a_0, b_0, c_0)z^{-k} q_1(z) = \frac{1}{2\sqrt{|c_0\widetilde{c}_0|}}(a_0\widetilde{a}_0 + b_0\widetilde{b}_0 + c_0\widetilde{c}_0)z^{-k}(1, 0, -1),$$

we can see that $q_1(z)$ reduces the length of the coefficient support of $A(z)$ by 1. Note that $A_1(z), A_2(z), \widetilde{A}_1(z)$, $\widetilde{A}_2(z)$ in (2.10) are all symmetric about origin and $A_3(z), \widetilde{A}_3(z)$ are both antisymmetric about origin. It is easy to see that the columns of $A(z)q_1(z)$ are symmetric/antisymmetric about $-1/2, 0, -1/2$, respectively. It is similar to $\widetilde{A}(z)\widetilde{q}_1(z)$.

Therefore, without loss of generality, we suppose that

$$\begin{aligned} A(z)q_1(z) &= (d_0, 0, f_0)z^{-k} + \cdots + (d_0, e_1, -f_0)z^{k-1}, \\ \widetilde{A}(z)\widetilde{q}_1(z) &= (f_0, 0, d_0)z^{-\widetilde{k}} + \cdots + (f_0, \widetilde{e}_1, -d_0)z^{\widetilde{k}-1}. \end{aligned}$$

Define

$$q_2(z) = \frac{1}{2\sqrt{|d_0 f_0|}} \begin{bmatrix} f_0(z+1) & 0 & -f_0(z-1) \\ 0 & 2d_0 & 0 \\ d_0(z-1) & 0 & -d_0(z+1) \end{bmatrix},$$

$$\widetilde{q}_2(z) = \operatorname{sgn}(d_0 f_0) \frac{1}{2\sqrt{|d_0 f_0|}} \begin{bmatrix} d_0(z+1) & 0 & -d_0(z-1) \\ 0 & 2f_0 & 0 \\ f_0(z-1) & 0 & -f_0(z+1) \end{bmatrix}. \tag{2.12}$$

Clearly, $q_2(z)$ and $\widetilde{q}_2(z)$ are biorthogonal, and the columns of $q_2(z)$ are symmetric/antisymmetric about $1/2, 0, 1/2$, respectively.

Since $(d_0, 0, f_0)z^{-k}q_2(z) = \frac{d_0 f_0}{\sqrt{|d_0 f_0|}} z^{-k+1}(1, 0, -1)$ and

$$(d_0, e_0, -f_0)z^{k-1}q_2(z) = \frac{1}{\sqrt{|d_0 f_0|}} z^{k-1}(d_0 f_0, d_0 e_0, d_0 f_0),$$

we have $q_2(z)$ reduces the length of the coefficient support of $A(z)q_1(z)$ by 1. Moreover, the columns of $A(z)q_1(z)q_2(z)$ are symmetric/antisymmetric to $0, 0, 0$, respectively. It is similar to $\widetilde{A}(z)\widetilde{q}_1(z)\widetilde{q}_2(z)$.

In summary, let $u(z) = q_1(z)q_2(z)$ and $\widetilde{u}(z) = \widetilde{q}_1(z)\widetilde{q}_2(z)$. Then $(u(z), \widetilde{u}(z))$ satisfy (a), (b) and (c). Moreover, $u(z)$ and $\widetilde{u}(z)$ reduce the lengths of the coefficient support of $A(z)$ and $\widetilde{A}(z)$ by 2, respectively. That is, $\operatorname{csupp}(A(z)u(z)) = [-k+1, k-1]$ and $\operatorname{csupp}(\widetilde{A}(z)\widetilde{u}(z)) \subseteq [-\widetilde{k}+1, \widetilde{k}-1]$, respectively.

According to Theorem 2.2, we have a matrix extension algorithm for the polyphase vectors, see Algorithm 1.

Algorithm 1 (1) Let $(H_0(z), \widetilde{H}_0(z))$ be a pair of dual scaling symbols satisfying (2.4), (2.6).

(2) Define

$$A_0(z) = z^{-[N_1/6]} \left(H_{0,0}(z), H_{0,1}(z), H_{0,2}(z)\right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

$$\widetilde{A}_0(z) = z^{-[N_2/6]} \left(\widetilde{H}_{0,0}(z), \widetilde{H}_{0,1}(z), \widetilde{H}_{0,2}(z)\right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}, \tag{2.13}$$

where $[x]$ denotes the largest integer less than or equal to x for $x \in R$, N_1 and N_2 are the lengths of the filters h_0 and $\widetilde{h}_0$, respectively. Then $(A_0(z), \widetilde{A}_0(z))$ satisfy (2.9) and (2.10).

(3) Recursively applying (2.11) and (2.12) until

$$A_K(z) = A_{K-1}(z)u_{K-1}(z), \widetilde{A}_K(z) = \widetilde{A}_{K-1}(z)\widetilde{u}_{K-1}(z), \tag{2.14}$$

and $\deg(A_K(z)) = 0$, $\deg(\widetilde{A}_K(z)) \geq 0$.

Moreover, $A_K(z) = (1, 0, 0)$, $\widetilde{A}_K(z)) = (1, p_1(z), p_2(z))$ for Laurent polynomials $p_1(z)$ and $p_2(z)$ with symmetry. Define

$$u_K(z) = \begin{bmatrix} 1 & 0 & 0 \\ p_1(z^{-1}) & 1 & 0 \\ p_2(z^{-1}) & 0 & 1 \end{bmatrix}, \widetilde{u}_K(z) = \begin{bmatrix} 1 & -p_1(z) & -p_2(z) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.15}$$

By direct calculation, it is obvious that $(u_K(z), \widetilde{u}_K(z))$ are biorthogonal and $A_K(z)u_K(z) = \widetilde{A}_K(z)\widetilde{u}_K(z) = (1, 0, 0)$.

(4) Define

$$U(z) = \widetilde{u}_K^T(z^{-1})\widetilde{u}_{K-1}^T(z^{-1}) \cdots \widetilde{u}_0^T(z^{-1}),$$
$$\widetilde{U}(z) = u_K^T(z^{-1})u_{K-1}^T(z^{-1}) \cdots u_0^T(z^{-1}).$$

The wavelet symbols can be derived by

$$\begin{aligned} \begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} &= z^{[N/6]}U(z^3)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}, \\ \begin{bmatrix} \widetilde{H}_0(z) \\ \widetilde{H}_1(z) \\ \widetilde{H}_2(z) \end{bmatrix} &= z^{[N/6]}\widetilde{U}(z^3)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}, \end{aligned} \tag{2.16}$$

where $N = \max\{N_1, N_2\}$.

3 Examples

In this section, some examples are given to verify this scheme. we only discuss the examples of odd lengths 3-band scaling filters.

Example 3.1 Let the lengths of scaling filters be $(15, 9)$. Assume that the scaling symbols $(H_0(z), \widetilde{H}_0(z))$ have the following form

$$\begin{aligned} H_0(z) &= \left(\tfrac{1+z+z^2}{3}\right)^5 Q(z), \\ \widetilde{H}_0(z) &= \left(\tfrac{1+z+z^2}{3}\right)^3 \widetilde{Q}(z), \end{aligned} \tag{3.1}$$

where $(Q(z), \widetilde{Q}(z))$ are symmetric Laurent polynomials with degree $(4, 2)$.

It is obvious that $H_0(z)$ is symmetric if and only if $Q(z)$ is symmetric. By (2.4) and (2.6), we can obtain the associated scaling filters as follows:

$$\begin{aligned} h_0 &\approx [0.0302708750, 0.0197271260, 0.0109853080, -0.12261759, 0.011382944, 0.24928687, \\ &\quad 0.83207454, 0.93777986], \\ \widetilde{h}_0 &\approx [-0.20140256, -0.090291448, 0.13193077, 1.0694718, 1.1805829] \end{aligned}$$

(whereas the other half is symmetric and so skipped). Thus, the polyphase decomposition of $H_0(z)$ and $\widetilde{H}_0(z)$ are as follows:

$$\begin{bmatrix} H_{0,0} & H_{0,1} & H_{0,2} \end{bmatrix}, \begin{bmatrix} \widetilde{H}_{0,0} & \widetilde{H}_{0,1} & \widetilde{H}_{0,2} \end{bmatrix},$$

where

$$\begin{aligned}
H_{0,0} &\approx 0.0063423705z^4 + 0.14392584z^3 + 0.48039846z^2 - 0.070793301z + 0.017476898,\\
H_{0,1} &\approx 0.011389462z^4 + 0.0065719459z^3 + 0.54142745z^2 + 0.0065719459z + 0.011389462,\\
H_{0,2} &\approx 0.017476898z^4 - 0.070793301z^3 + 0.48039846z^2 + 0.14392584z + 0.0063423705,\\
\widetilde{H}_{0,0} &\approx 0.076170268z^2 + 0.61745982z - 0.11627982,\\
\widetilde{H}_{0,1} &\approx -0.052129792z^2 + 0.68160985z - 0.052129792,\\
\widetilde{H}_{0,2} &\approx -0.11627982z^2 + 0.61745982z + 0.076170268.
\end{aligned}$$

Define

$$q_1(z) = \begin{bmatrix} 0.43323436(z^{-1}+1) & -0.34795513 & 0.43323436(z^{-1}-1) \\ 0.79629877(z^{-1}+1) & 0.51455697 & 0.79629877(z^{-1}-1) \\ -2.0787065(z^{-1}-1) & 0 & -2.0787065(z^{-1}+1) \end{bmatrix},$$

$$\widetilde{q}_1(z) = \begin{bmatrix} 0.25727848(z^{-1}+1) & -1.5925975 & 0.25727848(z^{-1}-1) \\ 0.17397756/z(z^{-1}+1) & 0.86646871 & 0.17397756(z^{-1}-1) \\ -0.12026710(z^{-1}-1) & 0 & -0.12026710(z^{-1}+1) \end{bmatrix}.$$

We have

$$\begin{aligned}
A_0(z)q_1(z) &\approx [0.37597811z + 0.43749957 + 0.43749957/z + 0.37597811/z^2,\\
&\quad -0.014611995z + 0.042199421 - 0.014611995/z,\\
&\quad -0.31051300z - 1.01344319 + 1.01344319/z + 0.31051300/z^2],\\
\widetilde{A}_0(z)\widetilde{q}_1(z) &\approx [0.31051300 + 0.31051300/z, -0.80009451, -0.37597811 + 0.37597811/z].
\end{aligned}$$

It follows that $d_0 = 0.31051300, f_0 = 0.37597811$ in (2.12). Thus,

$$\begin{aligned}
A_0(z)q_1(z)q_2(z) &\approx [1.51276001 - 0.017108015z - 0.017108015/z,\\
&\quad -0.016078699z + 0.046435260 - 0.016078699/z,\\
&\quad 0.70047056z - 0.70047056/z],\\
\widetilde{A}_0(z)\widetilde{q}_1(z)\widetilde{q}_2(z) &\approx [0.68336254, -0.72710963, 0].
\end{aligned}$$

Thus, there exist

$$v = \begin{bmatrix} 0.68336254 & -0.72710963 & 0 \\ -0.72710963 & -0.68336254 & 0 \\ 0 & 0 & 1 \end{bmatrix}, v^{-1} = \begin{bmatrix} 0.68633245 & -0.73026967 & 0 \\ -0.73026967 & -0.68633245 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

such that $A_0(z)q_1(z)q_2(z)v = (1, p_1(z), p_2(z))$, $\widetilde{A}_0(z)\widetilde{q}_1(z)\widetilde{q}_2(z)v^{-1} = (1, 0, 0)$, where

$$\begin{aligned}
p_1(z) &\approx -1.13167449285 + 0.0234269828611z + 0.0234269828611z^{-1}, \\
p_2(z) &\approx 0.700470556308z - 0.700470556308z^{-1}.
\end{aligned}$$

Therefore, we can obtain $(u_0(z), \widetilde{u}_0(z))$ in (2.14) and $(u_1(z), \widetilde{u}_1(z))$ in (2.15).

We derive the polyphase matrices of high-pass filter banks as follows:

$$\begin{bmatrix} H_{1,0} & H_{1,1} & H_{1,2} \\ H_{2,0} & H_{2,1} & H_{2,2} \end{bmatrix}, \begin{bmatrix} \widetilde{H}_{1,0} & \widetilde{H}_{1,1} & \widetilde{H}_{1,2} \\ \widetilde{H}_{2,0} & \widetilde{H}_{2,1} & \widetilde{H}_{2,2} \end{bmatrix},$$

where

$$\begin{aligned}
& H_{1,0} \approx -0.006778z^3 + 0.4355z^2 - 0.01868z, \\
& H_{1,1} \approx -0.01217z^3 - 0.7957z^2 - 0.01217z, H_{1,2} \approx -0.01868z^3 + 0.4355z^2 - 0.006778z, \\
& H_{2,0} \approx 0.009281z^3 + 0.1709z^2 - 0.02557z, H_{2,1} \approx 0.01667z^3 - 0.01667z, \\
& H_{2,2} \approx 0.02557z^3 - 0.1709z^2 - 0.009281z, \\
& \widetilde{H}_{1,0} \approx -0.001784z^4 - 0.009312z^3 + 0.4372z^2 - 0.02233z + 0.002724, \\
& \widetilde{H}_{1,1} \approx 0.001221z^4 - 0.0195z^3 - 0.7764z^2 - 0.0195z + 0.001221, \\
& \widetilde{H}_{1,2} \approx 0.002724z^4 - 0.02233z^3 + 0.4372z^2 - 0.009312z - 0.001784, \\
& \widetilde{H}_{2,0} \approx 0.05336z^4 + 0.544z^3 + 2.818z^2 - 0.2624z + 0.08145, \\
& \widetilde{H}_{2,1} \approx -0.03652z^4 + 0.4012z^3 - 0.4012z + 0.03652, \\
& \widetilde{H}_{2,2} \approx -0.08145z^4 + 0.2624z^3 - 2.818z^2 - 0.544z - 0.05336.
\end{aligned}$$

From (2.16), the wavelet filters h_1, h_2 and $\widetilde{h}_1, \widetilde{h}_2$ can be obtained as follows:

$$\begin{aligned}
h_1 &\approx [0.032348717, 0.021081228, 0.011739358, -0.75426796, 1.3781973], \\
h_2 &\approx 4[-0.044296947, -0.028867731, -0.016075373, 0.29594173, 0], \\
\widetilde{h}_1 &\approx [-0.0047182543, -0.0021152562, 0.0030907400, 0.038680809, 0.033766352, \\
&\quad 0.016128444, -0.75722871, 1.3447918], \\
\widetilde{h}_2 &\approx [0.14107656, 0.063246501, -0.092413623, -0.45441064, -0.69483538, \\
&\quad -0.94219467, 4.8815912, 0]/4
\end{aligned}$$

(whereas the other half is symmetric/antisymmetric and so skipped). See Fig. 1 for the graphs of the scaling functions and wavelets in this example.

Example 3.2 Let the lengths of scaling filters be $(21, 15)$. Assume that the scaling symbols $(H_0(z), \widetilde{H}_0(z))$ have the following form

$$\begin{aligned}
H_0(z) &= \left(\tfrac{1+z+z^2}{3}\right)^7 Q(z), \\
\widetilde{H}_0(z) &= \left(\tfrac{1+z+z^2}{3}\right)^5 \widetilde{Q}(z),
\end{aligned} \tag{3.2}$$

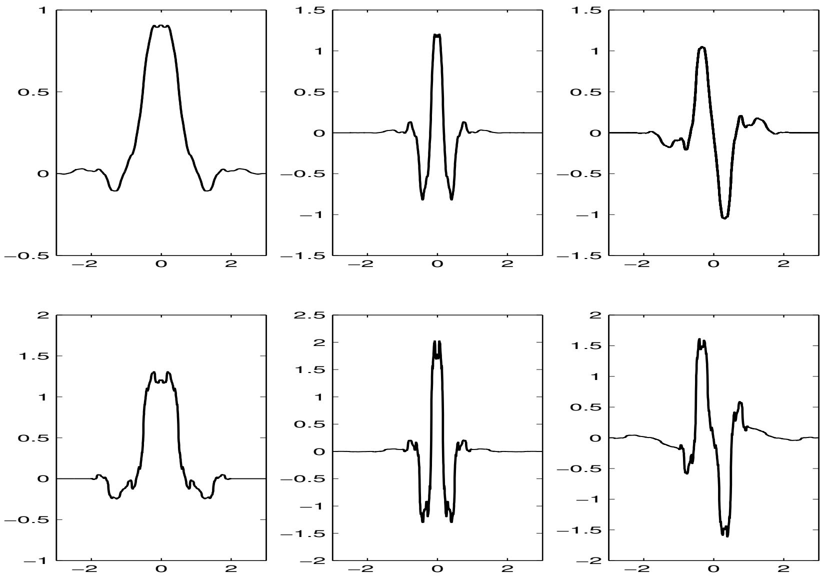

Figure 1: The graphs of ϕ, ψ_1, ψ_2(top row, from left to right) and the graphs of $\widetilde{\phi}, \widetilde{\psi}_1, \widetilde{\psi}_2$(bottom row, from left to right) in Example 3.1.

where $(Q(z), \widetilde{Q}(z))$ are symmetric Laurent polynomials with degree $(6,4)$. Thus,

$$\begin{aligned} h_0 &\approx [-0.0066877269, -0.0049799640, -0.0036014177, 0.041559784, 0.024297155, \\ &\quad -0.0036014177, -0.10184906, 0.0418949921, 0.28283863, 0.77648664, 0.87757563], \\ \widetilde{h}_0 &\approx [0.045518221, 0.014468319, -0.035285806, -0.30664390, -0.15213721, 0.18829749, \\ &\quad 1.1081140, 1.2753378]. \end{aligned}$$

We can obtain the polyphase decomposition of $H_0(z)$ and $\widetilde{H}_0(z)$ are as follows:

$$\begin{bmatrix} H_{0,0} & H_{0,1} & H_{0,2} \end{bmatrix}, \begin{bmatrix} \widetilde{H}_{0,0} & \widetilde{H}_{0,1} & \widetilde{H}_{0,2} \end{bmatrix},$$

where

$$\begin{aligned} &H_{0,0} \approx -0.002079z^6 + 0.006497z^5 + 0.1633z^4 + 0.4483z^3 - 0.0588z^2 + 0.02399z - 0.003861, \\ &H_{0,1} \approx -0.002875z^6 + 0.01403z^5 + 0.02419z^4 + 0.5067z^3 + 0.02419z^2 + 0.01403z - 0.002875, \\ &H_{0,2} \approx -0.003861z^6 + 0.02399z^5 - 0.0588z^4 + 0.4483z^3 + 0.1633z^2 + 0.006497z - 0.002079, \\ &\widetilde{H}_{0,0} \approx -0.02037z^4 + 0.1087z^3 + 0.6398z^2 - 0.1770z + 0.02628, \\ &\widetilde{H}_{0,1} \approx 0.008353z^4 - 0.08784z^3 + 0.7363z^2 - 0.08784z + 0.008353, \\ &\widetilde{H}_{0,2} \approx 0.02628z^4 - 0.1770z^3 + 0.6398z^2 + 0.1087z - 0.02037. \end{aligned}$$

Using the same method, we derive the polyphase matrices of high-pass filter banks as follows:

$$\begin{bmatrix} H_{1,0} & H_{1,1} & H_{1,2} \\ H_{2,0} & H_{2,1} & H_{2,2} \end{bmatrix}, \begin{bmatrix} \widetilde{H}_{1,0} & \widetilde{H}_{1,1} & \widetilde{H}_{1,2} \\ \widetilde{H}_{2,0} & \widetilde{H}_{2,1} & \widetilde{H}_{2,2} \end{bmatrix},$$

where

$$
\begin{aligned}
H_{1,0} &\approx 0.0008741z^5 - 0.008091z^4 + 0.3279z^3 - 0.02004z^2 + 0.001623z,\\
H_{1,1} &\approx 0.001209z^5 - 0.01331z^4 - 0.5803z^3 - 0.01331z^2 + 0.001209z,\\
H_{1,2} &\approx 0.001623z^5 - 0.02004z^4 + 0.3279z^3 - 0.008091z^2 + 0.0008741z,\\
H_{2,0} &\approx -0.002057z^5 + 0.008632z^4 + 0.1253z^3 - 0.02783z^2 + 0.003821z,\\
H_{2,1} &\approx -0.002845z^5 + 0.01693z^4 - 0.01693z^2 + 0.002845z,\\
H_{2,2} &\approx -0.003821z^5 + 0.02783z^4 - 0.1253z^3 - 0.008632z^2 + 0.002057z,\\
\widetilde{H}_{1,0} &\approx 0.0005821z^6 - 0.004607z^5 - 0.006087z^4 + 0.592z^3\\
&\quad -0.03672z^2 + 0.006994z - 0.0007509,\\
\widetilde{H}_{1,1} &\approx -0.0002387z^6 + 0.003125z^5 - 0.02922z^4 - 1.05z^3\\
&\quad -0.02922z^2 + 0.003125z - 0.0002387,\\
\widetilde{H}_{1,2} &\approx -0.0007509z^6 + 0.006994z^5 - 0.03672z^4 + 0.592z^3\\
&\quad -0.006087z^2 - 0.004607z + 0.0005821,\\
\widetilde{H}_{2,0} &\approx -0.02084z^6 + 0.09103z^5 + 0.9251z^4 + 3.782z^3 - 0.2686z^2 + 0.1551z - 0.02688,\\
\widetilde{H}_{2,1} &\approx 0.008544z^6 - 0.08157z^5 + 0.5993z^4 - 0.5993z^2 + 0.08157z - 0.008544,\\
\widetilde{H}_{2,2} &\approx 0.02688z^6 - 0.1551z^5 + 0.2686z^4 - 3.782z^3 - 0.9251z^2 - 0.09103z + 0.02084.
\end{aligned}
$$

From (2.16), the wavelet filters h_1, h_2 and $\widetilde{h}_1, \widetilde{h}_2$ can be obtained as follows:

$$
\begin{aligned}
h_1 &\approx [-0.0028113764, -0.0020934697, -0.0015139585, 0.034710876, 0.023051661,\\
&\quad 0.014014537, -0.56792580, 1.0051351],\\
h_2 &\approx 4[0.0066173484, 0.0049275573, 0.0035635181, -0.048208391, -0.029317970,\\
&\quad -0.014950592, 0.21698428, 0],\\
\widetilde{h}_1 &\approx [0.0013005736, 0.00041339737, -0.0010082069, -0.012113711, -0.0054124454,\\
&\quad 0.0079787011, 0.063598203, 0.050616897, 0.010543400, -1.0253619, 1.8188901],\\
\widetilde{h}_2 &\approx [-0.046554875, -0.014797828, 0.036089423, 0.26858835, 0.14128600, -0.15767139,\\
&\quad -0.46526530, -1.0379341, -1.6023583, 6.5501164, 0]/4
\end{aligned}
$$

(whereas the other half is antisymmetric and so skipped). The graphs of the scaling functions and wavelets are similar to Example 3.1.

References

[1] Daubechies I. Ten lectures on wavelets[M]. Philadelphia: SIAM Pub., 1992.

[2] Cohen A, Daubechies I, Feauveau J C. Biorthogonal bases of compactly supported wavelets[J]. Comm. on Pure and Applied Math., 1992, 45: 485–560.

[3] Chui C K, Lian J A. Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale=3[J]. Appl. Comput. Harmon. Anal., 1995, 45: 68–84.

[4] Han Bin. Symmetric orthonormal scaling functions and wavelets with dilation factor 4[J]. Adv. Comput. Math., 1998, 8: 221–247.

[5] Karoui A. Wavelet bases with a general integer dilation factor $d \geq 2$ and better regularity properties[J]. Appl. Math. Comput., 2009, 214(2): 557–568.

[6] Heller P. Rank M wavelets with N vanishing moments[J]. SIAM J. Math. Anal., 1995, 16(2): 502–519.

[7] Jiang Qingtang. Symmetric paraunitary matrix extension and parametrization of symmetric orthogonal multifilter banks[J]. SIAM J. Matrix Anal. Appl., 2001, 23: 167–186.

[8] Wang Guoqiu. Matrix methods of constructing wavelet filters and discrete hyper-wavelet transforms[J]. Optical Engineering, 2000, 39: 1080–1087.

[9] Wang Guoqiu. Four-bank compactly supported bi-symmetric orthonormal wavelets bases[J]. Optical Engineering, 2004, 43(10): 2362–2368.

[10] Soardi P M. Biorthogonal M-channel compactly supported wavelets[J]. Constr. Approx., 2000, 16: 283–311.

[11] Jiang Qingtang. Biorthogonal wavelets with 4-fold axial symmetry for quadrilateral surface multiresolution processing[J]. Adv. Comput. Math., 2011, 34: 127–165.

[12] Han Bin. Analysis and construction of optimal multivariate biorthogonal wavelets with compact support[J]. SIAM J. Math. Anal., 2000, 31(2): 274–304.

[13] Han Bin. Projectable multivariate refinable functions and biorthogonal wavelets[J]. Appl. Comput. Harmon. Anal., 2002, 13: 89–102.

[14] Cui Lihong. A method of construction for biorthogonal multiwavelets system with $2r$ multiplicity[J]. Appl. Math. Comput., 2005, 167: 907–918.

[15] Goh S S, Yap V B. Matrix extension and biorthogonal multiwavelet construction[J]. Linear Algebra Appl., 1998, 269: 139–157.

[16] Chui C K, Han Bin, Zhang Xiaosheng. A dual-chain approach for bottom-up construction of wavelet filters with any integer dilation[J]. Appl. Comput. Harmon. Anal., 2012, 33: 204–225.

[17] Han Bin, Zhuang Xiaosheng. Matrix extension with symmetry and its application to symmetric orthonormal multiwavelets[J]. SIAM J. Math. Anal., 2010, 42(5): 2297–2317.

[18] Zhuang Xiaosheng. Matrix extension with symmetry and construction of biorthogonal multiwavelets with any integer dilation[J]. Appl. Comput. Harmon. Anal., 2012, 33: 159–181.

具有高消失矩的3-进双正交对称小波的构造

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摘要: 本文用矩阵对称扩充来构造了具有高消失矩的3带对称双正交小波. 利用矩阵扩充, 获得了3维矩阵对称扩充方法和小波构造的算法, 并且, 该算法便于计算机程序化实现; 利用两个实例验证了相关的结论.

关键词: 双正交性; 对称性; 多相位矩阵; 矩阵扩充; 消失矩

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