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SURJECTIVE NUMERICAL RADIUS ISOMETRY ON $S(\mathcal{H}_n)$

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Abstract: In this article, we study the numerical radius isometry on matrix spaces. By using isometric embedding, we obtain surjective numerical radius isometry from the unit sphere of self-adjoint matrix space onto itself can be real-linear extended to the whole space, and give a method of Tingley isometric extension problem.

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1 Introduction

1.1 Linear Preserver Problems and Numerical Radius Isometry

Let $\mathcal{M}_n$ be the complex linear space of $n\times n$ ($n\geq 2$) matrices over complex field $\mathbb{C}$. $\mathcal{H}_n$ be the real linear space of $n\times n$ self-adjoint matrices over $\mathbb{C}$. $\Phi:\mathcal{M}_n\to\mathcal{M}_n$ is a map (may be non-linear). A general and interesting problem should been considered is as follows:

Problem 1.1 Find as few properties as possible that may be possessed by elements in $\mathcal{M}_n$ or the subset of $\mathcal{M}_n$ and that are enough to determine the structure of the map Φ if Φ takes these properties as invariants, i.e, if Φ preserves these properties.

When Φ is linear, the above problem is so-called linear preserver problems (LPP). The study of LPP can be traced back to the work of Frobenius in [7] and become one of the most active and fertile research fields in the matrix theory and operator theory during the past decades (see the survey paper [14]). Many results concerning LPP reveal the relations between linear structure and algebraic structure of operators algebras.

Problem 1.1 is also associated with the geometry of matrices whose study was initiated by Hua in the 1940s (see [10–13]). Consider the group of motions on $\mathcal{M}_n$ consisting of the following maps:

$$A\mapsto PAQ+R,\forall A\in\mathcal{M}_n \text{ or } A\mapsto PA^{tr}Q+R,\forall A\in\mathcal{M}_n,$$

where P and Q are $n\times n$ invertible matrices and R is an $n\times n$ matrix, A^{tr}denotes the transpose matrix of A. The fundamental problem of geometry of matrices is to characterize

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the group of motions by as few geometry invariants as possible [20]. Hua proved that "adjacency" (A and B are adjacent if $\mathrm{rank}(A-B)=1$) is such an invariant for $\mathcal{M}_n$.

The numerical range and numerical radius of $A \in \mathcal{M}_n$ are respectively defined as

$$W(A) = \{\langle Ax, x\rangle : x \in S(\mathbb{C}^n)\} = \{x^*Ax : x \in S(\mathbb{C}^n)\},$$
$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

These concepts and their generalizations were studied extensively because of their connections and applications to many different areas. In [15], Li and Šemrl fond that the geometry invariant "numerical radius distance" alone is sufficient to characterize the non-linear maps on upper triangular matrices. In [1], Bai and Hou fond that the geometry invariant "numerical radius distance" alone is sufficient to characterize the non-linear maps from $B(H)$ onto $B(K)$, where $B(H)$ is Banach space of all bounded linear operators on complex Hilbert space H. Precisely, it is shown that $\Phi : B(H) \to B(K)$ is a surjective numerical radius isometry, i.e, $\omega(\Phi(T)-\Phi(S)) = \omega(T-S)$ holds for all $T, S \in B(H)$, if and only if there exists complex unit μ and operator $R \in B(K)$ such that Φ takes one of the following forms:

(a) There exists a unitary or conjugate unitary operator $U : H \to K$ such that

$$\Phi(A) = \mu UAU^* + R, \forall A \in B(H);$$

(b) There exists a unitary or conjugate unitary operator $U : H \to K$ such that

$$\Phi(A) = \mu UA^*U^* + R, \forall A \in B(H).$$

1.2 Isometric Extension Between the Unit Spheres of Banach Spaces

Let E and F be Banach spaces, a mapping $T : E \to F$ is said to be isometric if $\|T(x) - T(y)\| = \|x - y\|$ for all $x, y \in E$.

In 1987, Tingley raised the following problem in [19]:

Problem 1.2 Let E and F be normed spaces with unit spheres $S_1(E)$ and $S_1(F)$. Suppose that $V_0 : S_1(E) \to S_1(F)$ is a surjective isometric mapping, is there a linear isometric mapping $V : E \to F$ such that $V\,|_{S_1(E)} = V_0$?

During the last two decades, many mathematicians have been engaged in Tingley's problem. This is a very nice mathematics problem since it is easy to understood but hard to solve. Some mathematicians asked whether the problem can be solved completely for finite dimensional Banach spaces. However, even in the case of two-dimensional Banach spaces, there is still no answer. In [19], Tingley only proved if E and F are finite-dimensional Banach spaces and $V_0 : S_1(E) \to S_1(F)$ is a surjective isometric mapping, then $V_0(-x) = -V_0(x)$ for all $x \in S_1(E)$.

During the last decade, many interesting results have been obtained on the above Tingley's problem. See [3, 4] and the survey paper [5]. All the results we mention above concern with isometries between the unit spheres of two Banach spaces of the same type. The first paper in which the isometric extension problem between Banach spaces of different types is consider in [2]. In that paper, isometric mappings from the unit sphere $S_1(E)$ to the unit sphere of $S_1(C(\Omega))$ (where E is a general Banach space and Ω is a compact Hausdorff space) is considered, and a condition concerning an inequality was given under which Tingley's problem has a positive answer. After two years, X.N. Fang in [6] obtained a generalization that the more general inequality:

$$\|T(x) - \lambda T(y)\| \leq \|x - \lambda y\| \ \ (\forall x, y \in S_1(E), \forall \lambda \in (0,1))$$

is a sufficient condition for having a positive answer of the Tingley's problem (where T is a surjective isometry from $S_1(E)$ onto $S_1(F)$).

Moreover, Fang and Wang obtained in [6] a positive answer in Tingley's problem between E and $C(\Omega)$ (where, Ω is a "metrizable" compact Hausdorff space), i.e., $\|T(x) - \lambda T(y)\| \leq \|x - \lambda y\|$ holds for all $x, y \in S_1(E)$ and $\lambda \in (0,1)$ when E is a Banach space and $F = C(\Omega)$. It is a beautiful result and obtained the above inequality by considering the unit spheres $S_1(E^*)$ and $S_1(F^*)$ of the dual spaces.

Tingley's problem doesn't make sense in complex spaces. A simple counterexample $E = F = \mathbb{C}, V_0(x) = \bar{x}, (\forall x \in \mathbb{C}, |x| = 1)$ shows there indeed exists a negative answer. However, any complex Banach space is also a real Banach space and any complex-linear isometry is also a real-linear, and therefore we can consider the real linear isometric extension between the real Banach spaces over the complex field $\mathbb{C}$.

Note that if Φ is a surjective numerical radius isometry on $\mathcal{H}_n$, let $\Psi(H) = \Phi(H) - \Phi(0)$, then $\omega(\Psi(H)) = \omega(\Phi(H))$ for every $H \in \mathcal{H}_n$ and $\Psi(0) = 0$. Since numerical radius is a norm on $\mathcal{H}_n$, from Mazur-Ulam Theorem [16] that Ψ is real-linear. Consequently, we can assume that Φ itself is real-linear.

In this paper, We try to characterize the group of motions (i.e., $H \mapsto PHQ$ *or* $H \mapsto PH^{\mathrm{tr}}Q$) on the unit sphere $S(\mathcal{H}_n) = \{H \in \mathcal{H}_n : \omega(H) = 1\}$ of $\mathcal{H}_n$ which preservers the "numerical radius distance" can determine the same type group of motions on the whole space $\mathcal{H}_n$ and also can preserver the "numerical radius distance". We obtain that surjective numerical radius isometry $\Phi : S(\mathcal{H}_n) \to S(\mathcal{H}_n)$ satisfying $\omega(\Phi(H_1) - \alpha\Phi(H_2)) \leq \omega(H_1 - \alpha H_2)$ for all $H_1, H_2 \in S(\mathcal{H}_n)$ and $\alpha \in (0, +\infty)$ can be real-linear extended to the whole space $\mathcal{H}_n$, and furthermore, there is a unitary matrix $U \in \mathcal{M}_n$ and a real number $\mu \in \{-1, 1\}$ such that one of the following is true:

(1) $\Phi(H) = \mu UHU^*$ for every $H \in S(\mathcal{H}_n)$;

(2) $\Phi(H) = \mu UH^{tr}U^*$ for every $H \in S(\mathcal{H}_n)$.

2 The Isometric Embedding Map Between $\mathcal{M}_n$ and $C(S(\mathbb{C}^n))$

Let $\mathcal{M}_n$ be the space of $n \times n$ $(n \geq 2)$ matrices over the complex field $\mathbb{C}$ and $S(\mathbb{C}^n) = \{x \in \mathbb{C}^n : \|x\| = 1\}$ be the unit sphere of $\mathbb{C}^n$. The numerical range (also know as the filed of values) and numerical radius of $A \in \mathcal{M}_n$ are defined, respectively, by

$$W(A) = \{\langle Ax, x\rangle : x \in S(\mathbb{C}^n)\} = \{x^*Ax : x \in S(\mathbb{C}^n)\},$$
$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

One may see [8] and chapter 1 in [9] for more information about numerical range and numerical radius.

Lemma 2.1 [18] Let $A \in \mathcal{M}_n$ and $\|A\| = \sup\{\|Ax\| : x \in S(\mathbb{C}^n)\}$ be the operator norm induced on $\mathcal{M}_n$. Then

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

Lemma 2.2 The numerical radius $\omega(\cdot)$ is a norm on $\mathcal{M}_n$ and $(\mathcal{M}_n, \omega(\cdot))$ is a Banach space.

Proof For every $A, B \in \mathcal{M}_n$ and $\lambda \in \mathbb{C}$, we have

(a) $\omega(A) \geq 0$ is trivial. We only to check that $\omega(A) = 0$ implies $A = 0$. By Lemma 2.1, $\|A\| \leq 2\omega(A) = 0$. So $\|A\| = 0$ and our claim follows.

(b) Absolute homogeneity.

$$\omega(\lambda A) = \sup_{\|x\|=1} |\langle \lambda Ax, x\rangle| = |\lambda| \sup_{\|x\|=1} |\langle Ax, x\rangle| = |\lambda|\omega(A).$$

(c) Subadditivity.

$$\begin{aligned}\omega(A+B) &= \sup_{\|x\|=1} |\langle (A+B)x, x\rangle| = \sup_{\|x\|=1} |\langle Ax, x\rangle + \langle Bx, x\rangle| \\ &\leq \sup_{\|x\|=1} (|\langle Ax, x\rangle| + |\langle Bx, x\rangle|) \leq \sup_{\|x\|=1} |\langle Ax, x\rangle| + \sup_{\|x\|=1} |\langle Bx, x\rangle| \leq \omega(A) + \omega(B).\end{aligned}$$

So the numerical radius $\omega(\cdot)$ is a norm on $\mathcal{M}_n$ and $(\mathcal{M}_n, \omega(\cdot))$ is a Banach space since it is finite-dimensional.

Since the unit sphere of $\mathbb{C}^n$, $S(\mathbb{C}^n) = \{x \in \mathbb{C}^n : \|x\| = 1\}$ is a compact metric space, $C(S(\mathbb{C}^n))$ is a Banach space of all the continuous complex-value functions on the compact metric space $S(\mathbb{C}^n)$.

Theorem 2.3 $(\mathcal{M}_n, \omega(\cdot))$ is isometrically isomorphic to the closed complex-linear subspace of $C(S(\mathbb{C}^n))$.

Proof For each $A \in \mathcal{M}_n$, define $f_A : S(\mathbb{C}^n) \to \mathbb{C}$ by

$$f_A(x) = x^*Ax, \forall x \in S(\mathbb{C}^n).$$

It is easy to see that f_A is continuous complex-value function on $S(\mathbb{C}^n)$. Then we get a map $A \to f_A$ from $\mathcal{M}_n$ to $C(S(\mathbb{C}^n))$.

First we show that $A \to f_A$ is a linear mapping from $\mathcal{M}_n$ to $C(S(\mathbb{C}^n))$.

In fact, For any $A, B \in \mathcal{M}_n$ and $\lambda \in \mathbb{C}$, we have

$$f_{A+B}(x) = x^*(A+B)x = x^*Ax + x^*Bx = f_A(x) + f_B(x) = (f_A + f_B)(x)$$

and

$$f_{\lambda A}(x) = x^*(\lambda A)x = \lambda x^* Ax = \lambda f_A(x) = (\lambda f_A)(x), \forall x \in S(\mathbb{C}^n)$$

Next we show that $A \to f_A$ is an isometric mapping from $\mathcal{M}_n$ to $C(S(\mathbb{C}^n))$.

$$\begin{aligned} & \|f_A - f_B\| = \sup\{|(f_A - f_B)(x)| : x \in S(\mathbb{C}^n)\} \\ = \ & \sup\{|f_A(x) - f_B(x)| : x \in S(\mathbb{C}^n)\} = \sup\{|x^* Ax - x^* Bx| : x \in S(\mathbb{C}^n)\} \\ = \ & \sup\{|x^*(A - B)x| : x \in S(\mathbb{C}^n)\} = \omega(A - B). \end{aligned}$$

Hence $\{f_A : f_A(x) = x^* Ax, \forall x \in S(\mathbb{C}^n), \forall A \in \mathcal{M}_n\}$ is a closed linear subspace of $C(S(\mathbb{C}^n))$ and $A \to f_A$ is an isometric isomorphism from $\mathcal{M}_n$ onto $\{f_A : f_A(x) = x^* Ax, \forall x \in S(\mathbb{C}^n), \forall A \in \mathcal{M}_n\}$.

We can easily get the similar conclusion on real-linear space $(\mathcal{H}_n, \omega(\cdot))$.

Corollary 2.4 $(\mathcal{H}_n, \omega(\cdot))$ is isometrically isomorphic to the closed real-linear subspace of $C(S(\mathbb{C}^n))$.

3 Numerical Radius Isometric Extension from $S(\mathcal{H}_n)$ onto Itself

3.1 The Properties of Numerical Radius Isometry from $S(\mathcal{M}_n)$ onto Itself

From Lemma 2.2, the space $(\mathcal{M}_n, \omega(\cdot))$ is a Banach space, we write $\mathcal{M}_n$ instead of $(\mathcal{M}_n, \omega(\cdot))$ for convenience. A mapping $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ is a numerical radius isometry if $\omega(\Phi(A) - \Phi(B)) = \omega(A - B)$ for all $A, B \in \mathcal{M}_n$. Denote by $S(\mathcal{M}_n) = \{A \in \mathcal{M}_n : \omega(A) = 1\}$ the unit sphere of $\mathcal{M}_n$.

Lemma 3.1 If $A, B \in S(\mathcal{M}_n)$ are real linearly independent such that $\alpha A + \beta B \in S(\mathcal{M}_n)$ holds for some $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$, then condition (a) and (b) are equivalent.

(a) There exists a complex unit μ such that $(A, B) = \mu(I, \pm iI)$.

(b) For any rank one matrix $A_1 \in S(\mathcal{M}_n)$, there are $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$ such that $\omega(\alpha A + \beta B + A_1) = 1 + \omega(A_1)$.

Proof It is obvious that (a) implies (b).

Now assume (b) holds. For any $x \in S(\mathbb{C}^n)$ and $\theta \in [0, 2\pi)$. Let $A_\theta = e^{i\theta} xx^*$, it follows from (b) that there are $\alpha_\theta, \beta_\theta \in \mathbb{R}$ with $\alpha_\theta^2 + \beta_\theta^2 = 1$ such that

$$\omega(\alpha_\theta A + \beta_\theta B + A_\theta) = 1 + \omega(A_\theta) = 2,$$

which implies that there exists $x_\theta \in S(\mathbb{C}^n)$ such that

$$|x_\theta^*(\alpha_\theta A + \beta_\theta B)x_\theta + x_\theta^* A_\theta x_\theta| = 2.$$

Since $\alpha_\theta A + \beta_\theta B \in S(\mathcal{M}_n)$ and $A_\theta \in S(\mathcal{M}_n)$, which implies that $|x^*(\alpha_\theta A + \beta_\theta B)x| \leq 1, |x^* A_\theta x| \leq 1$ for any $x \in S(\mathbb{C}^n)$. Hence

$$2 = |x_\theta^*(\alpha_\theta A + \beta_\theta B)x_\theta + x_\theta^* A_\theta x_\theta| \leq |x_\theta^*(\alpha_\theta A + \beta_\theta B)x_\theta| + |x_\theta^* A_\theta x_\theta| \leq 2.$$

So we have $|x_\theta^*(\alpha_\theta A+\beta_\theta B)x_\theta|=1$ and $|x_\theta^* A_\theta x_\theta|=1$. From which we obtain $x_\theta=x$ since $A_\theta=e^{i\theta}xx^*$. So $x_\theta^*(\alpha_\theta A+\beta_\theta B)x_\theta=e^{i\theta}$.

Suppose $x^*Ax=\alpha_1+i\alpha_2$ and $x^*Bx=\beta_1+i\beta_2$ with $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{R}$. Let $C=\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}\in\mathcal{M}_2(\mathbb{R})$. By above equation, for any $\theta\in[0,2\pi)$, we have

$$\begin{aligned} x^*(\alpha_\theta A+\beta_\theta B)x &= x^*(\alpha_\theta A)x+x^*(\beta_\theta B)x\\ &= \alpha_\theta(\alpha_1+i\alpha_2)+\beta_\theta(\beta_1+i\beta_2)\\ &= (\alpha_1\alpha_\theta+\beta_1\beta_\theta)+i(\alpha_2\alpha_\theta+\beta_2\beta_\theta)\\ &= e^{i\theta}. \end{aligned}$$

Now let $u_\theta=(\alpha_\theta,\beta_\theta)^{tr}$ then $u_\theta\in S(\mathbb{R}^2)$ and

$$Cu_\theta=\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}\begin{pmatrix}\alpha_\theta\\ \beta_\theta\end{pmatrix}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}=(\cos\theta,\sin\theta)^{tr}.$$

Hence C maps the unit ball in $\mathbb{R}^2$ onto itself, and thus C is an isometry on $\mathbb{R}^2$ with the form

$$C=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\in\mathcal{M}_2(\mathbb{R})\ \text{ or }\ C=\begin{pmatrix}\cos t & \sin t\\ \sin t & -\cos t\end{pmatrix}\in\mathcal{M}_2(\mathbb{R}).$$

It follows that $x^*Ax=\alpha_1+i\alpha_2$ is a complex unit and $x^*Bx=\pm ix^*Ax=x^*(\pm iA)x$.

Since $x\in S(\mathbb{C}^n)$ is arbitrary, we see that $B=\pm iA$, together with the convexity of $W(A)$, implies that $A=\mu I$ for some complex unit μ. Hence $(A,B)=\mu(I,\pm iI)$.

Lemma 3.2 Suppose $\Phi: S(\mathcal{M}_n)\to S(\mathcal{M}_n)$ is a surjective numerical radius isometry. If $\Phi(\alpha I+\beta iI)=\alpha\Phi(I)+\beta\Phi(iI)$ for all $\alpha,\beta\in\mathbb{R}$ with $\alpha^2+\beta^2=1$, then $(\Phi(I),\Phi(iI))=\mu(I,\pm iI))$ for some complex unit μ, and consequently, $\Phi(\lambda I)=\mu\lambda I$ or $\Phi(\lambda I)=\mu\bar\lambda I$ for all complex unit λ.

Proof It is easy to check that real linearly independent pair (I,iI) satisfying $\omega(\alpha I+\beta iI)=1$, i.e., $\alpha I+\beta iI\in S(\mathcal{M}_n)$ for all $\alpha,\beta\in\mathbb{R}$ with $\alpha^2+\beta^2=1$ and condition (b) of Lemma 3.1, that is $\omega(\alpha I+\beta iI+A_1)=1+\omega(A_1)$.

Since $\Phi: S(\mathcal{M}_n)\to S(\mathcal{M}_n)$ is a surjective numerical radius isometry, it follows that

$$\begin{aligned} \omega(\alpha\Phi(I)+\beta\Phi(iI)+\Phi(A_1)) &= \omega(\Phi(\alpha I+\beta iI)-\Phi(-A_1))\\ &= \omega(\alpha I+\beta iI+A_1)\\ &= 1+\omega(A_1)\\ &= 1+\omega(\Phi(A_1)). \end{aligned}$$

From Lemma 3.1, condition (a) holds, that is $(\Phi(I),\Phi(iI))=\mu(I,\pm iI)$.

Hence $\Phi(\lambda I)=\mu\lambda I$ or $\Phi(\lambda I)=\mu\bar\lambda I$ for all complex unit λ.

Theorem 3.3 Suppose $\Phi: S(\mathcal{M}_n)\to S(\mathcal{M}_n)$ is a surjective numerical radius isometry. If $\Phi(\alpha I+\beta iI)=\alpha\Phi(I)+\beta\Phi(iI)$ for all $\alpha,\beta\in\mathbb{R}$ with $\alpha^2+\beta^2=1$, then $W(\Phi(A))=W(\mu A),\forall A\in S(\mathcal{M}_n)$ for some complex unit μ.

Proof By Lemma 3.2, there have two cases.

Case 1 $\Phi(\lambda I) = \mu\lambda I$.

Assume that there exists $\gamma \in \mathbb{C}$ such that $\gamma \in W(\Phi(A)) \setminus W(\mu A)$. Then there is a circle with sufficient large radius and centered at a certain complex unit λ such that $W(A)$ lies inside the circle, but γ lies outside the circle. Hence

$$\omega(\mu A - \lambda I) < |\gamma - \lambda| \leq \omega(\Phi(A) - \lambda I) = \omega(\Phi(A) - \Phi(\bar{\mu}\lambda I)) = \omega(A - \bar{\mu}\lambda I) = \omega(\mu A - \lambda I),$$

which is a contradiction. So $W(\Phi(A)) \subseteq W(\mu A)$. Using the argument to Φ^{-1}, we obtain that $W(\mu A) \subseteq W(\Phi(A))$.

Case 2 $\Phi(\lambda I) = \mu\bar{\lambda} I$.

Assume that there exists $\gamma \in \mathbb{C}$ such that $\gamma \in W(\Phi(A)) \setminus W(\mu A)$. Then there is a circle with sufficient large radius and centered at a certain complex unit $\bar{\lambda}$ such that $W(A)$ lies inside the circle, but γ lies outside the circle. Hence

$$\omega(\mu A - \bar{\lambda} I) < |\gamma - \bar{\lambda}| \leq \omega(\Phi(A) - \bar{\lambda} I) = \omega(\Phi(A) - \Phi(\bar{\mu}\bar{\lambda} I)) = \omega(A - \bar{\mu}\bar{\lambda} I) = \omega(\mu A - \bar{\lambda} I).$$

Which is a contradiction. So $W(\Phi(A)) \subseteq W(\mu A)$. Using the argument to Φ^{-1}, we obtain that $W(\mu A) \subseteq W(\Phi(A))$. So $W(\Phi(A)) = W(\mu A)$ for all $A \in S(\mathcal{M}_n)$.

3.2 Numerical Radius Isometric Extension from $S(\mathcal{H}_n)$ onto Itself

In this section, we turn to consider real linear space $\mathcal{H}_n$ of self-adjoint matrices over complex field $\mathbb{C}$.

Let us start using some notation and terminology that will be used throughout the section. The space $(\mathcal{H}_n, \omega(\cdot))$ is a Banach space. We write $\mathcal{H}_n$ instead of $(\mathcal{H}_n, \omega(\cdot))$ for convenience. The unit sphere of $\mathcal{H}_n$ is $S(\mathcal{H}_n) = \{H \in \mathcal{H}_n : \omega(H) = 1\}$ and the unit ball of $\mathcal{H}_n$ is $B(\mathcal{H}_n) = \{H \in \mathcal{H}_n : \omega(H) \leq 1\}$. The dual space of $\mathcal{H}_n$ will be denoted by $(\mathcal{H}_n)^*$. Notice that the norm on $(\mathcal{H}_n)^*$ is defined as $\|f^*\| = \sup\{|f^*(H)| : H \in S(\mathcal{H}_n)\}$. Let $H \in S(\mathcal{H}_n)$, we set $St(H) = \{G \in S(\mathcal{H}_n) : \omega(H + G) = 2\}$.

To discuss the numerical radius isometry of $S(\mathcal{H}_n)$, we define the following relation $\lhd$ borrowed from [21].

Definition 3.4 [21] For $H_1, H_2 \in \mathcal{H}_n$, H_1 is said to be smaller than H_2 (denoted by $H_1 \lhd H_2$) if $\omega(H_1 + H) = \omega(H_1) + \omega(H)$ implies $\omega(H_2 + H) = \omega(H_2) + \omega(H)$ for all $H \in \mathcal{H}_n$.

The relation $\lhd$ has the following properties:

Lemma 3.5 [21] For any $H_1, H_2, H \in \mathcal{H}_n$, we have

(1) $H_1 \lhd H_2 \Longrightarrow \forall k_1, k_2 > 0, k_1 H_1 \lhd k_2 H_2$;

(2) $H_1 \lhd H_2 \Longrightarrow \omega(H_1 + H_2) = \omega(H_1) + \omega(H_2)$;

(3) $H_1 \lhd H_2 \Longleftrightarrow \omega(H_1 + H) = \omega(H_1) + 1$ implies $\omega(H_2 + H) = \omega(H_2) + 1, \forall H \in S(\mathcal{H}_n)$.

Lemma 3.6 [21] Suppose $\Phi : S(\mathcal{H}_n) \to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. Then for any $H_1, H_2 \in S(\mathcal{H}_n)$, $H_1 \lhd H_2 \Longleftrightarrow \Phi(H_1) \lhd \Phi(H_2)$.

Corollary 3.7 Suppose $\Phi : S(\mathcal{H}_n) \to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. Then for any $H_1, H_2 \in S(\mathcal{H}_n)$, $\omega(H_1 + H_2) = 2 \Longleftrightarrow \omega(\Phi(H_1) + \Phi(H_2)) = 2$.

Proof If $\omega(H_1+H_2)=2$, then there exists $x_0\in\mathbb{C}^n$ with $\|x_0\|=1$ such that

$$\omega(H_1)=|x_0^*H_1x_0|=\omega(H_2)=|x_0^*H_2x_0|=1.$$

For every $m\in\mathbb{N}$, put $G_m=(1-\frac{1}{m})H_1+\frac{1}{m}H_2$. Then $\omega(G_m)=\omega((1-\frac{1}{m})H_1+\frac{1}{m}H_2)\le 1$ and $|x_0^*G_mx_0|=|x_0^*((1-\frac{1}{m})H_1+\frac{1}{m}H_2)x_0|=1$. Hence $G_m\in S(\mathcal{H}_n)$ and $G_m\to H_1$ as $m\to\infty$, i.e., $\omega(G_m-H_1)\to 0$.

We assert that $G_m\lhd H_2$.

Suppose that $\omega(G_m+H)=\omega(G_m)+\omega(H)=2$ for some $H\in S(\mathcal{H}_n)$, then there exists $f^*\in S((\mathcal{H}_n)^*)$ such that $f^*(G_m+H)=\omega(G_m+H)=2$. Since $|f^*(G_m)|\le 1, |f^*(H)|\le 1$, we have $f^*(G_m)=f^*(H)=1$. It follows that $f^*(H_1)=f^*(H_2)=f^*(H)=1$. Hence $2=f^*(H_2+H)\le\omega(H_2+H)\le 2$, which implies that $\omega(H_2+H)=2$. By Lemma 3.5 (3), we obtain $G_m\lhd H_2$.

From Lemma 3.6, we have $\Phi(G_m)\lhd\Phi(H_2)$. Hence by Lemma 3.5 (2), we get $\omega(\Phi(G_m)+\Phi(H_2))=\omega(G_m)+\omega(H_2)=2$. Because $\omega(\Phi(G_m)-\Phi(H_1))=\omega(G_m-H_1)\to 0$ as $m\to\infty$, thus $\omega(\Phi(H_2)+\Phi(H_1))=\lim\limits_{m\to\infty}\omega(\Phi(H_2)+\Phi(G_m))=2$.

For the converse, we only to substitute Φ with Φ^{-1} in the above proof since Φ^{-1} also is an onto numerical radius isometry.

Definition 3.8 For $x,y\in\mathbb{C}^n$, we say x is equivalent to y (denoted by $x\sim y$) if $y=e^{i\theta}x$ for some θ in $[0,2\pi)$. The equivalent class of x is denoted by $[x]=\{e^{i\theta}x:\forall\theta\in[0,2\pi)\}$.

For each $x_0\in S(\mathbb{C}^n)$ and $\theta_0\in[0,2\pi)$, define three sets:

$$\begin{aligned}
&\mathcal{NRA}([x_0])=\{H\in S(\mathcal{H}_n):|x_0^*Hx_0|=1\},\\
&\mathcal{NRA}([x_0],1)=\{H\in S(\mathcal{H}_n):x_0^*Hx_0=1\},\\
&\mathcal{NRA}_1([x_0],1)=\{H\in S(\mathcal{H}_n):x_0^*Hx_0=1\,\text{and}\,|x^*Hx|<1,\forall x\in S(\mathbb{C}^n)\setminus[x_0]\}.
\end{aligned}$$

Next we give some properties of $\mathcal{NRA}([x_0],1)$ and $\mathcal{NRA}_1([x_0],1)$.

Remark 1 $S(\mathcal{H}_n)$ separates the points of $S(\mathbb{C}^n/\sim)$, i.e., If $x,y\in S(\mathbb{C}^n)$ and $[x]\ne[y]$, then there exists $H\in S(\mathcal{H}_n)$ such that $x^*Hx\ne y^*Hy$.

Proof If $x,y\in S(\mathbb{C}^n)$ and $[x]\ne[y]$, Then $x\ne y$. There must exists $1\le k\le n, k\in\mathbb{N}, x_k,y_k\in\mathbb{C}$ such that $x_k\ne y_k$ (where x_k and y_k is the kth component of x and y, respectively).

Case 1 If $|x_k|\ne|y_k|$, take $H=\mathrm{diag}(0,\cdots,0,\underset{kth}{1},0,\cdots,0)$, clearly $H\in S(\mathcal{H}_n)$ and $x^*Hx=|x_k|^2, y^*Hy=|y_k|^2$. Then $x^*Hx\ne y^*Hy$;

Case 2 If $|x_k|=|y_k|$, then there exists $\lambda\in\mathbb{C}$ with $|\lambda|=1$ such that $x_k=\lambda y_k$. Together with the hypothesis of $[x]\ne[y]$, there exists $1\le l\le n, l\in\mathbb{N}, l\ne k, x_l,y_l\in\mathbb{C}$ such that $x_l=\mu y_l$ with $\mu\ne\lambda$ (where x_l and y_l is the lth component of x and y, respectively).

Subcase 2a If $|x_l|\ne|y_l|$, Take $H=\mathrm{diag}(0,\cdots,0,\underset{lth}{1},0,\cdots,0)$;

Subcase 2b If $|x_l|=|y_l|$, then $|\mu|=1$ and $\mu\ne\lambda$ such that $x_l=\mu y_l$.

Take

$$G=\begin{pmatrix} \ddots & & & & \iddots \\ & 0 & \cdots & \bar{y_k}y_l & \\ & \vdots & & \vdots & \\ & y_k\bar{y_l} & \cdots & 0 & \\ \iddots & & & & \ddots \end{pmatrix}$$

(where $G=(g_{ij})$ is $n\times n$ matrix with $g_{kl}=y_k\bar{y_l}, g_{lk}=y_l\bar{y_k}$ and $g_{ij}=0,(i,j)\neq(k,l),(l,k)$).

Then $H=\frac{G}{\omega(G)}\in S(\mathcal{H}_n)$ and $x^*Hx=\frac{2}{\omega(G)}|y_ky_l|^2\Re(\lambda\bar{\mu})$ (where $\Re(\lambda\bar{\mu})$ denote the real part of $\lambda\bar{\mu}\in\lambda\bar{\mu}$), $y^*Hy=\frac{2}{\omega(G)}|y_ky_l|^2$. It easy to check that $x^*Hx\neq y^*Hy$ since $\lambda,\mu\in\mathbb{C},\lambda\neq\mu,|\lambda|=|\mu|=1$.

Hence if $[x]\neq[y]$, then there exists $H\in S(\mathcal{H}_n)$ such that $x^*Hx\neq y^*Hy$ for all $x\in[x]$ and $y\in[y]$.

Remark 2 $\mathcal{NRA}_1([x_0],1)\neq\emptyset$.

Proof For each $x_0\in S(\mathbb{C}^n)$, $x_0x_0^*\in\mathcal{NRA}_1([x_0],1)$.

Remark 3 Let $x_0\in S(\mathbb{C}^n)$, then for any $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1)$, we have $St(H_{[x_0]})=\mathcal{NRA}([x_0],1)$ and $\mathcal{NRA}([x_0],1)$ is a closed convex subset of $S(\mathcal{H}_n)$.

Proof Let $x_0\in S(\mathbb{C}^n)$ and $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1)$, then for any $H\in St(H_{[x_0]})$, we have

$$\begin{aligned}2&=\omega(H+H_{[x_0]})=\sup\{|x^*Hx+x^*H_{[x_0]}x|:x\in S(\mathbb{C}^n)\}\\&\leq\sup\{|x^*Hx|:x\in S(\mathbb{C}^n)\}+\sup\{|x^*H_{[x_0]}x|:x\in S(\mathbb{C}^n)\}=2.\end{aligned}$$

It follows that $x_0^*Hx_0=1$ and $H\in\mathcal{NRA}([x_0],1)$. Hence $St(H_{[x_0]})\subseteq\mathcal{NRA}([x_0],1)$.

Conversely, for any $H\in\mathcal{NRA}([x_0],1)$, we have $\omega(H+H_{[x_0]})=2$ for every $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1))$. It follows that $H\in St(H_{[x_0]})$. Hence $St(H_{[x_0]})\supseteq\mathcal{NRA}([x_0],1)$.

For any $H_1,H_2\in\mathcal{NRA}([x_0],1)$ and $t\in[0,1]$, we have $x_0^*((1-t)H_1+tH_2)x_0=1$ and $\omega((1-t)H_1+tH_2)\leq(1-t)\omega(H_1)+t\omega(H_2)=1$, which implies that $(1-t)H_1+tH_2\in\mathcal{NRA}([x_0],1)$. Hence $\mathcal{NRA}([x_0],1)$ is convex.

Suppose $\{G_m\}\subseteq\mathcal{NRA}([x_0],1)(m\in\mathbb{N})$ such that $G_m\to H$ as $m\to\infty$, i.e., $\omega(G_m-H)\to 0$, then $H\in S(\mathcal{H}_n)$ since $|\omega(G_m)-\omega(H)|\leq\omega(G_m-H)\to 0$ and $\omega(G_m)=1(\forall m\in\mathbb{N})$. Therefore, from $|x_0^*G_mx_0-x_0^*Hx_0|\leq\sup\{|x^*G_mx-x^*Hx|:x\in S(\mathcal{H}_n)\}=\omega(G_m-H)\to 0$, we have $x_0^*Hx_0=1$. Hence $H\in\mathcal{NRA}([x_0],1)$ and $\mathcal{NRA}([x_0],1)$ is closed.

Lemma 3.9 Suppose $\Phi:S(\mathcal{H}_n)\to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. Then for any $x_0\in S(\mathbb{C}^n)$ and $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1)$, we have

$$\Phi^{-1}(\mathcal{NRA}([x_0],1))=St(\Phi^{-1}(H_{[x_0]})).$$

Proof For any $H\in\Phi^{-1}(\mathcal{NRA}([x_0],1))$ and $\Phi(H)\in\mathcal{NRA}([x_0],1)$, we have

$$\omega(\Phi(H)+H_{[x_0]})=2.$$

By Corollary 3.7, we have

$$\omega(H+\Phi^{-1}(H_{[x_0]}))=2,$$

hence $H\in St(\Phi^{-1}(H_{[x_0]}))$.

Conversely, for any $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1)$ and $H\in St(\Phi^{-1}(H_{[x_0]}))$, we have $\omega(H+\Phi^{-1}(H_{[x_0]}))=2$. By Corollary 3.7, $\omega(\Phi(H)+H_{[x_0]})=2$, namely,

$$\Phi(H)\in St(H_{[x_0]})=\mathcal{NRA}([x_0],1).$$

Therefore, $H=\Phi^{-1}(\Phi(H))\in\Phi^{-1}(\mathcal{NRA}([x_0],1))$.

Thus $\Phi^{-1}(\mathcal{NRA}([x_0],1))=St(\Phi^{-1}(H_{[x_0]}))$ for any $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1)$.

Lemma 3.10 Suppose $\Phi:S(\mathcal{H}_n)\to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. Then for any $x_0\in S(\mathbb{C}^n)$, $\Phi^{-1}(\mathcal{NRA}([x_0],1))$ is a closed convex subset of $S(\mathcal{H}_n)$.

Proof Let $H_{[x_0]}\in\mathcal{NRA}_1([x_0],1)$ be fixed. By Lemma 3.9, we have

$$\omega(\Phi^{-1}(H_{[x_0]})+H_1)=\omega(\Phi^{-1}(H_{[x_0]})+H_2)=2$$

for any $H_1,H_2\in\Phi^{-1}(\mathcal{NRA}([x_0],1))$. Take $f_1^*\in S((\mathcal{H}_n)^*)$ such that

$$f_1^*(\Phi^{-1}(H_{[x_0]})+H_1)=\omega(\Phi^{-1}(H_{[x_0]})+H_1)=2.$$

Therefore, we have $f_1^*(\Phi^{-1}(H_{[x_0]}))=f_1^*(H_1)=1$ since $|f_1^*(\Phi^{-1}(H_{[x_0]}))|\le 1$ and $|f_1^*(H_1)|\le 1$. Hence

$$2=f_1^*(\Phi^{-1}(H_{[x_0]})+\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1))\le\omega(\Phi^{-1}(H_{[x_0]})+\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1))\le 2.$$

So $\omega(\Phi^{-1}(H_{[x_0]})+\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1))=2$. This means that

$$\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1)\in St(\Phi^{-1}(H_{[x_0]})=\Phi^{-1}(\mathcal{NRA}([x_0],1)).$$

So $\Phi(\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1))\in\mathcal{NRA}([x_0],1)$. From the assumption of $\Phi(H_2)\in\mathcal{NRA}([x_0],1)$ and the convexity of $\mathcal{NRA}([x_0],1)$, it is easy to get $\omega(\Phi(\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1))+\Phi(H_2))=2$. Thus from Corollary 3.7, we have

$$\omega(\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1)+H_2)=2.$$

Choose $f_2^*\in S((\mathcal{H}_n)^*)$ such that $f_2^*(\frac{1}{2}(\Phi^{-1}(H_{[x_0]})+H_1)+H_2)=2$. This implies that $f_2^*(\Phi^{-1}(H_{[x_0]}))=f_2^*(H_1)=f_2^*(H_2)=1$. Hence

$$2=f_2^*(\Phi^{-1}(H_{[x_0]})+\frac{1}{2}(H_1+H_2))\le\omega(\Phi^{-1}(H_{[x_0]})+\frac{1}{2}(H_1+H_2))\le 2.$$

So $\omega(\Phi^{-1}(H_{[x_0]})+\frac{1}{2}(H_1+H_2))=2$. It follows that

$$\frac{1}{2}(H_1+H_2)\in St(\Phi^{-1}(H_{[x_0]}))=\Phi^{-1}(\mathcal{NRA}([x_0],1)).$$

Therefore, $\Phi^{-1}(\mathcal{NRA}([x_0],1))$ is a convex subset.

Since Φ^{-1} is continuous and $\mathcal{NRA}([x_0],1)$ is closed, $\Phi^{-1}(\mathcal{NRA}([x_0],1))$ is a closed subset.

Lemma 3.11 Suppose $\Phi : S(\mathcal{H}_n) \to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. Then for any $x \in S(\mathbb{C}^n)$, there exists $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ such that $f^*_{[x]}(\Phi^{-1}(\mathcal{NRA}([x],1))) = 1$ (i.e., $f^*_{[x]}(H) = 1$, if $H \in \Phi^{-1}(\mathcal{NRA}([x],1))$).

Proof By Lemma 3.10, $\Phi^{-1}(\mathcal{NRA}([x],1))$ is a closed convex subset of the surface of unit ball $B(\mathcal{H}_n)$. Hence $\Phi^{-1}(\mathcal{NRA}([x],1))$ does not meet the interior of $B(\mathcal{H}_n)$. By Eidelheit Separation Theorem [17], there exists $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ such that $f^*_{[x]}(H) = 1$ for all H in $\Phi^{-1}(\mathcal{NRA}([x],1))$.

Now, for any $x \in S(\mathbb{C}^n)$, take $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ to be fixed as described in Lemma 3.10, then we obtain a map: $[x] \longrightarrow f^*_{[x]}, x \in S(\mathbb{C}^n)$.

Lemma 3.12 The map $[x] \longrightarrow f^*_{[x]}, x \in S(\mathbb{C}^n)$ is injective.

Proof Let $x, y \in S(\mathbb{C}^n), [x] \neq [y]$. Suppose $f^*_{[x]} = f^*_{[y]}$, then for any $H_{[x]} \in \mathcal{NRA}_1([x],1)$ and $H_{[y]} \in \mathcal{NRA}_1([y],1)$, we have

$$1 = f^*_{[x]}(\Phi^{-1}(H_{[x]})) = f^*_{[y]}(\Phi^{-1}(H_{[x]})) = f^*_{[y]}(\Phi^{-1}(H_{[y]})).$$

Hence

$$2 = f^*_{[y]}(\Phi^{-1}(H_{[x]}) + \Phi^{-1}(H_{[y]})) \leq \omega(\Phi^{-1}(H_{[x]}) + \Phi^{-1}(H_{[y]})) \leq 2.$$

It implies that $\omega(\Phi^{-1}(H_{[x]}) + \Phi^{-1}(H_{[y]})) = 2$. Which contradicts with

$$\omega(\Phi^{-1}(H_{[x]}) + \Phi^{-1}(H_{[y]})) = \omega(H_{[x]} + H_{[y]}) < 2$$

since $[x] \neq [y]$.

Lemma 3.13 Suppose $\Phi : S(\mathcal{H}_n) \to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. If $H \in \mathcal{NRA}([x])$, then there exists $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ such that $f^*_{[x]}(\Phi^{-1}(H)) = x^*Hx$.

Proof If $x^*Hx = 1$, then $H \in \mathcal{NRA}([x],1)$. By Lemma 3.11, there exists $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ such that $f^*_{[x]}(\Phi^{-1}(H)) = 1 = x^*Ax$.

If $x^*Hx = -1$, then $-H \in \mathcal{NRA}([x],1)$. Hence for any $H_{[x]} \in \mathcal{NRA}_1([x],1)$, we have

$$\omega(\Phi^{-1}(H) - \Phi^{-1}(H_{[x]})) = \omega(H - H_{[x]}) = \omega(-H + H_{[x]}) = 2.$$

It follows that $-\Phi^{-1}(H) \in St(\Phi^{-1}(H_{[x]}) = \Phi^{-1}(\mathcal{NRA}([x],1))$. By Lemma 3.11, there exists $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ such that $f^*_{[x]}(-\Phi^{-1}(H)) = 1$. Hence $f^*_{[x]}(\Phi^{-1}(H)) = -1 = x^*Hx$.

Theorem 3.14 Suppose $\Phi : S(\mathcal{H}_n) \to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. If $x \in S(\mathbb{C}^n)$ is an eigenvector of $H \in S(\mathcal{H}_n)$, then there exists $f^*_{[x]} \in S((\mathcal{H}_n)^*)$ such that

$$f^*_{[x]}(H) = x^*(\Phi(H))x$$

or

$$f^*_{[x]}(\Phi^{-1}(H)) = x^*Hx.$$

Proof Let $\{x_1, x_2, \cdots, x_n\} \subseteq S(\mathbb{C}^n)$ be pairwise orthogonal eigenvectors of H corresponding to eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ arranged in descending order. Then $\mu_i \in \mathbb{R}$ with $-1 \le \mu_i \le 1$ for all $1 \le i \le n$ and either $\mu_1 = 1$ or $\mu_n = -1$ for every self-adjoint matrix $H \in S(\mathcal{H}_n)$ (see [8, 9, 18]).

Assume x_0 is a eigenvector corresponding to eigenvalue μ_k $(1 \le k \le n)$ of H, then $\mu_k = x_0^* H x_0$. Take

$$G^{\pm} = (1 \mp \mu_k)x_k x_k^* \pm \sum_{i=1}^{n} \mu_i (x_i x_i^*)$$

and

$$U = (x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n),$$

where

$$G^{+} = (1 - \mu_k)x_k x_k^* + \sum_{i=1}^{n} \mu_i (x_i x_i^*)$$

and

$$G^{-} = (1 + \mu_k)x_k x_k^* - \sum_{i=1}^{n} \mu_i (x_i x_i^*).$$

It is easy to check that

$$\begin{aligned}
&G^{\pm} \in S(\mathcal{H}_n),\\
&x_0^* G^{\pm} x_0 = x_k^* G^{\pm} x_k = 1,\\
&U^* G^{\pm} U = \mathrm{diag}\{\pm\mu_1, \cdots, \pm\mu_{k-1}, 1, \pm\mu_{k+1}, \cdots, \pm\mu_n\},\\
&U^* H U = \mathrm{diag}\{\mu_1, \cdots, \mu_{k-1}, \mu_k, \mu_{k+1}, \cdots, \mu_n\}.
\end{aligned}$$

Hence $G^{\pm} \in \mathcal{NRA}([x_0], 1)$. By Lemma 3.13, we have $f^*_{[x_0]}(\Phi^{-1}(G^{\pm})) = 1$ and

$$\begin{aligned}
1 \mp f^*_{[x_0]}(\Phi^{-1}(H)) &= f^*_{[x_0]}(\Phi^{-1}(G^{\pm})) \mp f^*_{[x_0]}(\Phi^{-1}(H))\\
&= f^*_{[x_0]}(\Phi^{-1}(G^{\pm}) \mp \Phi^{-1}(H))\\
&\le \omega(\Phi^{-1}(G^{\pm}) \mp \Phi^{-1}(H))\\
&= \omega(G^{\pm} \mp H)\\
&= \omega(U^*(G^{\pm} \mp H)U)\\
&= 1 \mp \mu_k\\
&= 1 \mp x_0^* H x_0.
\end{aligned}$$

The above two inequalities imply $x_0^* H x_0 = f^*_{[x_0]}(\Phi^{-1}(H))$.

Therefore $G \in \mathcal{NRA}([x_0], 1)$. By Lemma 3.13, we have

$$f^*_{[x_0]}(\Phi^{-1}(G)) = 1$$

and

$$\begin{aligned} 1-f^*_{[x_0]}(\Phi^{-1}(H)) &= f^*_{[x_0]}(\Phi^{-1}(G))-f^*_{[x_0]}(\Phi^{-1}(H)) \\ &= f^*_{[x_0]}(\Phi^{-1}(G)-\Phi^{-1}(H)) \\ &\le \omega(\Phi^{-1}(G)-\Phi^{-1}(H)) \\ &= \omega(G-H) \\ &= \omega(U^*(G-H)U) \\ &= 1-x_0^*Hx_0. \end{aligned}$$

The above two inequalities imply $x_0^*Hx_0=f^*_{[x_0]}(\Phi^{-1}(H))$.

Lemma 3.15 Suppose $\Phi: S(\mathcal{H}_n)\to S(\mathcal{H}_n)$ is a surjective numerical radius isometry. If for any $H_1,H_2\in S(\mathcal{H}_n)$, we have $\omega(\Phi(H_1)-\alpha\Phi(H_2))\le\omega(H_1-\alpha H_2)$ for all $\alpha\in(0,+\infty)$, then Φ can be real linearly extended to numerical radius isometry $\widetilde{\Phi}$ of $\mathcal{H}_n$ onto itself.

Proof We first show that for any $H_1,H_2\in S(\mathcal{H}_n)$ and $\alpha\in(0,1)$, we have

$$\omega(H_1-\alpha H_2)=\sup\{\omega(H_1-H)-\omega(H-\alpha H_2): H\in S(\mathcal{H}_n)\}.$$

In fact, $\omega(H_1-\alpha H_2)\ge\omega(H_1-H)-\omega(H-\alpha H_2)$ for any $H\in S(\mathcal{H}_n)$. So

$$\omega(H_1-\alpha H_2)\ge\sup\{\omega(H_1-H)-\omega(H-\alpha H_2): H\in S(\mathcal{H}_n)\}.$$

Define $\phi(t)=\omega(tH_1+(1-t)\alpha H_2)$, $t\in(-\infty,0]$. Clearly, $\phi(0)=\alpha<1$ and

$$\phi(t)=\omega(t(H_1-\alpha H_2)+\alpha H_2)\ge|t|\omega(H_1-\alpha H_2)-\alpha\to+\infty,\ (t\to-\infty).$$

Then there exists $t_0<0$ such that $\phi(t_0)=1$, i.e., $G=t_0H_1+\alpha(1-t_0)H_2\in S(\mathcal{H}_n)$. Hence

$$\begin{aligned} &\omega(H_1-G)-\omega(G-\alpha H_2) \\ =\ &\omega(H_1-t_0H_1+\alpha(1-t_0)H_2)-\omega(t_0H_1+\alpha(1-t_0)H_2-\alpha H_2) \\ =\ &\omega(H_1-\alpha H_2). \end{aligned}$$

Thus $\omega(H_1-\alpha H_2)=\sup\{\omega(H_1-H)-\omega(H-\alpha H_2): H\in S(\mathcal{H}_n)\}$.

Since Φ is a surjective numerical radius isometry, we have

$$\begin{aligned} &\omega(H_1-\alpha H_2) \\ =\ &\sup\{\omega(H_1-H)-\omega(H-\alpha H_2): H\in S(\mathcal{H}_n)\} \\ \le\ &\sup\{\omega(\Phi(H_1)-\Phi(H))-\omega(\Phi(H)-\alpha\Phi(H_2)): H\in S(\mathcal{H}_n)\} \\ =\ &\omega(\Phi(H_1)-\alpha\Phi(H_2)). \end{aligned}$$

So $\omega(\Phi(H_1)-\alpha\Phi(H_2))=\omega(H_1-\alpha H_2)$ for all $H_1,H_2\in\mathcal{H}_n,\alpha\in(0,1)$.

For any $H\in\mathcal{H}_n$, define

$$\widetilde{\Phi}(H)=\begin{cases}\omega(H)\Phi(\frac{H}{\omega(H)}), & H\ne0;\\ 0, & H=0.\end{cases}$$

Clearly, $\omega(\widetilde{\Phi}(H))=\omega(H)$ and $\widetilde{\Phi}(\alpha^+H)=\alpha^+\widetilde{\Phi}(H),\alpha^+\in[0,+\infty)$. Thus, $\widetilde{\Phi}$ is surjective.

Finally, we show that for any $H_1,H_2\in\mathcal{H}_n$, we have $\omega(\widetilde{\Phi}(H_1)-\widetilde{\Phi}(H_2))=\omega(H_1-H_2)$.

If $H_1=0$ or $H_2=0$, it is clear that $\omega(\widetilde{\Phi}(H_1)-\widetilde{\Phi}(H_2))=\omega(H_1-H_2)$.

If $H_1,H_2\in\mathcal{H}_n$, $H_1\neq 0,H_2\neq 0$, without loss of generality we may assume that $\omega(H_1)\leq\omega(H_2)$, then

$$\begin{aligned}
&\omega(\widetilde{\Phi}(H_1)-\widetilde{\Phi}(H_2))\\
=\;&\omega(\omega(H_1)\Phi(\frac{H_1}{\omega(H_1)})-\omega(H_2)\Phi(\frac{H_2}{\omega(H_2)}))\\
=\;&\omega(H_2)\omega(\frac{\omega(H_1)}{\omega(H_2)}\Phi(\frac{H_1}{\omega(H_1)})-\Phi(\frac{H_2}{\omega(H_2)}))\\
=\;&\omega(H_2)\omega(\frac{\omega(H_1)}{\omega(H_2)}\frac{H_1}{\omega(H_1)}-\frac{H_2}{\omega(H_2)})\\
=\;&\omega(H_1-H_2).
\end{aligned}$$

Hence $\widetilde{\Phi}:\mathcal{H}_n\to\mathcal{H}_n$ is a surjective numerical radius isometry. From Mazur-Ulam Theorem [16], $\widetilde{\Phi}$ is a real linear numerical radius isometry of $\mathcal{H}_n$ onto itself with $\widetilde{\Phi}|_{S(\mathcal{H}_n)}=\Phi$ since $\widetilde{\Phi}(0)=0$.

Theorem 3.16 $\Phi:S(\mathcal{H}_n)\to S(\mathcal{H}_n)$ is a surjective numerical radius isometry satisfying $\omega(\Phi(H_1)-\alpha\Phi(H_2))\leq\omega(H_1-\alpha H_2)$ for all $H_1,H_2\in S(\mathcal{H}_n)$ and $\alpha\in(0,+\infty)$ if and only if there is a unitary matrix $U\in\mathcal{M}_n$ and a real number $\mu\in\{-1,1\}$ such that one of the following is true:

(1) $\Phi(H)=\mu UHU^*$ for every $H\in S(\mathcal{H}_n)$;

(2) $\Phi(H)=\mu UH^{tr}U^*$ for every $H\in S(\mathcal{H}_n)$.

Proof It is obvious that every map of the form (1) and (2) is a surjective numerical radius isometry satisfying $\omega(\Phi(H_1)-\alpha\Phi(H_2))\leq\omega(H_1-\alpha H_2)$ for all $H_1,H_2\in S(\mathcal{H}_n)$ and $\alpha\in(0,+\infty)$. So we only to check the "only if" part.

By Lemma 3.15, Φ can be real linearly extended to the whole space $\mathcal{H}_n$. Using Theorem 2 in [1], there is a unitary matrix $U\in\mathcal{M}_n$ and a real number $\mu\in\{-1,1\}$ such that one of the following is true:

(1) $\Phi(H)=\mu UHU^*$ for every $H\in S(\mathcal{H}_n)$;

(2) $\Phi(H)=\mu UH^{tr}U^*$ for every $H\in S(\mathcal{H}_n)$.

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$S(\mathcal{H}_n)$上的满数值半径等距

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摘要: 本文研究了矩阵空间到自身的满数值半径等距问题. 利用等距嵌入方法, 获得了自共轭矩阵空间单位球面到自身的满数值半径等距可实线性延拓至全空间上的满数值半径等距, 为Tingley等距延拓问题提供了一种方法.

关键词: 数值域; 数值半径; 等距

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