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SOME NEW EIGENVALUE BOUNDS FOR THE HADAMARD PRODUCT AND THE FAN PRODUCT OF MATRICES

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Abstract: A lower bound for the minimum eigenvalue for the Fan product of nonsingular M-matrices A and B and an upper bound for the spectral radius of Hadamard product of nonnegative matrices A and B are studied in this paper. By using the Brauer's theorem, some new estimating formulas of the bounds are obtained, which depend only on the entries and are easier to calculate. These bounds improve some results of [4].

Keywords: M-matrix; nonnegative matrix; Hadamard product; Fan product; spectral radius; minimum eigenvalue

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1 Introduction

For a positive integer n, N denotes the set $\{1, 2, \cdots, n\}$. The set of all $n \times n$ complex matrices is denoted by $C^{n\times n}$ and $R^{n\times n}$ denotes the set of all $n \times n$ real matrices.

We let Z_n denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An $n \times n$ matrix A is called an M-matrix if there exists an $n \times n$ nonnegative matrix B and a nonnegative real number λ such that $A = \lambda I - B$ and $\lambda \geq \rho(B)$, I is the identity matrix; if $\lambda > \rho(B)$, we call A a nonsingular M-matrix; if $\lambda = \rho(B)$, we call A a singular M-matrix. Denote by M_n the set of nonsingular M-matrices.

Let $A \in Z_n$ and $\tau(A)$ =min$\{Re(\lambda) : \lambda \in \sigma(A)\}$. Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [1]):

(1) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.

(2) If $A, B \in M_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$.

(3) If $A \in M_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

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Let $A, B \in C^{n\times n}$. The Fan product of A and B is denoted by $A\bigstar B \equiv C = (c_{ij}) \in C^{n\times n}$ and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

If $A, B \in M_n$, then so is $A\bigstar B$. In [1–3], the following bounds for $\tau(A\bigstar B)$ are given for two nonsingular M-matrices A and B, respectively.

$$\begin{aligned} &\tau(A\bigstar B) \geq \tau(A)\tau(B), \\ &\tau(A\bigstar B) \geq (1 - \rho(J_A)\rho(J_B)) \min_{1\leq i\leq n} (a_{ii}b_{ii}), \\ &\tau(A\bigstar B) \geq \min_{1\leq i\leq n} \{a_{ii}\tau(B) + b_{ii}\tau(A) - \tau(A)\tau(B)\}. \end{aligned}$$

Recently, Li [4] gave a sharper lower bound for $\tau(A\bigstar B)$, that is

$$\tau(A\bigstar B) \geq \min_{1\leq i\leq n} \{a_{ii}b_{ii} - m_i \sum_{k\neq i} \frac{|b_{ki}|}{h_k}\}.$$

Let $A = (a_{ij}) \in C^{n\times n}$ and $B = (b_{ij}) \in C^{n\times n}$. We write $A \geq B (> B)$ if $a_{ij} \geq b_{ij} (> b_{ij})$ for all $i, j \in \{1, 2, \cdots, n\}$. If 0 is the null matrix and $A \geq 0 (> 0)$, we say that A is a nonnegative (positive) matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A)$ is an eigenvalue of A.

An $n \times n$ matrix A is reducible if there exists a permutation matrix P such that

$$P^T AP = [\begin{array}{cc} B & 0 \\ C & D \end{array}],$$

where B, D are square matrices of order at least one. If A is not reducible, then it is called irreducible. Note that any 1×1 matrix is irreducible.

For two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the Hadamard product of A and B is $A \circ B = (a_{ij}b_{ij})$.

In [1–3], the following bounds for $\rho(A \circ B)$ are given for $A, B \geq 0$, respectively.

$$\begin{aligned} &\rho(A \circ B) \leq \rho(A)\rho(B), \\ &\rho(A \circ B) \leq (1 + \rho(J'_A\rho(J'_B))) \max_{1\leq i\leq n} a_{ii}b_{ii}, \\ &\rho(A \circ B) \leq \max_{1\leq i\leq n} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\}. \end{aligned}$$

Recently, Li [4] gave a sharper upper bound for $\rho(A \circ B)$, that is

$$\rho(A \circ B) \leq \max_{1\leq i\leq n} \{a_{ii}b_{ii} + m_i \sum_{k\neq i} \frac{b_{ki}}{h_k}\}.$$

In this paper, we give a new lower bound on $\tau(A\bigstar B)$ for two matrices $A, B \in M_n$ in Section 2 and a new upper bound on $\rho(A \circ B)$ for two nonnegative matrices A and B in Section 3. Some examples are given to illustrate our results.

In the following, we will need the notations:

$$R_i = \sum_{k\neq i} |a_{ik}|, \qquad d_i = \frac{R_i}{|a_{ii}|}, \quad i \in N;$$

$$m_{ji} = |a_{ji}|h_j, \quad m_i = \max_{j\neq i}\{m_{ji}\}, \quad i,j \in N, \quad h_j = \begin{cases} d_j, & d_j \neq 0, \\ 1, & d_j = 0. \end{cases}$$

2 A Lower Bound for the Minimum Eigenvalue of the Fan Product of M-Matrices

In this section, we will give a lower bound for $\tau(A\bigstar B)$. In order to prove our results, we first give some lemmas.

Lemma 2.1 [5] Let $A = (a_{ij}) \in C^{n\times n}$. Then all the eigenvalues of A lie in the region:

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in C : |z - a_{ii}||z - a_{jj}| \leq \sum_{k\neq i} |a_{ki}| \sum_{l\neq j} |a_{lj}|\}.$$

Lemma 2.2 Let $A = (a_{ij}) \in C^{n\times n}$ and let $x_1, x_2, \cdots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region:

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in C : |z - a_{ii}||z - a_{jj}| \leq (x_i \sum_{k\neq i} \frac{1}{x_k}|a_{ki}|)(x_j \sum_{l\neq j} \frac{1}{x_l}|a_{lj}|)\}.$$

Proof Let $x_1, x_2, \cdots, x_n$ be positive real numbers, and define $X = \mathrm{diag}(x_1, x_2, \cdots, x_n), B = (b_{ij}) = X^{-1}AX$. Then we have

$$B = (b_{ij}) = X^{-1}AX = \begin{bmatrix} a_{11} & \frac{x_2}{x_1}a_{12} & \cdots & \frac{x_n}{x_1}a_{1n} \\ \frac{x_1}{x_2}a_{21} & a_{22} & \cdots & \frac{x_n}{x_2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1}{x_n}a_{n1} & \frac{x_2}{x_n}a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Since $B = X^{-1}AX$, we have that $\sigma(A) = \sigma(B)$. By Lemma 2.1, there exists a pair(i, j) for positive integers with $i \neq j$ such that

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in C : |z - b_{ii}||z - b_{jj}| \leq \sum_{k\neq i} |b_{ki}| \sum_{l\neq j} |b_{lj}|\}.$$

Observe that

$$a_{ii} = b_{ii}, \quad a_{jj} = b_{jj}, \quad b_{ki} = \frac{x_i}{x_k}a_{ki}, \quad b_{lj} = \frac{x_j}{x_l}a_{lj}.$$

Thus, we have

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{z \in C : |z - a_{ii}||z - a_{jj}| \leq (\sum_{k\neq i} \frac{x_i}{x_k}|a_{ki}|)(\sum_{l\neq j} \frac{x_j}{x_l}|a_{lj}|)\}.$$

That is

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n}\{z\in C:|z-a_{ii}||z-a_{jj}|\leq (x_i\sum_{k\neq i}\frac{1}{x_k}|a_{ki}|)(x_j\sum_{l\neq j}\frac{1}{x_l}|a_{lj}|)\}.$$

Theorem 2.1 Let $A, B \in R^{n\times n}$ be two nonsingular M-matrices. Then

$$\tau(A\bigstar B)\geq \min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}. \tag{2.1}$$

Proof It is evident that (2.1) is an equality for $n=1$.

We next assume that $n\geq 2$.

If $A\bigstar B$ is irreducible, then A and B are irreducible. Let λ be an eigenvalue of $A\bigstar B$ and satisfy $\tau(A\bigstar B)=\lambda$, so that $0<\lambda<a_{ii}b_{ii}$, $\forall i\in N$. Thus, by Lemma 2.2, there is a pair (i,j) of positive integers with $i\neq j$ such that

$$\begin{aligned}|\lambda-a_{ii}b_{ii}||\lambda-a_{jj}b_{jj}| &\leq (m_i\sum_{k\neq i}\frac{1}{m_k}|a_{ki}b_{ki}|)(m_j\sum_{l\neq j}\frac{1}{m_l}|a_{lj}b_{lj}|)\\ &\leq (m_i\sum_{k\neq i}\frac{|a_{ki}b_{ki}|}{|a_{ki}|h_k})(m_j\sum_{l\neq j}\frac{|a_{lj}b_{lj}|}{|a_{lj}|h_l})\\ &= (m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l}).\end{aligned}$$

Thus, we have

$$\lambda \geq \frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}.$$

That is

$$\begin{aligned}&\tau(A\bigstar B)\geq \frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}\\ &\geq \min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}.\end{aligned}$$

Now, assume that $A\bigstar B$ is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive(see condition (E17) of Theorem 6.2.3 of [6]). If we denote by $D=(d_{ij})$ the $n\times n$ permutation matrix with $d_{12}=d_{23}=\cdots=d_{n-1,n}=d_{n1}=1$, the remaining d_{ij} zero, then both $A-tD$ and $B-tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A-tD$ and $B-tD$ are positive. Now we substitute $A-tD$ and $B-tD$ for A and B, respectively in the previous case, and then letting $t\longrightarrow 0$, the result follows by continuity.

Theorem 2.2 Let $A, B \in R^{n\times n}$ be two nonsingular M-matrices. Then

$$\begin{aligned}
&\min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}\\
&\geq \min_{1\leq i\leq n}\{a_{ii}b_{ii}-m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k}\}.
\end{aligned}$$

Proof Without loss of generality, for $i\neq j$, assume that

$$a_{ii}b_{ii}-m_i\sum_{k\neq i}\frac{|b_{ki}}{h_k}\leq a_{jj}b_{jj}-m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l}. \tag{2.2}$$

Thus, (2.2) is equivalent to

$$m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l}\leq m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k}+a_{jj}b_{jj}-a_{ii}b_{ii}. \tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned}
&\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\\
\geq\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\
&+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k}+a_{jj}b_{jj}-a_{ii}b_{ii})]^{\frac{1}{2}}\}\\
=\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\
&+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(a_{jj}b_{jj}-a_{ii}b_{ii})]^{\frac{1}{2}}\}\\
=\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{jj}b_{jj}-a_{ii}b_{ii}+2m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})^2]^{\frac{1}{2}}\}\\
=\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-(a_{jj}b_{jj}-a_{ii}b_{ii}+2m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})\}\\
=\ &a_{ii}b_{ii}-m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\tau(A\bigstar B)\ &\geq\ \min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}\\
&\geq\ \min_{1\leq i\leq n}\{a_{ii}b_{ii}-m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k}\}.
\end{aligned}$$

Example 2.1 Let

$$A=\begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}, \qquad B=\begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{bmatrix}.$$

Then

$$A\bigstar B=\begin{bmatrix} 4 & -0.5 & 0 & 0 \\ -1 & 5 & -0.5 & 0 \\ 0 & -1 & 4 & -0.5 \\ 0 & 0 & -0.5 & 4 \end{bmatrix}.$$

By calculating with Matlab 7.0, we have $\tau(A\bigstar B)=3.2296$. By Theorem 3.1 in [4], we have

$$\tau(A\bigstar B)\geq \min_i\{a_{ii}b_{ii}-m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k}\}=2.4333.$$

By Theorem 2.1 in this paper, we have

$$\begin{aligned}\tau(A\bigstar B) \quad \geq \quad & \min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ & +(m_i\sum_{k\neq i}\frac{|b_{ki}|}{h_k})(m_j\sum_{l\neq j}\frac{|b_{lj}|}{h_l})]^{\frac{1}{2}}\}=2.9779.\end{aligned}$$

This numerical example shows that the result in Theorem 2.1 is better than that in Theorem 3.1 in [4].

3 An Upper Bound for the Spectral Radius of the Hadamard Product of Nonnegative Matrices

In this section, we will give an upper bound for $\rho(A\circ B)$.

Theorem 3.1 Let $A,B\in R^{n\times n}$, $A\geq 0$ and $B\geq 0$. Then

$$\begin{aligned}\rho(A\circ B) \quad \leq \quad & \max_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ & +(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})(m_j\sum_{l\neq j}\frac{b_{lj}}{h_l})]^{\frac{1}{2}}\}.\end{aligned} \tag{3.1}$$

Proof It is evident that (3.1) is an equality for $n=1$.

We next assume that $n\geq 2$.

If $A\circ B$ is irreducible, then A and B are irreducible. Let λ be an eigenvalue of $A\circ B$ and satisfy $\rho(A\circ B)=\lambda$, so that $\rho(A\circ B)\geq a_{ii}b_{ii}$, $\forall i\in N$. Thus, by Lemma 2.2, there is a pair (i,j) of positive integers with $i \neq j$ such that

$$\begin{aligned} |\lambda - a_{ii}b_{ii}||\lambda - a_{jj}b_{jj}| &\leq (m_i \sum_{k\neq i} \frac{1}{m_k}|a_{ki}b_{ki}|)(m_j \sum_{l\neq j} \frac{1}{m_l}|a_{lj}b_{lj}|) \\ &\leq (m_i \sum_{k\neq i} \frac{|a_{ki}b_{ki}|}{a_{ki}h_k})(m_j \sum_{l\neq j} \frac{|a_{lj}b_{lj}|}{a_{lj}h_l}) \\ &= (m_i \sum_{k\neq i} \frac{b_{ki}}{h_k})(m_j \sum_{l\neq j} \frac{b_{lj}}{h_l}). \end{aligned}$$

Thus, we have

$$\lambda \leq \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(m_i \sum_{k\neq i} \frac{b_{ki}}{h_k})(m_j \sum_{l\neq j} \frac{b_{lj}}{h_l})]^{\frac{1}{2}}\}.$$

That is

$$\begin{aligned} &\rho(A\circ B) \leq \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(m_i \sum_{k\neq i} \frac{b_{ki}}{h_k})(m_j \sum_{l\neq j} \frac{b_{lj}}{h_l})]^{\frac{1}{2}}\} \\ \leq\ &\max_{i\neq j} \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(m_i \sum_{k\neq i} \frac{b_{ki}}{h_k})(m_j \sum_{l\neq j} \frac{b_{lj}}{h_l})]^{\frac{1}{2}}\}. \end{aligned}$$

Now, assume that $A \circ B$ is reducible. If we denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{12} = d_{23} = \cdots = d_{n-1,n} = d_{n1} = 1$, the remaining d_{ij} zero, then both $A + tD$ and $B + tD$ are nonsingular irreducible matrices for any chosen positive real number t. Now we substitute $A + tD$ and $B + tD$ for A and B, respectively in the previous case, and then let $t \longrightarrow 0$, the result follows by continuity.

Theorem 3.2 Let $A, B \in R^{n\times n}$, $A \geq 0$ and $B \geq 0$. Then

$$\begin{aligned} &\max_{i\neq j} \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(m_i \sum_{k\neq i} \frac{b_{ki}}{h_k})(m_j \sum_{l\neq j} \frac{b_{lj}}{h_l})]^{\frac{1}{2}}\} \\ &\leq \max_{1\leq i\leq n}\{a_{ii}b_{ii} + m_i \sum_{k\neq i} \frac{b_{ki}}{h_k}\}. \end{aligned}$$

Proof Without loss of generality, for $i \neq j$, assume that

$$a_{ii}b_{ii} + m_i \sum_{k\neq i} \frac{b_{ki}}{h_k} \geq a_{jj}b_{jj} + m_j \sum_{l\neq j} \frac{b_{lj}}{h_l}. \tag{3.2}$$

Thus, (3.2) is equivalent to

$$m_j \sum_{l\neq j} \frac{b_{lj}}{h_l} \leq m_i \sum_{k\neq i} \frac{b_{ki}}{h_k} + a_{ii}b_{ii} - a_{jj}b_{jj}. \tag{3.3}$$

From (3.1) and (3.3), we have

$$\begin{aligned}
&\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})(m_j\sum_{l\neq j}\frac{b_{lj}}{h_l})]^{\frac{1}{2}}\}\\
\leq\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\
&+4(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k}+a_{ii}b_{ii}-a_{jj}b_{jj})]^{\frac{1}{2}}\}\\
=\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\
&+4(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})^2+4(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})(a_{ii}b_{ii}-a_{jj}b_{jj})]^{\frac{1}{2}}\}\\
=\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj}+2m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})^2]^{\frac{1}{2}}\}\\
=\ &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+(a_{ii}b_{ii}-a_{jj}b_{jj}+2m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})\}\\
=\ &a_{ii}b_{ii}+m_i\sum_{k\neq i}\frac{b_{ki}}{h_k}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\rho(A\circ B)\ &\leq\ \max_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(m_i\sum_{k\neq i}\frac{b_{ki}}{h_k})(m_j\sum_{l\neq j}\frac{b_{lj}}{h_l})]^{\frac{1}{2}}\}\\
&\leq\ \max_{1\leq i\leq n}\{a_{ii}b_{ii}+m_i\sum_{k\neq i}\frac{b_{ki}}{h_k}\}.
\end{aligned}$$

Example 3.1 Let

$$A=\begin{bmatrix}4&1&1&1\\2&5&1&1\\0&2&4&1\\1&1&1&4\end{bmatrix},\qquad B=\begin{bmatrix}1&1&0&0\\1&3&2&0\\0&1&4&3\\0&0&1&5\end{bmatrix}.$$

Then

$$A\circ B=\begin{bmatrix}4&1&0&0\\2&15&2&0\\0&2&16&3\\0&0&1&20\end{bmatrix}.$$

By calculating with Matlab 7.0, we have $\rho(A\circ B)=20.7439$. By Theorem 4.1 in [4], we have

$$\rho(A\circ B)\leq\max_i\{a_{ii}b_{ii}+m_i\sum_{k\neq i}\frac{b_{ki}}{h_k}\}=23.2.$$

By Theorem 3.1 in this paper, we have

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(m_i \sum_{k \neq i} \frac{b_{ki}}{h_k})(m_j \sum_{l \neq j} \frac{b_{lj}}{h_l})]^{\frac{1}{2}}\} = 21.865.$$

This numerical example shows that the result in Theorem 3.1 is better than that in Theorem 4.1 in [4].

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矩阵Hadamard积和Fan积的特征值界的一些新估计式

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摘 要: 本文研究了非奇异M-矩阵A 与B 的Fan积的最小特征值下界和非负矩阵A与B的Hadamard积的谱半径上界的估计问题. 利用Brauer定理, 得到了一些只依赖于矩阵的元素且易于计算的新估计式, 改进了文献[4]现有的一些结果.

关键词: M-矩阵; 非负矩阵; Hadamard积; Fan积; 谱半径; 最小特征值

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