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RUIN PROBABILITIES CAUSED BY DIFFERENT CLASSES OF CLAIMS FOR A CORRELATED AGGREGATE CLAIMS MODEL

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Abstract: This paper considers the same risk model as in [1]. The risk model involves two correlated classes of insurance business and the two claim number processes related to Poisson and Erlang processes. Asymptotic results for ruin probabilities caused by different classes of claims are obtained by renewal argument. Explicit expressions for ruin probabilities caused by different classes of claims are derived when the original claim sizes are exponentially distributed. So the relevant results in [1] is improved.

Keywords: correlated aggregate claims; ruin probability; Erlang process; renewal theorem

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1 Introduction

In this paper we consider a correlated aggregated claims model which was introduced in [1]. We start with the description of the risk model involving two dependent classes of insurance business. Let $\{X_n, n \geq 1\}$ be claim size random variables for the first class with common distribution function F_1 and mean μ_1, and $\{Y_n, n \geq 1\}$ be those for the second class with common distribution function F_2 and mean μ_2. Then the risk model generated from the two correlated classes of business is given by

$$U(t) = u + ct - \sum_{i=1}^{K_1(t)} X_i - \sum_{i=1}^{K_2(t)} Y_i, \tag{1.1}$$

where u is the amount of initial surplus , c is the rate of premium, and $\{K_i(t), t \geq 0\}$ is the claim number process for class $i (i = 1, 2)$. It is assumed that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$

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are independent claim size random variables, and that they are independent of $\{K_1(t), t \geq 0\}$ and $\{K_2(t), t \geq 0\}$. The two claim number processes are correlated in the way that

$$K_1(t) = N_1(t) + N_3(t) \quad \text{and} \quad K_2(t) = N_2(t) + N_3(t),$$

with $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N_3(t), t \geq 0\}$ being three independent renewal processes. The ultimate ruin probability is $\psi(u) = P(U(t) < 0$ (for some $t \geq 0$) and ultimate survival probability is $\phi(u) = 1 - \psi(u)$. Let $\psi_j(u)(j = 1, 2, 3)$ be the ultimate ruin probability caused by the jump of $\{N_j(t), t \geq 0\}$ $(j = 1, 2, 3)$, respectively. $\psi_j(u)$ $(j = 1, 2, 3)$ are useful variables if the insurer wants to know the impact of different classes of claims. It is obvious that $\psi(u) = \psi_1(u) + \psi_2(u) + \psi_3(u)$.

As we know that the correlation in (1.1) comes from the incorporation of the common component $N_3(t)$ into the two claim number processes. In reality, the common shock $N_3(t)$ can depict the effect of a natural disaster that causes various kinds of insurance claims.

Bai Xiaodong and Song Lixin [2] considered the model (1.1) with a constant force of interest, and assumed that the claim-size distributions were heavy-tailed and $\{N_j(t), t \geq 0\}, j = 1, 2, 3$ were three independent general renewal processes. Under this setting, the paper investigated the tail behavior of the sum of the two correlated classes of discounted aggregate claims, and obtained the uniform asymptotic formulas for some subclass of subexponential distributions. Yuen et al. [1] and Liu Yan [3] considered the model (1.1) with $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ being Poisson processes and $\{N_3(t), t \geq 0\}$ being Erlag(2) process. Yuen et al. [1] derived explicit expressions for the ultimate survival probabilities under the assumed model when the original claim sizes were exponentially distributed and also examined the asymptotic property of the ruin probability for this special risk process with general claim size distributions. Liu Yan [3] further discussed some other ruin functions such as the distribution of the surplus immediately before ruin, the distribution of the surplus immediately after ruin and the joint distribution of the surplus immediately before and after ruin. Lv Tonglin et al. [4] considered the model (1.1) with $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N_3(t), t \geq 0\}$ being all Poisson processes. The asymptotic results for the deficit at ruin caused by different classes of claims were obtained. The explicit expression for the deficit at ruin caused by different classes of claims were given when the original claim sizes were exponentially distributed.

In our paper, motivated by the work in [4], we further improve the work of Yuen et al. [1], and consider the ultimate ruin probability $\psi_j(u)$ caused by the jump of $\{N_j(t), t \geq 0\}$ $(j = 1, 2, 3)$ for the model (1.1) with $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ being Poisson processes and $\{N_3(t), t \geq 0\}$ being Erlag(2) process. Let the parameters of the two Poisson processes, $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$, be λ_1 and λ_2, respectively. Assume that $\{N_3(t), t \geq 0\}$ is an Erlang(2) process with parameter $\tilde{\lambda}$. That is, the claim inter arrival times for $\{N_3(t), t \geq 0\}$ follow Erlang distribution with density function $f(t) = \tilde{\lambda}^2 t \exp\{-\tilde{\lambda} t\}$ for $t > 0$.

2 Model Transformation

For investigating the probability of ruin for $U(t)$ in model (1.1), we make use of the following transformed surplus process:

$$U^{'}(t)=u+ct-\sum_{i=1}^{N_1(t)}X_i^{'}-\sum_{i=1}^{N_2(t)}Y_i^{'}-\sum_{i=1}^{N_3(t)}Z_i^{'},$$

where $\{X_n^{'},n\geq 1\}$ and $\{Y_n^{'},n\geq 1\}$ are independent claim size random variables, and their common distribution functions are F_1 and F_2, respectively. $\{Z_n^{'},n\geq 1\}$ are independent claim size random variables with common distribution function $F_3=F_1*F_2$, the notation F_1*F_2 stands for the convolution of F_1 and F_2 . Furthermore $\{X_n^{'},n\geq 1\}$, $\{Y_n^{'},n\geq 1\}$, $\{Z_n^{'},n\geq 1\}$, $\{N_1(t),t\geq 0\}$, $\{N_2(t),t\geq 0\}$ and $\{N_3(t),t\geq 0\}$ are independent. It is easy to see that the transformed process $\{U^{'}(t),t\geq 0\}$ and the original process $\{U(t),t\geq 0\}$ are identically distributed. Hence, the process $\{U(t),t\geq 0\}$ can be examined via $\{U^{'}(t),t\geq 0\}$. Let $T_1=T_{11}+T_{12},T_2=T_{21}+T_{22},\cdots$ be the inter arrival times for $\{N_3(t),t\geq 0\}$, where $T_{11},T_{12},T_{21},T_{22},\cdots$ are independent exponential random variables with mean $\tilde{\lambda}^{-1}$. Since $\lambda_1\mu_1$, $\lambda_2\mu_2$ and $\frac{1}{2}\tilde{\lambda}(\mu_1+\mu_2)$ are the expected aggregate claims associated with $\{N_1(t),t\geq 0\}$, $\{N_2(t),t\geq 0\}$ and $\{N_3(t),t\geq 0\}$, respectively, over a unit time interval, the positive relative security loading condition implies that $c>\lambda_1\mu_1+\lambda_2\mu_2+\frac{1}{2}\tilde{\lambda}(\mu_1+\mu_2)$.

We now make a slight change to the transformed process and introduce the the following surplus process:

$$\tilde{U}(t)=u+ct-\sum_{i=1}^{N_1(t)}X_i^{'}-\sum_{i=1}^{N_2(t)}Y_i^{'}-\sum_{i=1}^{\tilde{N}_3(t)}Z_i^{'},\tag{2.1}$$

where $\{\tilde{N}_3(t),t\geq 0\}$ is a delayed renewal processes with the inter arrival times $T_{11},T_{21}+T_{22},T_{31}+T_{32},\cdots$. The corresponding ruin probabilities and survival probability for the model (2.1) are denoted by $\tilde{\psi}(u),\tilde{\psi}_1(u),\tilde{\psi}_2(u),\tilde{\psi}_3(u),\tilde{\phi}(u)$.

Define $h_i(r)=\int_0^\infty e^{rx}dF_i(x)-1,i=1,2,3$, and set $\overline{F}_i(x)=1-F_i(x),i=1,2,3$.

3 Asymptotic Results for General Claim Sizes

The main result in this section is the following theorem.

Theorem 1 Assume that there exist $r_1>0$ and $r_2>0$ such that $h_1(r)\uparrow\infty$ as $r\uparrow r_1$ and $h_2(r)\uparrow\infty$ as $r\uparrow r_2$, then we have

$$\lim_{u\to\infty}e^{Ru}\frac{\psi_i(u)+\tilde{\psi}_i(u)}{2}\leq\frac{\lambda_i\left(\frac{h_i(R)}{R}-\mu_i\right)}{\lambda_1h_1^{'}(R)+\lambda_2h_2^{'}(R)+\frac{\tilde{\lambda}}{2}h_3^{'}(R)-c},\quad i=1,2$$

and

$$\lim_{u\to\infty}e^{Ru}\frac{\psi_3(u)+\tilde{\psi}_3(u)}{2}\leq\frac{\frac{\tilde{\lambda}}{2}\left(\frac{h_3(R)}{R}-\mu_1-\mu_2\right)}{\lambda_1h_1^{'}(R)+\lambda_2h_2^{'}(R)+\frac{\tilde{\lambda}}{2}h_3^{'}(R)-c},$$

where R is the positive solution of the equation $\lambda_1h_1(r)+\lambda_2h_2(r)+\frac{\tilde{\lambda}}{2}h_3(r)=cr$.

Proof We first consider $\{U^{'}(t), t \geq 0\}$ in a small time interval $(0, t]$. Noting that $P(N_3(t) \geq 1) = o(t)$, we separate the five possible cases as follows:

(1) $N_1(t) = 0, N_2(t) = 0$ and $t < T_{11}$;

(2) $N_1(t) = 0, N_2(t) = 0$ and $T_{11} < t < T_{11} + T_{12}$;

(3) $N_1(t) = 1, N_2(t) = 0$ and $t < T_{11}$;

(4) $N_1(t) = 0, N_2(t) = 1$ and $t < T_{11}$;

(5) all other cases.

The probabilities of the above several cases are $p_1 = 1-(\lambda_1+\lambda_2+\tilde{\lambda})t+o(t)$, $p_2 = \tilde{\lambda}t+o(t)$, $p_3 = \lambda_1 t + o(t)$, $p_4 = \lambda_2 t + o(t)$, and $p_5 = o(t)$ respectively. Then by the total probability formula we get

$$\begin{aligned}\psi_1(u) &= \left[1-(\lambda_1+\lambda_2+\tilde{\lambda})t\right]\psi_1(u+ct) + \tilde{\lambda}t\tilde{\psi}_1(u+ct) \\ &\quad +\lambda_1 t\int_0^{u+ct}\psi_1(u+ct-x)dF_1(x) + \lambda_1 t\overline{F}_1(u+ct) \\ &\quad +\lambda_2 t\int_0^{u+ct}\psi_1(u+ct-x)dF_2(x) + o(t), \qquad (3.1)\end{aligned}$$

$$\begin{aligned}\psi_2(u) &= \left[1-(\lambda_1+\lambda_2+\tilde{\lambda})t\right]\psi_2(u+ct) + \tilde{\lambda}t\tilde{\psi}_2(u+ct) + \lambda_1 t\int_0^{u+ct}\psi_2(u+ct-x)dF_1(x) \\ &\quad +\lambda_2 t\int_0^{u+ct}\psi_2(u+ct-x)dF_2(x) + \lambda_1 t\overline{F}_2(u+ct) + o(t) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\psi_3(u) &= \left[1-(\lambda_1+\lambda_2+\tilde{\lambda})t\right]\psi_3(u+ct) + \tilde{\lambda}t\tilde{\psi}_3(u+ct) \\ &\quad +\lambda_1 t\int_0^{u+ct}\psi_3(u+ct-x)dF_1(x) \\ &\quad +\lambda_2 t\int_0^{u+ct}\psi_3(u+ct-x)dF_2(x) + o(t). \qquad (3.3)\end{aligned}$$

Using (3.1), provided $\psi_1(u)$ is differentiable, we can get

$$\begin{aligned}c\psi_1^{(1)}(u) &= (\lambda_1+\lambda_2+\tilde{\lambda})\psi_1(u) - \tilde{\lambda}\tilde{\psi}_1(u) \\ &\quad -\lambda_1\int_0^u\psi_1(u-x)dF_1(x) - \lambda_1\overline{F}_1(u) - \lambda_2\int_0^u\psi_1(u-x)dF_2(x). \qquad (3.4)\end{aligned}$$

Integrating (3.4) over $(0, u)$ yields

$$\begin{aligned}c\psi_1(u) &= c\psi_1(0) + \int_0^u\psi_1(u-x)\left(\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)\right)dx \\ &\quad +\tilde{\lambda}\int_0^u\left(\psi_1(x)-\tilde{\psi}_1(x)\right)dx - \lambda_1\int_0^u\overline{F}_1(x)dx.\end{aligned}$$

Let $u \to \infty$ yields

$$c\psi_1(0) = \lambda_1\int_0^\infty\overline{F}_1(x)dx - \tilde{\lambda}\int_0^\infty\left(\psi_1(x)-\tilde{\psi}_1(x)\right)dx,$$

which in turn implies that

$$\begin{aligned} c\psi_1(u) &= \int_0^u \psi_1(u-x)\left(\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)\right)dx \\ &\quad -\tilde{\lambda}\int_u^\infty \left(\psi_1(x)-\tilde{\psi}_1(x)\right)dx+\lambda_1\int_u^\infty \overline{F}_1(x)dx. \end{aligned} \tag{3.5}$$

Next we consider $\{\tilde{U}(t), t \geq 0\}$ in a small time interval $(0, t]$ and separate the five possible cases as follows:

(1) $\{N_1(t)=0, N_2(t)=0$ and $\tilde{N}_3(t)=0$;

(2) $\{N_1(t)=1, N_2(t)=0$ and $\tilde{N}_3(t)=0$;

(3) $\{N_1(t)=0, N_2(t)=1$ and $\tilde{N}_3(t)=0$;

(4) $\{N_1(t)=0, N_2(t)=0$ and $T_{11}<t<T_{11}+T_{21}$;

(5) all other cases.

The probabilities of the above several cases are $p_1=1-(\lambda_1+\lambda_2+\tilde{\lambda})t+o(t)$, $p_2=\lambda_1 t+o(t), p_3=\lambda_2 t+o(t)$, $p_4=\tilde{\lambda}t+o(t)$, and $p_5=o(t)$ respectively. Then by the total probability formula we get

$$\begin{aligned} &\tilde{\psi}_1(u)=\left[1-(\lambda_1+\lambda_2+\tilde{\lambda})t\right]\tilde{\psi}_1(u+ct) \\ &+\lambda_1 t\int_0^{u+ct}\tilde{\psi}_1(u+ct-x)dF_1(x)+\lambda_1 t\overline{F}_1(u+ct) \\ &+\lambda_2 t\int_0^{u+ct}\tilde{\psi}_1(u+ct-x)dF_2(x)+\tilde{\lambda}t\int_0^{u+ct}\psi_1(u+ct-x)dF_3(x)+o(t), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\tilde{\psi}_2(u)=\left[1-(\lambda_1+\lambda_2+\tilde{\lambda})t\right]\tilde{\psi}_2(u+ct)+\lambda_1 t\int_0^{u+ct}\tilde{\psi}_2(u+ct-x)dF_1(x) \\ &+\lambda_2 t\int_0^{u+ct}\tilde{\psi}_2(u+ct-x)dF_2(x)+\lambda_2 t\overline{F}_2(u+ct) \\ &+\tilde{\lambda}t\int_0^{u+ct}\psi_2(u+ct-x)dF_3(x)+o(t) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \tilde{\psi}_3(u) &= \left[1-(\lambda_1+\lambda_2+\tilde{\lambda})t\right]\tilde{\psi}_3(u+ct)+\lambda_1 t\int_0^{u+ct}\tilde{\psi}_3(u+ct-x)dF_1(x) \\ &\quad +\lambda_2 t\int_0^{u+ct}\tilde{\psi}_3(u+ct-x)dF_2(x)+\tilde{\lambda}t\int_0^{u+ct}\psi_3(u+ct-x)dF_3(x) \\ &\quad +\tilde{\lambda}t\overline{F}_3(u+ct)+o(t). \end{aligned} \tag{3.8}$$

Based on (3.6), using similar argument by which we deduce (3.5), we can obtain

$$\begin{aligned} c\tilde{\psi}_1(u) &= \int_0^u \tilde{\psi}_1(u-x)\left(\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)\right)dx+\tilde{\lambda}\int_0^u \psi_1(u-x)\overline{F}_3(x)dx \\ &\quad +\tilde{\lambda}\int_u^\infty \left(\psi_1(s)-\tilde{\psi}_1(s)\right)ds+\lambda_1\int_u^\infty \overline{F}_1(s)ds. \end{aligned} \tag{3.9}$$

From (3.5) and (3.9), it is easy to see that

$$\begin{aligned}\frac{\psi_1(u)+\tilde{\psi}_1(u)}{2} &= \frac{\lambda_1}{c}\int_u^\infty \overline{F}_1(x)dx+\frac{\tilde{\lambda}}{2c}\int_0^u \psi_1(u-x)\overline{F}_3(x)dx\\ &\quad+\int_0^u \frac{\psi_1(u-x)+\tilde{\psi}_1(u-x)}{2}\frac{\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)}{c}dx. \qquad (3.10)\end{aligned}$$

Noting $\psi_1(u)\le\tilde{\psi}_1(u)$, we get from (3.10) that

$$\begin{aligned}&\frac{\psi_1(u)+\tilde{\psi}_1(u)}{2}\le\frac{\lambda_1}{c}\int_u^\infty \overline{F}_1(x)dx\\ &+\int_0^u \frac{\psi_1(u-x)+\tilde{\psi}_1(u-x)}{2}\frac{\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)+\frac{\tilde{\lambda}}{2}\overline{F}_3(x)}{c}dx. \qquad (3.11)\end{aligned}$$

The positive relative security loading condition implies that

$$\int_0^\infty \frac{\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)+\frac{\tilde{\lambda}}{2}\overline{F}_3(x)}{c}dx<1,$$

so inequality (3.11) is a defective renewal type of inequality. Assume that there exist $r_1>0$ and $r_2>0$ such that $h_1(r)\uparrow\infty$ as $r\uparrow r_1$ and $h_2(r)\uparrow\infty$ as $r\uparrow r_2$. Then there exists $R>0$ such that $\lambda_1h_1(R)+\lambda_2h_2(R)+\frac{\tilde{\lambda}}{2}h_3(R)=cR$, that is,

$$\int_0^\infty e^{Rx}\frac{\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)+\frac{\tilde{\lambda}}{2}\overline{F}_3(x)}{c}dx=1.$$

Then multiplication of (3.11) by e^{Ru} yields the following renewal type of inequality

$$\begin{aligned}&e^{Ru}\frac{\psi_1(u)+\tilde{\psi}_1(u)}{2}\le\frac{\lambda_1}{c}e^{Ru}\int_u^\infty \overline{F}_1(x)dx\\ &+\int_0^u e^{R(u-x)}\frac{\psi_1(u-x)+\tilde{\psi}_{\cdot}(u-x)}{2}e^{Rx}\frac{\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)+\frac{\tilde{\lambda}}{2}\overline{F}_3(x)}{c}dx. \qquad (3.12)\end{aligned}$$

From the renewal theorem[5], it then follows that

$$\begin{aligned}\lim_{u\to\infty} e^{Ru}\frac{\psi_1(u)+\tilde{\psi}_1(u)}{2} &\le \frac{\int_0^\infty \frac{\lambda_1}{c}e^{Ru}\int_u^\infty \overline{F}_1(x)dxdu}{\int_0^\infty xe^{Rx}\frac{\lambda_1\overline{F}_1(x)+\lambda_2\overline{F}_2(x)+\frac{\tilde{\lambda}}{2}\overline{F}_3(x)}{c}dx}\\ &= \frac{\lambda_1\left(\frac{h_1(R)}{R}-\mu_1\right)}{\lambda_1h_1'(R)+\lambda_2h_2'(R)+\frac{\tilde{\lambda}}{2}h_3'(R)-c}.\end{aligned}$$

Similarly, (3.2) and (3.7) lead to the result for ψ_2, (3.3) and (3.8) lead to the result for ψ_3. Thus we complete the proof of Theorem 1.

Remark 1 Noting that $\psi(u)=\psi_1(u)+\psi_2(u)+\psi_3(u)$ and $\tilde{\psi}(u)=\tilde{\psi}_1(u)+\tilde{\psi}_2(u)+\tilde{\psi}_3(u)$, based on Theorem 1, we can get the asymptotic result for $\frac{\psi(u)+\tilde{\psi}(u)}{2}$ in [1].

4 Ruin Probabilities for Exponential Claim Sizes

In this section, we will consider the case of exponential claims.

Theorem 2 Suppose that $F_1(x), F_2(x)$ are exponential distributed with equal mean $\mu_1 = \mu_2 = \mu$. Then $\psi_i(u) = C_{i1}q(z_1)e^{z_1 u} + C_{i2}q(z_2)e^{z_2 u}$, $i = 1, 2, 3$, where

$$\begin{aligned}
&z_1 = \frac{\lambda\mu - c}{c\mu}, z_2 = \frac{1}{2c\mu}\left(\lambda\mu - c - \left(8c\mu\tilde{\lambda} + (c-\lambda\mu)^2\right)^{1/2}\right),\\
&q(z) = 1 + \frac{\mu}{\tilde{\lambda}}(\lambda + \tilde{\lambda} - \frac{c}{\mu})z + \frac{\mu^2}{\tilde{\lambda}}(\lambda - \frac{2c}{\mu})z^2 - \frac{c\mu^2}{\tilde{\lambda}}z^3,\\
&\lambda = \lambda_1 + \lambda_2 + \tilde{\lambda},
\end{aligned}$$

C_{i1}, C_{i2} (and C_{i3}), $i = 1, 2, 3$ satisfy

$$\left\{\begin{array}{ll}
(cz_1 - \lambda)C_{i1} + (cz_2 - \lambda)C_{i2} - (\frac{c}{\mu} + \lambda)C_{i3} = -\lambda_i, & \\
\left((cz_1 - \lambda)q(z_1) + \tilde{\lambda}\right)C_{i1} + \left((cz_2 - \lambda)q(z_2) + \tilde{\lambda}\right)C_{i2} + \tilde{\lambda}C_{i3} = -\lambda_i, & i = 1, 2\\
\left(cz_1^2 - (\lambda - \frac{c}{\mu})z_1 - \frac{\tilde{\lambda}}{\mu})\right)C_{i1} + \left(cz_2^2 - (\lambda - \frac{c}{\mu})z_2 - \frac{\tilde{\lambda}}{\mu})\right)C_{i2} + \frac{\lambda_1+\lambda_2}{\mu}C_{i3} = 0, &
\end{array}\right.$$

and

$$\left\{\begin{array}{l}
(cz_1 - \lambda)C_{31} + (cz_2 - \lambda)C_{32} - (\frac{c}{\mu} + \lambda)C_{33} = 0,\\
\left((cz_1 - \lambda)q(z_1) + \tilde{\lambda}\right)C_{31} + \left((cz_2 - \lambda)q(z_2) + \tilde{\lambda}\right)C_{32} + \tilde{\lambda}C_{33} = -\tilde{\lambda},\\
\left(cz_1^2 - (\lambda - \frac{c}{\mu})z_1 - \frac{\tilde{\lambda}}{\mu})\right)C_{31} + \left(cz_2^2 - (\lambda - \frac{c}{\mu})z_2 - \frac{\tilde{\lambda}}{\mu})\right)C_{33} + \frac{\lambda_1+\lambda_2}{\mu}C_{33} = -\frac{\tilde{\lambda}}{\mu}.
\end{array}\right.$$

Proof We firstly consider $\psi_1(u)$. Using (3.6), we can get

$$\begin{aligned}
c\tilde{\psi}_1^{(1)}(u) \quad = \quad & \lambda\tilde{\psi}_1(u) - \lambda_1\int_0^u \tilde{\psi}_1(u-x)dF_1(x) - \lambda_1\overline{F}_1(u)\\
& -\lambda_2\int_0^u \tilde{\psi}_1(u-x)dF_2(x) - \tilde{\lambda}\int_0^u \psi_1(u-x)dF_3(x). \qquad (4.1)
\end{aligned}$$

Since $F_1(x), F_2(x)$ are exponential distributed with equal mean $\mu_1 = \mu_2 = \mu$, $F_3(x)$ follows an Erlang distribution with density $\mu^{-2}x\exp\{-\mu^{-1}x\}$ for $x > 0$. In this case (3.4) and (4.1) become

$$c\psi_1^{(1)}(u) = \lambda\psi_1(u) - \tilde{\lambda}\tilde{\psi}_1(u) - \frac{\lambda_1+\lambda_2}{\mu}\int_0^u \psi_1(u-x)e^{-\frac{x}{\mu}}dx - \lambda_1 e^{-\frac{u}{\mu}}$$

and

$$\begin{aligned}
c\tilde{\psi}_1^{(1)}(u) \quad = \quad & \lambda\tilde{\psi}_1(u) - \frac{\lambda_1+\lambda_2}{\mu}\int_0^u \tilde{\psi}_1(u-x)e^{-\frac{x}{\mu}}dx - \lambda_1 e^{-\frac{u}{\mu}}\\
& -\tilde{\lambda}\int_0^{u+ct} \psi_1(u-x)\mu^{-2}x\exp\{-\mu^{-1}x\}dx.
\end{aligned}$$

Differentiation leads to

$$\begin{aligned} c\psi_1^{(2)}(u) &= \lambda\psi_1^{(1)}(u) - \tilde{\lambda}\tilde{\psi}_1^{(1)}(u) - \frac{\lambda_1+\lambda_2}{\mu}\left[\psi_1(u) - \frac{1}{\mu}\int_0^u \psi_1(x)e^{-\frac{u-x}{\mu}}dx\right] + \frac{\lambda_1}{\mu}e^{-\frac{u}{\mu}} \\ &= \lambda\psi_1^{(1)}(u) - \tilde{\lambda}\tilde{\psi}_1^{(1)}(u) + \frac{1}{\mu}\left[(\lambda_1+\lambda_2+\tilde{\lambda})\psi_1(u) - \tilde{\lambda}\tilde{\psi}_1(u) - c\psi_1^{(1)}(u)\right] \\ &= (\lambda - \frac{c}{\mu})\psi_1^{(1)}(u) + \frac{\tilde{\lambda}}{\mu}\psi_1(u) - \tilde{\lambda}\tilde{\psi}_1^{(1)}(u) - \frac{\tilde{\lambda}}{\mu}\tilde{\psi}_1(u), \end{aligned} \tag{4.2}$$

and $c\tilde{\psi}_1^{(2)}(u) = (\lambda - \frac{c}{\mu})\tilde{\psi}_1^{(1)}(u) + \frac{\tilde{\lambda}}{\mu}\tilde{\psi}_1(u) - \frac{\tilde{\lambda}}{\mu^2}\int_0^u \psi_1(x)e^{-\frac{u-x}{\mu}}dx$.

Furthermore,

$$c\tilde{\psi}_1^{(3)}(u) = (\lambda - \frac{2c}{\mu})\tilde{\psi}_1^{(2)}(u) + \left(\frac{\lambda+\tilde{\lambda}}{\mu} - \frac{c}{\mu^2}\right)\tilde{\psi}_1^{(1)}(u) + \frac{\tilde{\lambda}}{\mu^2}\tilde{\psi}_1(u) - \frac{\tilde{\lambda}}{\mu^2}\psi_1(u). \tag{4.3}$$

Hence, (4.2) and (4.3) form a linear differential system with boundary conditions

$$\begin{cases} \psi_1(\infty) = 0, \tilde{\psi}_1(\infty) = 0, \\ c\psi_1^{(1)}(0) = \lambda\psi_1(0) - \tilde{\lambda}\tilde{\psi}_1(0) - \lambda_1, \\ c\tilde{\psi}_1^{(1)}(0) = \lambda\tilde{\psi}_1(0) - \lambda_1, \\ c\tilde{\psi}_1^{(2)}(0) = (\lambda - \frac{c}{\mu})\tilde{\psi}_1^{(1)}(0) + \frac{\tilde{\lambda}}{\mu}\tilde{\psi}_1(0). \end{cases} \tag{4.4}$$

Using (4.2) and (4.3),we obtain

$$\begin{aligned} &c^2\mu^2\tilde{\psi}_1^{(5)}(u) + c\mu(3c - 2\lambda\mu)\tilde{\psi}_1^{(4)}(u) + \left((\lambda\mu - c)(\lambda\mu - 3c) - 2c\mu\tilde{\lambda}\right)\tilde{\psi}_1^{(3)}(u) \\ &+ \left((\lambda\mu - c)(\lambda + 2\tilde{\lambda} - \frac{c}{\mu}) - 2c\tilde{\lambda}\right)\tilde{\psi}_1^{(2)}(u) + 2\tilde{\lambda}\left(\lambda - \frac{c}{\mu}\right)\tilde{\psi}_1^{(1)}(u) = 0. \end{aligned}$$

Its characteristic equation

$$\begin{aligned} &c^2\mu^2 z^5 + c\mu(3c - 2\lambda\mu)z^4 + \left((\lambda\mu - c)(\lambda\mu - 3c) - 2c\mu\tilde{\lambda}\right)z^3 \\ &+ \left((\lambda\mu - c)(\lambda + 2\tilde{\lambda} - \frac{c}{\mu}) - 2c\tilde{\lambda}\right)z^2 + 2\tilde{\lambda}\left(\lambda - \frac{c}{\mu}\right)z = 0 \end{aligned}$$

has five roots, namely,

$$\begin{aligned} &z_1 = \frac{\lambda\mu - c}{c\mu}, z_2 = \frac{1}{2c\mu}\left(\lambda\mu - c - \left(8c\mu\tilde{\lambda} + (c - \lambda\mu)^2\right)^{1/2}\right), z_3 = -\frac{1}{\mu}, \\ &z_4 = \frac{1}{2c\mu}\left(\lambda\mu - c + \left(8c\mu\tilde{\lambda} + (c - \lambda\mu)^2\right)^{1/2}\right), z_5 = 0. \end{aligned}$$

The positive relative security loading condition, $c > \lambda\mu$, implies that only z_4 is positive. Together with $\tilde{\psi}_1(\infty) = 0$, the general solution for $\tilde{\psi}_1(u)$ is

$$\tilde{\psi}_1(u) = C_{11}e^{z_1 u} + C_{12}e^{z_2 u} + C_{13}e^{z_3 u}. \tag{4.5}$$

From (4.3) and (4.5), we have $\psi_1(u) = C_{11}q(z_1)e^{z_1 u} + C_{12}q(z_2)e^{z_2 u} + C_{13}q(z_3)e^{z_3 u}$, where

$$q(z) = 1 + \frac{\mu}{\tilde{\lambda}}(\lambda + \tilde{\lambda} - \frac{c}{\mu})z + \frac{\mu^2}{\tilde{\lambda}}(\lambda - \frac{2c}{\mu})z^2 - \frac{c\mu^2}{\tilde{\lambda}}z^3.$$

Noting that $q(z_3)=0$, we get

$$\psi_1(u)=C_{11}q(z_1)e^{z_1u}+C_{12}q(z_2)e^{z_2u}. \tag{4.6}$$

(4.5), (4.6) and boundary conditions (4.4) leads to

$$\begin{cases} (cz_1-\lambda)C_{11}+(cz_2-\lambda)C_{12}-(\frac{c}{\mu}+\lambda)C_{13}=-\lambda_1, \\ \left((cz_1-\lambda)q(z_1)+\tilde{\lambda}\right)C_{11}+\left((cz_2-\lambda)q(z_2)+\tilde{\lambda}\right)C_{12}+\tilde{\lambda}C_{13}=-\lambda_1, \\ \left(cz_1^2-(\lambda-\frac{c}{\mu})z_1-\frac{\tilde{\lambda}}{\mu}\right)\right)C_{11}+\left(cz_2^2-(\lambda-\frac{c}{\mu})z_2-\frac{\tilde{\lambda}}{\mu}\right)\right)C_{12}+\frac{\lambda_1+\lambda_2}{\mu}C_{13}=0. \end{cases}$$

Similarly, for $i=2,3$, we have

$$c\psi_i^{(2)}(u)=(\lambda-\frac{c}{\mu})\psi_i^{(1)}(u)+\frac{\tilde{\lambda}}{\mu}\psi_i(u)-\tilde{\lambda}\tilde{\psi}_i^{(1)}(u)-\frac{\tilde{\lambda}}{\mu}\tilde{\psi}_i(u)$$

and

$$c\tilde{\psi}_i^{(3)}(u)=(\lambda-\frac{2c}{\mu})\tilde{\psi}_i^{(2)}(u)+\left(\frac{\lambda+\tilde{\lambda}}{\mu}-\frac{c}{\mu^2}\right)\tilde{\psi}_i^{(1)}(u)+\frac{\tilde{\lambda}}{\mu^2}\tilde{\psi}_i(u)-\frac{\tilde{\lambda}}{\mu^2}\psi_i(u)$$

with boundary conditions

$$\begin{cases} \psi_2(\infty)=0,\tilde{\psi}_2(\infty)=0, \\ c\psi_2^{(1)}(0)=\lambda\psi_2(0)-\tilde{\lambda}\tilde{\psi}_2(0)-\lambda_2, \\ c\tilde{\psi}_2^{(1)}(0)=\lambda\tilde{\psi}_2(0)-\lambda_2, \\ c\tilde{\psi}_2^{(2)}(0)=(\lambda-\frac{c}{\mu})\tilde{\psi}_2^{(1)}(0)+\frac{\tilde{\lambda}}{\mu}\tilde{\psi}_2(0) \end{cases}$$

and

$$\begin{cases} \psi_3(\infty)=0,\tilde{\psi}_3(\infty)=0, \\ c\psi_3^{(1)}(0)=\lambda\psi_3(0)-\tilde{\lambda}\tilde{\psi}_3(0), \\ c\tilde{\psi}_3^{(1)}(0)=\lambda\tilde{\psi}_3(0)-\tilde{\lambda}, \\ c\tilde{\psi}_3^{(2)}(0)=(\lambda-\frac{c}{\mu})\tilde{\psi}_3^{(1)}(0)+\frac{\tilde{\lambda}}{\mu}\tilde{\psi}_3(0)-\frac{\tilde{\lambda}}{\mu}. \end{cases}$$

Further, we can get the results for ψ_2 and ψ_3 in Theorem 2.

Example For $\lambda_1=\lambda_2=1,\tilde{\lambda}=1,\mu=1$ and $c=6$, the ruin probability are

$$\psi_1(u)=0.203832e^{-0.333333u}+0.059671e^{-0.767592u},$$

$$\psi_3(u)=0.167713e^{-0.333333u}-0.087518e^{-0.767592u}.$$

Noting that $\psi(u)=\psi_1(u)+\psi_2(u)+\psi_3(u)$, $\tilde{\psi}(u)=\tilde{\psi}_1(u)+\tilde{\psi}_2(u)+\tilde{\psi}_3(u)$ and

$$\phi(u)=1-\psi(u),\tilde{\phi}(u)=1-\tilde{\psi}(u).$$

We can get the following linear differential system

$$c\phi^{(2)}(u)=(\lambda-\frac{c}{\mu})\phi^{(1)}(u)+\frac{\tilde{\lambda}}{\mu}\phi(u)-\tilde{\lambda}\tilde{\phi}^{(1)}(u)-\frac{\tilde{\lambda}}{\mu}\tilde{\phi}(u)$$

and

$$c\tilde{\phi}^{(3)}(u) = (\lambda - \frac{2c}{\mu})\tilde{\phi}^{(2)}(u) + \left(\frac{\lambda + \tilde{\lambda}}{\mu} - \frac{c}{\mu^2}\right)\tilde{\phi}^{(1)}(u) + \frac{\tilde{\lambda}}{\mu^2}\tilde{\phi}(u) - \frac{\tilde{\lambda}}{\mu^2}\phi(u)$$

with boundary conditions

$$\begin{cases} \phi(\infty) = 1, \tilde{\phi}(\infty) = 1, \\ c\phi^{(1)}(0) = \lambda\phi(0) - \tilde{\lambda}\tilde{\phi}(0), \\ c\tilde{\phi}^{(1)}(0) = \lambda\tilde{\phi}(0), \\ c\tilde{\phi}^{(2)}(0) = (\lambda - \frac{c}{\mu})\tilde{\phi}^{(1)}(0) + \frac{\tilde{\lambda}}{\mu}\tilde{\phi}(0). \end{cases}$$

These are just (3.5)–(3.7), respectively in [1].

Using methods similar to the proof of Theorem 2, we can get following theorem and remark.

Theorem 3 Suppose that $F_1(x), F_2(x)$ are exponential distributed with unequal mean $\mu_1 \neq \mu_2$. Then we can obtain, for $i = 1, 2, 3$,

$$\begin{aligned} c\psi_i^{(3)}(u) &= \left(\lambda - \frac{c}{\mu_1} - \frac{c}{\mu_2}\right)\psi_i^{(2)}(u) + \left(\frac{\lambda - \lambda_1}{\mu_1} + \frac{\lambda - \lambda_2}{\mu_2} - \frac{c}{\mu_1\mu_2}\right)\psi_i^{(1)}(u) \\ &+ \frac{\tilde{\lambda}}{\mu_1\mu_2}\psi_i(u) - \tilde{\lambda}\tilde{\psi}_i^{(2)}(u) - \tilde{\lambda}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\tilde{\psi}_i^{(1)}(u) - \frac{\tilde{\lambda}}{\mu_1\mu_2}\tilde{\psi}_i(u) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} c\tilde{\psi}_i^{(3)}(u) &= \left(\lambda - \frac{c}{\mu_1} - \frac{c}{\mu_2}\right)\tilde{\psi}_i^{(2)}(u) + \left(\frac{\lambda - \lambda_1}{\mu_1} + \frac{\lambda - \lambda_2}{\mu_2} - \frac{c}{\mu_1\mu_2}\right)\tilde{\psi}_i^{(1)}(u) \\ &+ \frac{\tilde{\lambda}}{\mu_1\mu_2}\tilde{\psi}_i(u) - \frac{\tilde{\lambda}}{\mu_1\mu_2}\psi_i(u) \end{aligned} \tag{4.8}$$

with boundary conditions

$$\begin{cases} \psi_i(\infty) = 0, \tilde{\psi}_i(\infty) = 0, \\ c\psi_i^{(1)}(0) = \lambda\psi_i(0) - \tilde{\lambda}\tilde{\psi}_i(0) - \lambda_i, \\ c\tilde{\psi}_i^{(1)}(0) = \lambda\tilde{\psi}_i(0) - \lambda_i, \\ c\psi_i^{(2)}(0) = \lambda\psi_i^{(1)}(0) - \left(\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}\right)\psi_i(0) - \tilde{\lambda}\tilde{\psi}_i^{(1)}(0) + \frac{\lambda_i}{\mu_i}, \\ c\tilde{\psi}_i^{(2)}(0) = \lambda\tilde{\psi}_i^{(1)}(0) - \left(\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}\right)\tilde{\psi}_i(0) + \frac{\lambda_i}{\mu_i}, \end{cases} \quad i = 1, 2 \tag{4.9}$$

and

$$\begin{cases} \psi_3(\infty) = 0, \tilde{\psi}_3(\infty) = 0, \\ c\psi_3^{(1)}(0) = \lambda\psi_3(0) - \tilde{\lambda}\tilde{\psi}_3(0), \\ c\tilde{\psi}_3^{(1)}(0) = \lambda\tilde{\psi}_3(0) - \tilde{\lambda}, \\ c\psi_3^{(2)}(0) = \lambda\psi_3^{(1)}(0) - \left(\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}\right)\psi_3(0) - \tilde{\lambda}\tilde{\psi}_3^{(1)}(0), \\ c\tilde{\psi}_3^{(2)}(0) = \lambda\tilde{\psi}_3^{(1)}(0) - \left(\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}\right)\tilde{\psi}_3(0). \end{cases} \tag{4.10}$$

Remark 2 Based on (4.7) and (4.8), we can get the following linear differential equation

$$A_6\tilde{\psi}_i^{(6)}(u)+A_5\tilde{\psi}_i^{(5)}(u)+A_4\tilde{\psi}_i^{(4)}(u)+A_3\tilde{\psi}_i^{(3)}(u)+A_2\tilde{\psi}_i^{(2)}(u)+A_1\tilde{\psi}_i^{(1)}(u)=0,$$

where

$$\begin{aligned}
A_1 &= 2\tilde{\lambda}\left(\frac{\lambda-\lambda_1}{\mu_1}+\frac{\lambda-\lambda_2}{\mu_2}-\frac{c}{\mu_1\mu_2}\right)-\tilde{\lambda}^2\left(\frac{c}{\mu_1}+\frac{c}{\mu_2}\right),\\
A_2 &= \mu_1\mu_2\left(\frac{\lambda-\lambda_1}{\mu_1}+\frac{\lambda-\lambda_2}{\mu_2}-\frac{c}{\mu_1\mu_2}\right)^2+2\tilde{\lambda}\left(\lambda-\frac{c}{\mu_1}-\frac{c}{\mu_2}\right)-\tilde{\lambda}^2,\\
A_3 &= 2\left(\mu_1\mu_2\left(\lambda-\frac{c}{\mu_1}-\frac{c}{\mu_2}\right)\left(\frac{\lambda-\lambda_1}{\mu_1}+\frac{\lambda-\lambda_2}{\mu_2}-\frac{c}{\mu_1\mu_2}\right)-c\tilde{\lambda}\right),\\
A_4 &= \mu_1\mu_2\left(\left(\lambda-\frac{c}{\mu_1}-\frac{c}{\mu_2}\right)^2-2c\left(\frac{\lambda-\lambda_1}{\mu_1}+\frac{\lambda-\lambda_2}{\mu_2}-\frac{c}{\mu_1\mu_2}\right)\right),\\
A_5 &= -2c\mu_1\mu_2\left(\lambda-\frac{c}{\mu_1}-\frac{c}{\mu_2}\right),\\
A_6 &= c^2\mu_1\mu_2.
\end{aligned}$$

Given the parameter values, together with the boundary conditions, we can solve for $\tilde{\psi}_i, i=1,2,3$ (and hence $\psi_i, i=1,2,3$) using computer software.

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一类相依两险种风险模型的分类破产概率

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摘 要: 本文考虑文[1]中引入的一类索赔达到计数过程相关的两险种风险模型. 利用更新方法, 获得了该风险模型的分类破产概率的渐进结果, 并给出了指数索赔情形下分类破产概率的表达式, 从而改进了文[1]中的相关结果.

关键词: 相关总索赔; 破产概率; Erlang过程; 更新定理

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