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# EXISTENCE OF SOLUTIONS FOR A SECOND ORDER DIFFERENCE BOUNDARY VALUE PROBLEM

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**Abstract:** This paper studies the existence of solutions for a second order difference boundary value problem. Using critical point theory and variational method, some existence theorems of solutions are established. The results obtained here improve some existing results in the literature.

**Keywords:** difference equation; variational method; critical point theory; solution

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## 1 Introduction

Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ be the set of all natural numbers, integers and real numbers, respectively. $\mathbb{R}^N$ is the real linear space with dimension $N$. For $a, b \in \mathbb{Z}$ with $a \le b$, $[a, b]$ denotes the discrete interval $\{a, a+1, \cdots, b\}$.

In this paper, we shall investigate the existence of solutions for the second order difference boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = \lambda f(t, u(t)), \ t \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where $T \ge 3$ is a fixed positive integer, $\lambda$ is a positive parameter, $\Delta u(t) = u(t+1) - u(t)$ is the forward operator, $\Delta^2 u(t) = \Delta(\Delta u(t))$, and $f \in C([1, T] \times \mathbb{R}, \mathbb{R})$ with $f(t, 0) = 0$.

The study of discrete boundary value problems has captured special attention in the last years. The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis. We refer to the results [1–7] and the references therein.

In [1], Wang et al. considered the problem (1.1) without the parameter $\lambda$, and obtained the existence of non-trivial solutions of problem (1.1) with resonance at both infinity and zero.

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The methods used here are based on combining the minimax methods and the Morse theory especially some new observations on the critical groups of a local linking-type degenerate critical point.

In [2], Alexandru Kristály et al considered (1.1) with the autonomous nonlinearity and $\lambda = 1$. By a direct variational method, they showed that (1.1) has a sequence of non-negative, distinct solutions which converges to 0 (respectively $+\infty$) in the sup-norm whenever $f$ oscillates at the origin (respectively at infinity).

In our paper, under the case of dependence on our parameter $\lambda$, we shall discuss the existence of solutions for (1.1). The features of this paper mainly include the following aspects. First, we assume that the nonlinearity $f$ satisfies the Lipschitz condition to ensure (1.1) has precisely one solution by virtue of Browder theorem (see [8, Theorem 5.3.22]). Second, we respectively utilize the least action principle, Linking theorem, Clark theorem to obtain the existence of one solution and multiple solutions for (1.1).

## 2 Variational Structure and Lemmas

We must need to define a Hilbert space so that we can give the variational formulation of (1.1). Let $H = \{u = (u(0), \cdots, u(T+1)) : u(0) = u(T+1) = 0\}$. Equipped with the inner product and the induced norm

$$(u, v) = \sum_{t=1}^{T+1} \Delta u(t-1)\Delta v(t-1)$$

and norm

$$\|u\| = \left(\sum_{t=1}^{T+1} |\Delta u(t-1)|^2\right)^{\frac{1}{2}},$$

$H$ is a $T$-dimensional Hilbert space.

We first consider the linear boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = \lambda u(t), \ t \in [1, T], \\ u(0) = u(T+1) = 0. \end{cases} \tag{2.1}$$

As known to all, $\lambda_i := 4\sin^2 \frac{i\pi}{2(T+1)} > 0, i \in [1, T]$ are the distinct eigenvalues of (2.1). Let $\phi_i, i \in [1, T]$ be the corresponding orthogonal eigenvectors, where $\phi_i(j) = \sin \frac{ij\pi}{T+1}, j \in [1, T]$. Corresponding to the eigenvalues $\lambda_i$ we have the splitting $H = X \bigoplus W$, where $X = \mathrm{span}\{\phi_1, \cdots, \phi_i\}$, $W = \mathrm{span}\{\phi_{i+1}, \cdots, \phi_T\}$. Moreover, the following inequalities hold:

$$\lambda_1 |u|^2 \le \|u\|^2 \le \lambda_T |u|^2, \forall u \in H, \tag{2.2}$$

$$\lambda_1 |u|^2 \le \|u\|^2 \le \lambda_i |u|^2, \forall u \in X, \tag{2.3}$$

$$\lambda_{i+1} |u|^2 \le \|u\|^2 \le \lambda_T |u|^2, \forall u \in W, \tag{2.4}$$

where $|u|^2 = \sum\limits_{t=1}^{T+1} |u(t)|^2$.

We define an energy functional $J : H \to \mathbb{R}$ as

$$J(u) = \frac{1}{2} \sum_{t=1}^{T+1} |\Delta u(t-1)|^2 - \lambda \sum_{t=1}^{T} F(t, u(t)), u \in H, \tag{2.5}$$

where $F(t, x) = \int_0^x f(t, s)\mathrm{d}s$. Then it is clear that $J$ is of $C^1(H, \mathbb{R})$ with the fréchet derivatives given by

$$(J'(u), v) = \sum_{t=1}^{T+1} \Delta u(t-1)\Delta v(t-1) - \lambda \sum_{t=1}^{T+1} f(t, u(t))v(t), u, v \in H, \tag{2.6}$$

and the solutions of (1.1) are exactly the critical points of $J$ in $H$.

**Definition 2.1** Let $E$ be a Banach space. For $I \in C^1(E, \mathbb{R})$, we say $I$ satisfies the Palais-Smale condition if any sequence $\{x_n\} \subset E$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

**Definition 2.2** (see [8, p303]) Let $E$ be a reflexive real Banach space. We say $\mathscr{L} : E \to E^*$ is demicontinuous if $\mathscr{L}$ maps strongly convergent sequences in $E$ to weakly convergent sequences in $E^*$.

**Lemma 2.1** (Browder Theorem, see [8, Theorem 5.3.22]) Let $E$ be a reflexive real Banach space. Suppose that $\mathscr{L} : E \to E^*$ be an operator satisfying the conditions

(i) $\mathscr{L}$ is bounded, demicontinuous;

(ii) $\lim\limits_{\|u\|\to\infty} \frac{(\mathscr{L}(u),u)}{\|u\|} = +\infty$;

(iii) $\mathscr{L}$ is monotone on the space $E$, i.e., for all $u, v \in E$, we have

$$(\mathscr{L}(u) - \mathscr{L}(v), u - v) \geq 0. \tag{2.7}$$

Then the equation

$$\mathscr{L}(u) = f^* \tag{2.8}$$

has at least one solution $u \in E$ for every $f^* \in E^*$. If, moreover, inequality (2.7) is strict for all $u, v \in E$, $u \neq v$, then equation (2.8) has precisely one solution $u \in E$ for every $f^* \in E^*$.

**Lemma 2.2** (see [9]) Let $E$ be a reflexive real Banach space and $I \in C^1(E, \mathbb{R})$ weakly lower semicontinuous and coercive, i.e., $\lim\limits_{\|u\|\to\infty} I(u) = +\infty$. Then there exists $u_0 \in E$ such that $I(u_0) = \inf\limits_{E} I(u)$.

**Lemma 2.3** (Linking theorem, see [10]) Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ a functional satisfying (PS) condition and bounded from below. Suppose that $I$ has a local linking at the origin $\theta$, i.e., there is a decomposition $E = X \bigoplus W$ and a positive number $\rho$ such that $\dim X < \infty$, $I(y) < I(\theta)$ for $y \in X$ with $0 < \|y\| \leq \rho$; $I(y) \geq I(\theta)$ for $y \in W$ with $0 < \|y\| \leq \rho$. Then $I$ has at least three critical points.

**Lemma 2.4** (Clark theorem, see [11]) Let $E$ be a real Banach space, $I \in C^1(E, \mathbb{R})$ with $I$ even, bounded from below, and satisfying (PS) condition. Suppose that $I(\theta) = 0$, there is a set $K \subset E$ such that $K$ is homeomorphic to $S^{j-1}$ (the $j-1$ dimensional unit sphere) by an odd map and $\sup_K I < 0$. Then $I$ has at least $j$ distinct pairs of critical points.

## 3 Main Results

Now, we list our assumptions on $f$ and $F$.

(H1) There exists a positive constant $c_1$ such that

$$|f(t,u) - f(t,v)| \le c_1|u-v|,\ \forall u, v \in H \text{ and } t \in [1,T].$$

(H2) There exist $a > 0$ and $\mu \in (0,2)$ such that $\limsup\limits_{|u|\to\infty} \frac{F(t,u)}{|u|^\mu} \le a,\ \forall u \in H$ and $t \in [1,T]$.

(H3) There is a positive constant $q_1$ such that $\lim\limits_{|u|\to 0} \frac{F(t,u)}{|u|^2} = q_1, \forall u \in H$ and $t \in [1,T]$.

(H4) There is a positive constant $q_2$ such that $\liminf\limits_{|u|\to 0} \frac{F(t,u)}{|u|^2} \ge q_2, \forall u \in H$ and $t \in [1,T]$.

(H5) $F(t,u)$ is even with respect to $u$, i.e., $F(t,-u) = F(t,u)$ for $u \in H$ and $t \in [1,T]$.

**Theorem 3.1** Suppose that (H1) holds. Then (1.1) has precisely one solution for $\lambda \in (0, \frac{\lambda_1}{c_1})$.

**Proof** Put

$$(\mathscr{L}_1(u), v) = \sum_{t=1}^{T+1} \Delta u(t-1)\Delta v(t-1),\ (\mathscr{L}_2(u), v) = \sum_{t=1}^{T+1} f(t,u(t))v(t),\ \text{for every } u, v \in H.$$

Clearly, $\mathscr{L}_1 : H \to H$ is a linear bounded operator, so demicontinuous. By (H1), (2.2) and Cauchy-Schwarz inequality, we see

$$\begin{aligned}|(\mathscr{L}_2(u_1) - \mathscr{L}_2(u_2), v)| &\le \sum_{t=1}^{T+1} |f(t,u_1(t)) - f(t,u_2(t))||v(t)| \\ \le \sum_{t=1}^{T+1} c_1|u_1(t) - u_2(t)||v(t)| &\le \lambda_1^{-1}c_1\|u_1 - u_2\|\|v\|, \forall u_1, u_2, v \in H.\end{aligned} \tag{3.1}$$

Consequently, $\mathscr{L}_2 : H \to H$ is continuous, so demicontinuous. Therefore, $\mathscr{L} := \mathscr{L}_1 - \lambda\mathscr{L}_2$ form $H$ to $H$ is demicontinuous. From (H1) and $f(t,0) = 0$, we see there is a $c_2 > 0$ such that

$$|f(t,u)| \le c_1|u| + c_2,\ \forall u \in H \text{ and } t \in [1,T]. \tag{3.2}$$

It follows from (2.2), (3.2) and Cauchy-Schwarz inequality that

$$\begin{aligned}|(\mathscr{L}_2(u), v)| &\le \sum_{t=1}^{T+1} |f(t,u(t))||v(t)| \le \left(\sum_{t=1}^{T+1} |f(t,u(t))|^2\right)^{\frac{1}{2}} \left(\sum_{t=1}^{T+1} |v(t)|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{t=1}^{T+1} (2c_1^2|u(t)|^2 + 2c_2^2)\right)^{\frac{1}{2}} \left(\sum_{t=1}^{T+1} |v(t)|^2\right)^{\frac{1}{2}} \\ &\le \sqrt{\frac{2}{\lambda_1}} \left(c_1\lambda_1^{-\frac{1}{2}}\|u\| + c_2\sqrt{T+1}\right)\|v\| < \infty.\end{aligned} \tag{3.3}$$

Therefore, $\mathscr{L}$ is bounded. By (3.2) again, we get

$$(\mathscr{L}(u),u)=\sum_{t=1}^{T+1}|\Delta u(t-1)|^2-\lambda\sum_{t=1}^{T+1}f(t,u(t))u(t)$$
$$\geq\|u\|^2-\lambda\sum_{t=1}^{T+1}(c_1|u(t)|+c_2)|u(t)|\geq(1-\frac{\lambda c_1}{\lambda_1})\|u\|^2-\frac{\lambda c_2}{\sqrt{\lambda_1}}\sqrt{T+1}\|u\|.$$

Therefore, $\lim\limits_{\|u\|\to\infty}\frac{(\mathscr{L}(u),u)}{\|u\|}=+\infty$. Finally, by (H1), we arrive at

$$\begin{aligned}&(\mathscr{L}(u)-\mathscr{L}(v),u-v)\\=&\sum_{t=1}^{T+1}|\Delta u(t-1)-\Delta v(t-1)|^2-\lambda\sum_{t=1}^{T+1}(f(t,u(t))-f(t,v(t)))(u(t)-v(t))\\\geq&\|u-v\|^2-\frac{\lambda c_1}{\lambda_1}\|u-v\|^2>0,\ \forall u,v\in H\text{ and }u\neq v.\end{aligned}$$

Hence, Lemma 2.1 implies the equation $\mathscr{L}(u)=0$ has precisely one solution $u\in H$. Equivalently, (1.1) has only a solution $u\in H$.

**Theorem 3.2** Suppose that (H2) holds. Then (1.1) has a solution for all $\lambda>0$.

**Proof** By (H2), we see there is a $b>0$ such that

$$F(t,u)\leq a|u|^\mu+b,\forall u\in H\text{ and }t\in[1,T].\tag{3.4}$$

Consequently, we have by Hölder inequality

$$\begin{aligned}J(u)&=\frac{1}{2}\sum_{t=1}^{T+1}|\Delta u(t-1)|^2-\lambda\sum_{t=1}^{T}F(t,u(t))\\&\geq\frac{1}{2}\|u\|^2-\lambda\sum_{t=1}^{T}(a|u(t)|^\mu+b)\geq\frac{1}{2}\|u\|^2-\lambda a\left(\sum_{t=1}^{T}(|u(t)|^\mu)^{\frac{2}{\mu}}\right)^{\frac{\mu}{2}}\left(\sum_{t=1}^{T}1\right)^{1-\frac{\mu}{2}}-\lambda bT\\&\geq\frac{1}{2}\|u\|^2-\lambda a\lambda_1^{-\frac{\mu}{2}}T^{\frac{2-\mu}{2}}\|u\|^\mu-\lambda bT.\end{aligned}\tag{3.5}$$

Therefore, $\mu\in(0,2)$ implies $J(u)$ is coercive, i.e., $\lim\limits_{\|u\|\to\infty}J(u)=+\infty$. Since $H$ is a real reflexive finite dimensional Banach space and $J\in C^1(H,\mathbb{R})$, so $J$ is weakly lower semi-continuous on $H$. Lemma 2.2 leads to there is a $u_0\in H$ such that $J(u_0)=\inf\limits_H J(u)$ and $J'(u_0)=0$, hence (1.1) has at least one solution.

**Theorem 3.3** Suppose that (H2) and (H3) hold. Then (1.1) has at least three solutions for $\lambda\in(\frac{\lambda_i}{2q_1},\frac{\lambda_{i+1}}{2q_1})$ $(i=1,2,\cdots,T-1)$.

**Proof** Recall that in the finite dimensional setting, it is well-known that a coercive functional satisfies (PS) condition. By (H2), we see $J$ is coercive, i.e., $\lim\limits_{\|u\|\to\infty}J(u)=+\infty$. So (PS) condition is spontaneously satisfied for $J$.

By (H3), for $\varepsilon \in (0, q_1)$, there exists $\rho > 0$ such that

$$(q_1 - \varepsilon)|u|^2 \leq F(t, u(t)) \leq (q_1 + \varepsilon)|u|^2, \text{ for } |u| \leq \rho, t \in [1, T]. \tag{3.6}$$

For $u \in X = \text{span}\{\phi_1, \cdots, \phi_i\}$ with $0 < \|u\| \leq \rho$, we see

$$\begin{aligned} J(u) &= \frac{1}{2} \sum_{t=1}^{T+1} |\Delta u(t-1)|^2 - \lambda \sum_{t=1}^{T} F(t, u(t)) \\ &\leq \frac{1}{2} \|u\|^2 - \lambda \sum_{t=1}^{T} (q_1 - \varepsilon)|u(t)|^2 \leq \left( \frac{1}{2} - \frac{\lambda(q_1 - \varepsilon)}{\lambda_i} \right) \|u\|^2. \end{aligned} \tag{3.7}$$

Thus, for $\lambda > \frac{\lambda_i}{2(q_1 - \varepsilon)}$, we have $J(u) < 0$ for $u \in X$ with $0 < \|u\| \leq \rho$.

On the other hand, for $u \in W = \text{span}\{\phi_{i+1}, \cdots, \phi_T\}$ with $0 < \|u\| \leq \rho$,

$$\begin{aligned} J(u) &= \frac{1}{2} \sum_{t=1}^{T+1} |\Delta u(t-1)|^2 - \lambda \sum_{t=1}^{T} F(t, u(t)) \\ &\geq \frac{1}{2} \|u\|^2 - \lambda \sum_{t=1}^{T} (q_1 + \varepsilon)|u(t)|^2 \geq \left( \frac{1}{2} - \frac{\lambda(q_1 + \varepsilon)}{\lambda_{i+1}} \right) \|u\|^2, \end{aligned} \tag{3.8}$$

the for $\lambda < \frac{\lambda_{i+1}}{2(q_1 + \varepsilon)}$, we have $J(u) > 0$ for $u \in W$ with $0 < \|u\| \leq \rho$. So, by Lemma 2.3, for $\varepsilon \in (0, q_1)$, if $\lambda \in (\frac{\lambda_i}{2(q_1 - \varepsilon)}, \frac{\lambda_{i+1}}{2(q_1 + \varepsilon)})$, $J$ has at least three critical points. By the arbitrariness of $\varepsilon$, we get for $\lambda \in (\frac{\lambda_i}{2q_1}, \frac{\lambda_{i+1}}{2q_1})(i = 1, 2, \cdots, T-1)$, (1.1) possesses at least three solutions.

**Theorem 3.4** Suppose that (H2), (H4) and (H5) hold. Then (1.1) has $i (i = 1, 2, \cdots, T)$ pairs of nontrivial solutions for $\lambda \in (0, \frac{\lambda_i}{2q_2})$.

**Proof** $J(u)$ is an even functional on $H$ by (H5). As in Theorem 3.3, $J$ satisfies (PS) condition from (H2). If we choose $K = X \cap \partial B_\rho$, then $K$ is homeomorphic to $S^{i-1}$ by an odd map. By (H4), there is a $\rho > 0$ such that $F(t, u) \geq q_2|u|^2$ for $|u| \leq \rho$ and $t \in [1, T]$. So for $x \in K$, we see

$$\begin{aligned} J(u) &= \frac{1}{2} \sum_{t=1}^{T+1} |\Delta u(t-1)|^2 - \lambda \sum_{t=1}^{T} F(t, u(t)) \\ &\leq \frac{1}{2} \|u\|^2 - \lambda \sum_{t=1}^{T} q_2|u(t)|^2 \leq \left( \frac{1}{2} - \frac{\lambda q_2}{\lambda_i} \right) \rho^2, \end{aligned} \tag{3.9}$$

so for $\lambda < \frac{\lambda_i}{2q_2}$, we have $\sup_K J(u) < 0$. Therefore, by Lemma 2.4, $J$ has at least $i$ pairs of nontrivial critical points, i.e., (1.1) has $i$ pairs of nontrivial solutions.

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# 一类二阶差分方程边值问题解的存在性

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**摘 要**: 本文研究了一个二阶差分方程边值问题解的存在性问题. 利用临界点理论和变分方法, 获得了几个解的存在性结果, 推广了一些现有的结果.

**关 键 词**: 差分方程; 变分法; 临界点理论; 解

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