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MAPS PRESERVING ZERO LIE BRACKETS ON A MAXIMAL NILPOTENT SUBALGEBRA OF THE ORTHOGONAL LIE AlGEBRA OF D_m TYPE

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Abstract: Let F be a field, $l_{2m}(F)$ a maximal nilpotent subalgebra of the orthogonal Lie algebra of D_m type. The aim of this paper is to characterize every linear map of $l_{2m}(F)$ which preserves zero Lie brackets in both directions when $m \geq 5$. By using the main theorem of the paper [7] and the skill of matrix computation, it is proved that a linear map φ of $l_{2m}(F)$ preserves zero Lie brackets in both directions if and only if φ is the product of an inner automorphism, a graph automorphism, a generalized diagonal automorphism, a central map, a sub-central automorphism, an extremal map and a scalar multiplication. This extends the main result of the paper [7].

Keywords: maximal nilpotent subalgebra; zero Lie brackets; D_m type orthogonal Lie algebra

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1 Introduction

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem (LPP). The earliest paper on such problem dates back to 1897 (see [1]), and a great deal of effort were devoted to the study of this type of question since then. One may consult the survey paper [2–3] for details. It is one of the important linear preserver problems to classify commutativity preserving linear maps on matrix spaces or algebras. A linear map φ on an algebra or a matrix space $\mathcal{A}$ is said to be commutativity preserving in both directions when the condition $ab = ba$ holds if and only if $\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$. Commutativity preserving linear maps on spaces of matrices or operators were considered by several authors, see [4–12]. There are several motivations to study this kind of

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maps. Problems concerning commutativity preserving maps are closely related to the study of Lie homomorphisms. Every associative algebra $\mathcal{A}$ becomes a Lie algebra if we introduce the Lie bracket $[a,b]$ by $[a,b]=ab-ba$ for $a,b\in\mathcal{A}$. A linear map $\phi:\mathcal{A}\to\mathcal{B}$ is called a Lie homomorphism if $\phi([a,b])=[\phi(a),\phi(b)]$ for every pair $a,b\in\mathcal{A}$. It is clear that every Lie homomorphism preserves commutativity. The assumption of preserving commutativity can be reformulated as the assumption of preserving zero Lie brackets. Let L be a Lie algebra over a field, φ a linear map of L. We say that φ preserves zero Lie brackets in both directions if for every pair $x,y\in L$, we have $[x,y]=0$ if and only if $[\varphi(x),\varphi(y)]=0$. Choi et al.[6] mentioned that the results on linear maps preserving commutativity can be viewed in the content of Lie algebra, where one assumes that the linear map preserves zero products and the conclusion is that the map "essentially" preserves all products. Marcoux and Sourour [8] also pointed out that the linear maps that preserve zero Lie brackets in both directions differ only slightly from those that preserve all Lie brackets. But this assertion is not true here. In this paper, we obtain three types of linear maps which preserve zero Lie brackets in both directions but fail to preserve all Lie brackets.

Let F be an arbitrary field and F^* the group consisting of all non-zero elements of F. Let $F^{m\times n}$ denote the set of all $m\times n$ matrices over F, $E^{(n)}$ the $n\times n$ identity matrix ($E^{(m)}$ is abbreviated to E), $gl(n,F)$ the general linear Lie algebra over F. For $A\in F^{n\times n}$, $A^{'}$ denotes the transpose of A. Let $T(n,F)$ (resp., $S(n,F)$) be the subalgebra of $gl(n,F)$ consisting of all upper triangular (resp., strictly upper triangular) matrices, $T^*(n,F)$ the group consisting of all invertible elements in $T(n,F)$. Set $I=\begin{bmatrix} 0 & E\\ E & 0\end{bmatrix}$. The orthogonal algebra $o(2m,F)$ is defined to be the subalgebra of $gl(2m,F)$ consisting of all $X\in gl(2m,F)$ satisfying $X'I=-IX$. Let

$$l_{2m}(F)=\left\{\begin{bmatrix} A & B\\ 0 & -A'\end{bmatrix} \mid A\in S(m,F), B\in F^{m\times m},\ B'=-B\right\}.$$

It is a maximal nilpotent subalgebra of $o(2m,F)$. In this paper, by using the main theorem of [7], we shall describe all the linear maps preserving zero Lie brackets in both directions of $l_{2m}(F)$ when $m\geq 5$. The main idea of this paper is to reduce the problem on $l_{2m}(F)$ to that on $S(m,F)$.

2 Preliminaries

For $1\leq i,j\leq m$, let E_{ij} denote the $2m\times 2m$ matrix whose (i,j)-entry is 1 and all other entries are 0; $E_{i,-j}$ the $2m\times 2m$ matrix whose $(i,j+m)$-entry is 1 and all other entries are 0; $E_{-i,j}$ the $2m\times 2m$ matrix whose $(i+m,j)$-entry is 1 and all other entries are 0; $E_{-j,-i}$ the $2m\times 2m$ matrix whose $(j+m,i+m)$-entry is 1 and all other entries are 0. For $1\leq i,j\leq m$, let e_{ij} denote the $m\times m$ matrix whose (i,j)-entry is 1 and all other entries are 0. For $a \in F,\ 1 \le i < j \le m$, set

$$T_{ij}(a) = a(E_{ij} - E_{-j,-i}),\ \ T_{ij} = \{T_{ij}(a) \mid a \in F\};$$
$$T_{i,-j}(a) = a(E_{i,-j} - E_{j,-i}),\ \ T_{i,-j} = \{T_{i,-j}(a) \mid a \in F\}.$$

Let $l_{2m}^{(1)}(F) = [l_{2m}(F), l_{2m}(F)]$, $l_{2m}^{(2)}(F) = [l_{2m}(F), l_{2m}^{(1)}(F)], \cdots,\ l_{2m}^{(k)}(F) = [l_{2m}(F), l_{2m}^{(k)}(F)]$, $\cdots$, and denote

$$\begin{aligned} p(F) &= T_{1m} + \sum_{1\le i<j\le m, i+j\le m+1} T_{i,-j}, & u(F) &= \sum_{1\le i<j\le m, i+j\le m} T_{i,-j},\\ x(F) &= \sum_{1\le i<j\le m-1} T_{i,-j}, & y(F) &= x(F) + T_{1m} + T_{1,-m},\\ q(F) &= y(F) + T_{2m} + T_{2,-m}, & z(F) &= \sum_{1\le i<j\le m} T_{i,-j} + \sum_{1\le i\le m-1} T_{im},\\ s(F) &= z(F) + T_{1,m-1}. \end{aligned}$$

Let $t(F) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \middle| A \in S(m,F) \right\}$, $w(F) = \left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \mid B \in F^{m\times m}, B' = -B \right\}$, $v(F) = \left\{ \mathrm{diag}(A, 0, -A', 0) | A \in S(m-1, F) \right\}$. Then $l_{2m}(F) = t(F) + w(F)$.

Let L be a Lie algebra. The center of L is $z(L) = \{z \in L \mid [x, z] = 0 \text{ for all } x \in L\}$, the centralizer of a subset X of L is $C_L(X) = \{x \in L \mid [x, X] = 0\}$. It is easy to know that the center of $l_{2m}(F)$ is $l_{2m}^{(2m-4)}(F)$ which is equal to $T_{1,-2}$, and the center of $q(F)$ is $x(F)$. The centralizer of $p(F)$ (resp., $u(F)$; resp., $x(F)$) in $l_{2m}(F)$ is $y(F)$ (resp., $s(F)$; resp., $z(F)$).

Denote by $\mathcal{T}$ the set of all linear maps of $l_{2m}(F)$ that preserve zero Lie brackets in both directions and by $\mathcal{T}'$ the set of all bijections in $\mathcal{T}$. Denote by 1 the identity map on $l_{2m}(F)$. It is clear that for $\varphi \in \mathcal{T}$ and a linear function f from $l_{2m}(F)$ to F, the map $\varphi + f : X \mapsto \varphi(X) + f(X)T_{1,-2}(1)$ is in $\mathcal{T}$.

The following lemma is obvious.

Lemma 2.1 (i) If $\varphi \in \mathcal{T}$, then $Ker\varphi \subseteq T_{1,-2}$.

(ii) $\varphi \in \mathcal{T}'$ if and only if $\varphi(T_{1,-2}(1)) \neq 0$.

(iii) If $\varphi \in \mathcal{T}'$, then $\varphi(T_{1,-2}(1)) = T_{1,-2}(c)$ for some $c \in F^*$.

3 Standard Maps of $t(F)$

It is obvious that $t(F)$ is isomorphic to $S(m, F)$. Cao et al.[7] have described the linear maps preserving commutativity in both directions on $S(m, F)$. We now transfer them to $t(F)$ for later use. $t(F)$ has the following standard maps that preserve zero Lie brackets in both directions.

(a) $\psi_{t,c} : X \mapsto cX$, where c is a constant in F^*.

(b) $\mathrm{Int}_t P : X \mapsto P^{-1}XP$, where $P = \begin{bmatrix} A & 0 \\ 0 & A'^{-1} \end{bmatrix}$ with $A \in T^*(m, F)$.

(c) $\eta_{t,f} : X \mapsto X + f(X)T_{1m}(1)$, where $f : t(F) \to F$ is a linear function.

(d) $\omega = 1$ or $\omega : X = \begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix} \mapsto \begin{bmatrix} -RA'R & 0 \\ 0 & RAR \end{bmatrix}$ with $R = e_{1m} + e_{2,m-1} + \cdots + e_{m-1,2} + e_{m1}$.

(e) $\mu_{t,b}^{(ij)}$ for $b \in F$, $i = 1, m$ and $j = 1, 2$ are defined by

$$\mu_{t,b}^{(11)} : X = \sum_{1 \le i < j \le m} T_{ij}(a_{ij}) \mapsto X + T_{2m}(ba_{12});$$
$$\mu_{t,b}^{(m1)} : X = \sum_{1 \le i < j \le m} T_{ij}(a_{ij}) \mapsto X + T_{1,m-1}(ba_{m-1,m});$$
$$\mu_{t,b}^{(12)} : X = \sum_{1 \le i < j \le m} T_{ij}(a_{ij}) \mapsto X + T_{2m}(ba_{13}) + T_{3m}(ba_{12});$$
$$\mu_{t,b}^{(m2)} : X = \sum_{1 \le i < j \le m} T_{ij}(a_{ij}) \mapsto X + T_{1,m-2}(ba_{m-1,m}) + T_{1,m-1}(ba_{m-2,m}).$$

We call the linear maps of types (a)–(e) defined above standard maps of $t(F)$. By Lemma 2.2 [7], 2.3 [7] and Theorem 1.1 [7], we have the following theorem.

Theorem 3.1 Let $m \ge 5$. Then a linear map φ of $t(F)$ preserves commutativity in both directions if and only if φ is of the form

$$\varphi \quad = \quad \psi_{t,c}\mathrm{Int}_t T \omega \mu_{t,b_4}^{(m2)} \mu_{t,b_3}^{(12)} \mu_{t,b_2}^{(m1)} \mu_{t,b_1}^{(11)} \eta_{t,f},$$

where $\psi_{t,c}$, $\mathrm{Int}_t T$, ω, $\mu_{t,b_4}^{(m2)}$, $\mu_{t,b_3}^{(12)}$, $\mu_{t,b_2}^{(m1)}$, $\mu_{t,b_1}^{(11)}$, $\eta_{t,f}$ are the standard maps of $t(F)$.

4 Standard Maps of $l_{2m}(F)$

It is obvious that $\mathcal{T}'$ forms a group under multiplication of maps. We now define some standard maps of $l_{2m}(F)$ which preserve zero Lie brackets in both directions, then we use them to prove the main theorem of this paper. It is easy to check that the following linear maps of $l_{2m}(F)$ are all in $\mathcal{T}$ when $m \ge 5$.

(1) Inner automorphisms

For $A \in T^*(m, F)$ and $B' = -B \in F^{m \times m}$, set $X = \begin{bmatrix} A & AB \\ 0 & A'^{-1} \end{bmatrix}$. The map $\mathrm{Int}X : Y \mapsto X^{-1}YX$ is an automorphism of $l_{2m}(F)$, called the inner automorphism of $l_{2m}(F)$ induced by X.

(2) Graph automorphisms

Let $\pi = E^{(2m)} - E_{mm} - E_{-m,-m} + E_{m,-m} + E_{-m,m}$. The map $\mu_\pi : Y \mapsto \pi Y \pi$ is an automorphism of $l_{2m}(F)$, called the graph automorphism of $l_{2m}(F)$.

(3) Generalized diagonal automorphisms

For $g \in F^*$, let $G = \mathrm{diag}(E, gE)$. The map $\xi_g : l_{2m}(F) \to l_{2m}(F)$ defined by $Y \mapsto G^{-1}YG$ is an automorphism of $l_{2m}(F)$, called the generalized diagonal automorphism of $l_{2m}(F)$ induced by G.

(4) Central maps

Let f be a linear map from $l_{2m}(F)$ to F, we define the map $\eta_f : l_{2m}(F) \to l_{2m}(F)$ by $\eta_f(Y) = Y + f(Y)T_{1,-2}(1)$. It is easy to check that $\eta_f \in \mathcal{T}$, called the central map. If f satisfies the additional conditions: $f([X,Y]) = 0$ for any $X, Y \in l_{2m}(F)$ and $1+f(T_{1,-2}(1)) \neq 0$, then η_f is also a Lie automorphism of $l_{2m}(F)$.

(5) Sub-central automorphisms

For $h \in F$, $Y = \sum\limits_{1\leq i<j\leq m} T_{ij}(a_{ij}) + \sum\limits_{1\leq k<l\leq m} T_{k,-l}(a_{k,-l}) \in l_{2m}(F)$. The map λ_h : $l_{2m}(F) \to l_{2m}(F)$ defined by $\lambda_h(Y) = Y + T_{1,-3}(ha_{23})$ is an automorphism of $l_{2m}(F)$, called the sub-central automorphism of $l_{2m}(F)$ induced by h.

(6) Extremal maps

For $a, b, c \in F$, $Y = \sum\limits_{1\leq i<j\leq m} T_{ij}(a_{ij}) + \sum\limits_{1\leq k<l\leq m} T_{k,-l}(a_{k,-l}) \in l_{2m}(F)$, we define the map $\rho_{a,b,c} : l_{2m}(F) \to l_{2m}(F)$ by

$$\rho_{a,b,c}(Y) = Y + T_{1,-3}(aa_{13} + ca_{24}) + T_{1,-4}(ba_{13} + ca_{23}) - T_{2,-3}(aa_{23} + ba_{24}).$$

It is easy to check that $\rho_{a,b,c} \in \mathcal{T}'$, but it usually fails to be an automorphism of $l_{2m}(F)$. We call it the extremal map of type I.

(7) Scalar multiplication

For $c \in F^*$, we define the map $\psi_c : l_{2m}(F) \to l_{2m}(F)$ by $\psi_c(X) = cX$. Clearly, $\psi_c \in \mathcal{T}'$, called the multiple map. It is easy to see that ψ_c is an automorphism of $l_{2m}(F)$ if and only if $c = 1$.

Lemma 4.1 (i) $\xi_g \mathrm{Int}T_1 = \mathrm{Int}T_2\xi_g$, where $T_2 = \xi_g(T_1)$.

(ii) $\xi_g\lambda_{h_1} = \lambda_{h_2}\xi_g$, where $h_2 = gh_1$.

(iii) $\xi_g\rho_{a_1,b_1,c_1} = \rho_{a_2,b_2,c_2}\xi_g$, where $a_2 = ga_1, b_2 = gb_1, c_2 = gc_1$.

(iv) $\mathrm{Int}T\mu_\pi = \mu_\pi\mathrm{Int}T$ for $T = \mathrm{diag}(A, 1, A^{'-1}, 1)$ with $A \in T^*(m-1, F)$.

(v) ψ_c commutes with every linear map on $l_{2m}(F)$. In particular, ψ_c commutes with every standard map.

Proof The proof is trivial, we omit it.

5 Lemmas and the Proof of Main Theorem

Let φ be a linear map preserving zero Lie brackets in both directions. Throughout this section, without loss of generality, we assume that φ is bijective and $m \geq 5$. In fact, if φ is not bijective, we have $\varphi(T_{1,-2}(1)) = 0$ by Lemma 2.1. Let f be a linear function from $l_{2m}(F)$ to F such that $f(T_{1,-2}(1)) \neq 0$, then $\varphi + f \in \mathcal{T}'$ again by Lemma 2.1. Thus φ can be replaced with $\varphi + f$. For $X \in l_{2m}(F)$, we denote by $C(X)$ the centralizer of X in $l_{2m}(F)$, i.e., $C(X) = \{Y \in l_{2m}(F) \mid [X,Y] = 0\}$. In order to prove the main result in this paper, we need to give some lemmas first.

Lemma 5.1 Let $\varphi \in \mathcal{T}'$, then $p(F)$ and $y(F)$ are stable under φ.

Proof If we can prove that $p(F)$ is invariant under φ, then $y(F)$, being the centralizer of $p(F)$ in $l_{2m}(F)$, is also invariant under φ. So for our goal, it suffices to prove that $p(F)$ is invariant under φ. It is clear that the set

$$\mathcal{B}=\{T_{i,-j}(1),T_{1m}(1)|1\le i<j\le m,i+j\le m+1\}$$

is the canonical basis of $p(F)$, so we only need to show that $\varphi(X)\in p(F)$ for all $X\in\mathcal{B}$. It is not difficult to check that dim $C(X)\ge(m-1)m-(m-2)$ for any $X\in\mathcal{B}$. Since φ is bijective and preserves zero Lie brackets in both directions, we have

$$\dim C(\varphi(X))=\dim C(X)\ge(m-1)m-(m-2). \tag{5.1}$$

In the following, we prove that $\varphi(X)\in p(F)$ for all $X\in\mathcal{B}$ by two steps.

Step 1 $\varphi(X)\in w(F)+T_{1m}$ for all $X\in\mathcal{B}$.

If there exists some $X\in\mathcal{B}$ such that $\varphi(X)\notin w(F)+T_{1m}$, then we can assume that $\varphi(X)=\sum_{1\le i<j\le m}T_{ij}(a_{ij})+W$ with some $a_{st}\ne0$ for $1\le s<t\le m,(s,t)\ne(1,m)$ and $W\in w(F)$. Let i_0,j_0 be such that $a_{i_0j_0}\ne0,(i_0,j_0)\ne(1,m)$ and $a_{i_0,k}=0$ for all $k<j_0$ and $a_{k,j_0}=0$ for all $k>i_0$. Set

$$\begin{aligned}M_1&=E^{(2m)}-\sum_{k=1}^{m-j_0}T_{j_0,j_0+k}(a_{i_0j_0}^{-1}a_{i_0,j_0+k}),\\ M_2&=E^{(2m)}+\sum_{k=1}^{i_0-1}T_{ki_0}(a_{i_0j_0}^{-1}a_{kj_0}),\\ v_0&=\sum_{k=1}^{m-j_0}T_{j_0,j_0+k}+\sum_{l=1}^{i_0-1}T_{l,-j_0}+\sum_{l=i_0+1}^{j_0-1}T_{l,-j_0}+\sum_{h=1}^{m-j_0}T_{j_0,-(j_0+h)}+\sum_{k=1}^{i_0-1}T_{ki_0}.\end{aligned}$$

Then v_0 is a subspace of $l_{2m}(F)$ and $v_0\cap C((M_1M_2)^{-1}\varphi(X)M_1M_2)=\{0\}$. It is clear that dim $v_0\ge m-1$. So

$$\dim C(\varphi(X))=\dim C((M_1M_2)^{-1}\varphi(X)M_1M_2)\le m(m-1)-(m-1). \tag{5.2}$$

In contradiction with (5.1). So $\varphi(X)\in w(F)+T_{1m}$ for all $X\in\mathcal{B}$.

Step 2 $\varphi(X)\in p(F)$ for all $X\in\mathcal{B}$.

By Step 1, we know that $\varphi(X)\in w(F)+T_{1m}$ for all $X\in\mathcal{B}$. If there exists some $X\in\mathcal{B}$ such that $\varphi(X)\in w(F)+T_{1m}$ but $\varphi(X)\notin p(F)$, then we may write $\varphi(X)=\sum_{1\le i<j\le m}T_{i,-j}(b_{i,-j})+T_{1m}(b_{1m})$, and there exist some $b_{p,-q}\ne0$ with $p+q>m+1$ and $p<q$. Let p_0,q_0 be such that $b_{p_0,-q_0}\ne0$ and $b_{p_0,-k}=0$ for all $k>q_0$ and $b_{k,-q_0}=0$ for all $k>p_0$. Set

$$\begin{aligned}N_1&=E^{(2m)}+\sum_{k=1}^{p_0-1}T_{k,p_0}(b_{p_0,-q_0}^{-1}b_{k,-q_0}),\\ N_2&=E^{(2m)}-\sum_{k=1}^{p_0-1}T_{k,q_0}(b_{p_0,-q_0}^{-1}b_{k,-p_0})+\sum_{k=p_0+1}^{q_0-1}T_{k,q_0}(b_{p_0,-q_0}^{-1}b_{p_0,-k}).\end{aligned}$$

Then the entries of $N_2^{-1}N_1^{-1}\varphi(X)N_1N_2$ in the p_0-th row, q_0-th row, (p_0+m)-th column and (q_0+m)-th column are all zero except the (p_0, q_0+m)-entry and (q_0, p_0+m)-entry. Let

$$v_1 = T_{1p_0} + \cdots + T_{p_0-1,p_0} + T_{1q_0} + \cdots + T_{p_0-1,q_0} + T_{p_0+1,q_0} + \cdots + T_{q_0-1,q_0}.$$

Then v_1 is a subspace of $l_{2m}(F)$. One may check that $v_1 \cap C(N_2^{-1}N_1^{-1}\varphi(X)N_1N_2) = \{0\}$ and $\dim v_1 = p_0 - 1 + q_0 - 2 > m-2$. So

$$\dim C(\varphi(X)) = \dim C(N_2^{-1}N_1^{-1}\varphi(X)N_1N_2) < m(m-1) - (m-2). \tag{5.3}$$

In contradiction with (5.1). So $\varphi(X) \in p(F)$ for all $X \in \mathcal{B}$.

Since $\mathcal{B}$ is a basis of $p(F)$ and φ is a bijective linear map, we have $\varphi(p(F)) = p(F)$. That is to say $p(F)$ is stable under φ.

Lemma 5.2 Let $\varphi \in \mathcal{T}'$, then $l_{2m}^{(k)}(F)$ is invariant under φ for every $1 \le k \le 2m-4$. Furthermore, $s(F)$, being the centralizer of $u(F)$ which exactly is $l_{2m}^{(m-1)}(F)$, is also invariant under φ.

Proof The process, being similar to Lemma 5.1, is omitted.

Lemma 5.3 Let $\varphi \in \mathcal{T}'$, then $q(F)$, $x(F)$ and $z(F)$ are invariant under φ.

Proof If we can prove that $q(F)$ is invariant under φ, then $x(F)$, being the center of $q(F)$, is invariant under φ. Furthermore, $z(F)$, being the centralizer of $x(F)$ in $l_{2m}(F)$, is also invariant under φ. For our goal, we only need to prove that $q(F)$ is invariant under φ. Let $\varphi \in \mathcal{T}'$. Since $\varphi(y(F)) = y(F) \subseteq q(F)$, it suffices to prove that $\varphi(T_{2m}(1))$ and $\varphi(T_{2,-m}(1))$ are all contained in $q(F)$. Since $T_{2m}(1) \in l_{2m}^{(m-3)}(F)$ and $T_{m-2,-(m-1)}(1) \in y(F)$, we may assume that

$$\varphi(T_{2m}(1)) \equiv T_{1,m-1}(a_{1,m-1}) + T_{2m}(a_{2m}) + T_{2,-m}(a_{2,-m}) \pmod{y(F)},$$

$$\varphi(T_{m-2,-(m-1)}(1)) = \sum_{1\le i<j\le m-1} T_{i,-j}(b_{i,-j}) + T_{1,-m}(b_{1,-m}) + T_{1m}(b_{1m}),$$

where $b_{m-2,-(m-1)} \ne 0$. By considering the action of φ on $[T_{2m}(1), T_{m-2,-(m-1)}(1)] = 0$, we have $a_{1,m-1}b_{m-2,-(m-1)} = 0$, so $a_{1,m-1} = 0$. That is to say $\varphi(T_{2m}(1)) \in q(F)$. We can also prove that $\varphi(T_{2,-m}(1)) \in q(F)$ in the similar way. So $\varphi(q(F)) \subseteq q(F)$. Since φ is invertible, we have $\varphi(q(F)) = q(F)$.

Now we give the main result of this paper.

Theorem 5.4 Let $m \ge 5$. A linear map φ of $l_{2m}(F)$ preserves zero Lie brackets in both directions if and only if φ is of the form

$$\varphi = \psi_c(\mu_\pi)^\delta \mathrm{Int}X \lambda_h \rho_{a,b,c} \phi_d \xi_g \eta_f, \tag{5.4}$$

where $\psi_c, \mu_\pi, \mathrm{Int}X, \lambda_h, \rho_{a,b,c}, \phi_d, \xi_g, \eta_f$ are the standard maps preserving zero Lie brackets in both directions, $\delta = 0$ or 1.

Proof The "if" part of the theorem is clear. For the "only if" part, we will give the proof by steps. In the following, we assume that $m \ge 5$ and $\varphi \in \mathcal{T}'$.

Step 1 There exist $X_1 = \mathrm{diag}(A, 1, A^{'-1}, 1)$ with $A \in T^*(m-1, F)$ and $c_1 \in F^*$ such that $\mathrm{Int}^{-1} X_1 \psi_{c_1}^{-1} \varphi(T_{ij}(1)) \equiv T_{ij}(1) \pmod{s(F)}$ for $1 \le i < j \le m-1$.

Since $z(F)$ is an ideal of $l_{2m}(F)$ and stable under φ, then φ induces a linear map $\overline{\varphi}$ of $l_{2m}(F)/z(F)$ by $\overline{\varphi}(\overline{Y}) = \overline{\varphi(Y)}$, where $\overline{Y} = Y + z(F) \in l_{2m}(F)/z(F)$, $Y \in l_{2m}(F)$. It can be proved that $\overline{\varphi}$ is invertible and preserves zero Lie brackets in both directions. Since $l_{2m}(F)/z(F)$ is isomorphic to $v(F)$, we may directly view $l_{2m}(F)/z(F)$ as $v(F)$. Thus by Theorem 3.1, $\overline{\varphi}$ can be written in the form:

$$\overline{\varphi} = \psi_{v,c_1} \mathrm{Int}_v X_1 \omega \mu_{v,b_4}^{(m2)} \mu_{v,b_3}^{(12)} \mu_{v,b_2}^{(m1)} \mu_{v,b_1}^{(11)} \eta_{v,f},$$

where ψ_{v,c_1}, $\mathrm{Int}_v X_1$, ω, $\mu_{v,b_4}^{(m2)}$, $\mu_{v,b_3}^{(12)}$, $\mu_{v,b_2}^{(m1)}$, $\mu_{v,b_1}^{(11)}$, $\eta_{v,f}$ are the standard maps of $v(F)$. It is easy to see that $\psi_{v,c_1} = \overline{\psi_{c_1}}$, $\mathrm{Int}_v X_1 = \overline{\mathrm{Int} X_1}$. So $\overline{\mathrm{Int}^{-1} X_1 \psi_{c_1}^{-1} \varphi} = \omega \mu_{v,b_4}^{(m2)} \mu_{v,b_3}^{(12)} \mu_{v,b_2}^{(m1)} \mu_{v,b_1}^{(11)} \eta_{v,f}$. Denote $\mathrm{Int}^{-1} X_1 \psi_{c_1}^{-1} \varphi$ by φ_1.

We are now ready to prove that $\omega = 1$, $b_4 = b_3 = b_2 = b_1 = 0$. Since $x(F)$ and $p(F)$ are stable under φ_1, $T_{2,-(m-1)}(1) \in x(F) \cap p(F)$, we may write

$$\varphi_1(T_{2,-(m-1)}(1)) = \sum_{1 \le i < j \le m-1, i+j \le m+1} T_{i,-j}(a_{i,-j}),$$

where $a_{2,-(m-1)} \neq 0$. If $\omega \neq 1$, we have

$$\varphi_1(T_{13}(1)) \equiv -T_{1,m-2}(b_3) - f(T_{13}(1)) T_{1,m-1}(1) - T_{m-2,m-1}(1) \pmod{z(F)}.$$

From $[T_{13}(1), T_{2,-(m-1)}(1)] = 0$, we have $[\varphi_1(T_{13}(1)), \varphi_1(T_{2,-(m-1)}(1))] = 0$, which implies that $a_{2,-(m-1)} = 0$, a contradiction. So $\omega = 1$.

Since $x(F)$ is stable under φ_1, we may write

$$\varphi_1(T_{m-2,-(m-1)}(1)) = \sum_{1 \le i < j \le m-1} T_{i,-j}(b_{i,-j}),$$

where $b_{m-2,-(m-1)} \neq 0$. By considering the action of φ_1 on $[T_{12}(1), T_{m-2,-(m-1)}(1)] = 0$, we have $b_{m-2,-(m-1)} b_3 = b_{m-2,-(m-1)} b_1 = 0$, which implies that $b_1 = b_3 = 0$.

Because $x(F)$ and $u(F)$ are stable under φ_1, $T_{2,-(m-2)} \in x(F) \cap u(F)$, we may write

$$\varphi_1(T_{2,-(m-2)}(1)) = \sum_{1 \le i < j \le m-1, i+j \le m} T_{i,-j}(c_{i,-j}).$$

Since $[\varphi_1(T_{2,-(m-2)}(1)), \varphi_1(T_{1,m-2}(1))] \neq 0$, we have $c_{2,-(m-2)} \neq 0$. By applying φ_1 on $[T_{m-3,m-1}(1), T_{2,-(m-2)}(1)] = 0$ and $[T_{m-2,m-1}(1), T_{2,-(m-2)}(1)] = 0$, respectively, we have $b_4 c_{2,-(m-2)} = b_2 c_{2,-(m-2)} = 0$, which implies that $b_4 = b_2 = 0$. So $\overline{\varphi_1} = \eta_{v,f}$, as desired.

Step 2 There exist a graph automorphism μ_π and a generalized diagonal automorphism ξ_g such that $\xi_g^{-1} (\mu_\pi^{-1})^\delta \varphi_1(T_{1,-m}(1)) \equiv T_{1,-m}(1) \pmod{x(F)}$, where $\delta = 0$ or $\delta = 1$.

Since $y(F)$ is stable under φ_1 and $T_{1,-m}(1) \in y(F)$, we may assume that

$$\varphi_1(T_{1,-m}(1)) \equiv T_{1m}(a) + T_{1,-m}(b) \pmod{x(F)}, \text{ where } a, b \in F.$$

It is easy to see that a and b can not be zero simultaneously. Since $q(F)$, $z(F)$ and $l_{2m}^{(m-4)}(F)$ are stable under φ_1, $T_{2,-m}(1) \in q(F)$, $T_{3,-m}(1) \in z(F) \cap l_{2m}^{(m-4)}(F)$, we may write

$$\varphi_1(T_{2,-m}(1)) \equiv T_{2m}(c) + T_{2,-m}(d) \pmod{y(F)},$$
$$\varphi_1(T_{3,-m}(1)) \equiv T_{3m}(e) + T_{3,-m}(f) \pmod{q(F)},$$

where $c, d, e, f \in F$. By applying φ_1 on $[T_{1,-m}(1), T_{2,-m}(1)] = 0$, $[T_{2,-m}(1), T_{3,-m}(1)] = 0$ and $[T_{1,-m}(1), T_{3,-m}(1)] = 0$, respectively, we get $ad + bc = 0$, $cf + de = 0$ and $af + be = 0$. If $ab \neq 0$, then $cd \neq 0$ and $ef \neq 0$. So we have

$$\begin{cases} \frac{a}{b} = -\frac{c}{d}, \\ \frac{a}{b} = -\frac{e}{f}, \\ \frac{e}{f} = -\frac{c}{d}, \end{cases}$$

which implies that $\frac{c}{d} = -\frac{c}{d}$. That is to say $cd = 0$. This contradiction shows that $ab = 0$.

If $a \neq 0$ and $b = 0$, let $g = a$ and $\delta = 1$; if $b \neq 0$ and $a = 0$, let $g = b$ and $\delta = 0$. Then

$$\xi_g^{-1}(\mu_\pi^{-1})^\delta \varphi_1(T_{1,-m}(1)) \equiv T_{1,-m}(1) \pmod{x(F)},$$

where $(\mu_\pi^{-1})^1 = \mu_\pi^{-1}$, $(\mu_\pi^{-1})^0 = 1$. Denote $\xi_g^{-1}(\mu_\pi^{-1})^\delta\varphi_1$ by φ_2.

Step 3 There exists $X_2 = E^{(2m)} + P_2$, $P_2 \in \sum_{2 \leq k \leq m-1} T_{1k}$ such that $w(F)$ is stable under $\mathrm{Int}^{-1}X_2\varphi_2$. In particular, $\mathrm{Int}^{-1}X_2\varphi_2(T_{i,-m}(1)) \equiv T_{i,-m}(a_{i,-m}^{(i)}) \pmod{x(F)}$, where $a_{1,-m}^{(1)} = 1, a_{i,-m}^{(i)} \in F^*$ for $2 \leq i \leq m-1$.

If we can prove the latter assertion, then we shall get the former one by the fact that $\mathrm{Int}^{-1}X_2\varphi_2(x(F)) = x(F)$ and the latter assertion. Since $z(F)$ is stable under φ_2 and $T_{i,-m}(1) \in z(F)$, we may assume that

$$\varphi_2(T_{i,-m}(1)) \equiv \sum_{k=1}^{m-1}(T_{km}(a_{km}^{(i)}) + T_{k,-m}(a_{k,-m}^{(i)})) \pmod{x(F)},$$

where $2 \leq i \leq m-1$. For $2 \leq k \neq i \leq m-1$, by applying φ_2 on $[T_{k-1,k}(1), T_{i,-m}(1)] = 0$, we have $a_{km}^{(i)} = a_{k,-m}^{(i)} = 0$. By considering the action of φ_2 on $[T_{1,-m}(1), T_{i,-m}(1)] = 0$, and $[T_{i,-m}(1), T_{m-1,-m}(1)] = 0$, we obtain that $a_{im}^{(i)} = 0$ and $a_{1m}^{(i)} = a_{1m}^{(m-1)} = 0$. Since $T_{i,-m}(1) \notin y(F)$, we have $a_{i,-m}^{(i)} \neq 0$. Set $X_2 = E^{(2m)} + T_{12}(a_{1,-m}^{(2)}(a_{2,-m}^{(2)})^{-1}) + T_{13}(a_{1,-m}^{(3)}(a_{3,-m}^{(3)})^{-1}) + \cdots + T_{1,m-1}(a_{1,-m}^{(m-1)}(a_{m-1,-m}^{(m-1)})^{-1})$ and denote $\mathrm{Int}^{-1}X_2\varphi_2$ by φ_3. We see that $\varphi_3(T_{i,-m}(1)) \equiv T_{i,-m}(a_{i,-m}^{(i)}) \pmod{x(F)}$, where $a_{1,-m}^{(1)} = 1$ and $a_{i,-m}^{(i)} \in F^*$ for $2 \leq i \leq m-1$.

Step 4 There exist $X_3 = \mathrm{diag}(A, A'^{-1})$ with $A \in T^*(m, F)$ and $c_2 \in F^*$ such that $\psi_{c_2}^{-1}X_3\varphi_3(T_{ij}(1)) \equiv T_{ij}(1) \pmod{w(F) + T_{1m}}$ for $1 \leq i < j \leq m$.

By Step 3, we know that $w(F)$ is stable under φ_3, so φ_3 induces a linear map $\widetilde{\varphi_3}$ of $l_{2m}(F)/w(F)$ by $\widetilde{\varphi_3}(\widetilde{X}) = \widetilde{\varphi_3(X)}$ for $\widetilde{X} = X + w(F) \in l_{2m}(F)/w(F)$. It is not difficult to prove that $\widetilde{\varphi_3}$ is bijective and preserves zero Lie brackets in both directions. Since $l_{2m}(F)/w(F)$ is isomorphic to $t(F)$, we may directly view $l_{2m}(F)/w(F)$ as $t(F)$. Thus by Theorem 3.1, $\widetilde{\varphi_3}$ can be written in the form:

$$\widetilde{\varphi_3} = \psi_{t,c_2}\mathrm{Int}_t X_3 \omega \mu_{t,b_4}^{(m2)} \mu_{t,b_3}^{(12)} \mu_{t,b_2}^{(m1)} \mu_{t,b_1}^{(11)} \eta_{t,f},$$

where $\psi_{t,c_2}, \mathrm{Int}_t X_3, \omega, \mu_{t,b_4}^{(m2)}, \mu_{t,b_3}^{(12)}, \mu_{t,b_2}^{(m1)}, \mu_{t,b_1}^{(11)}, \eta_{t,f}$ are the standard maps of $t(F)$. It is easy to know that $\widetilde{\psi_{c_2}} = \psi_{t,c_2}$, $\widetilde{\mathrm{Int}X_3} = \mathrm{Int}_t X_3$. So $\mathrm{Int}^{-1}\widetilde{X_3\psi_{c_2}^{-1}}\varphi_3 = \omega\mu_{t,b_4}^{(m2)}\mu_{t,b_3}^{(12)}\mu_{t,b_2}^{(m1)}\mu_{t,b_1}^{(11)}\eta_{t,f}$. Denote $\mathrm{Int}^{-1}X_3\psi_{c_2}^{-1}\varphi_3$ by φ_4.

If $\omega \neq 1$, then $\varphi_4(T_{1,m-1}(1)) \equiv -f(T_{1,m-1}(1))T_{1m}(1) - T_{2m}(1) \pmod{w(F)}$. It is easy to see that $\varphi_4(T_{1,-m}(1)) = T_{1,-m}(a) + X_0$ for some $X_0 \in x(F)$ and $a \in F^*$. By applying φ_4 on $[T_{1,m-1}(1), T_{1,-m}(1)] = 0$, we get $a = 0$, a contradiction. So $\omega = 1$.

Since $w(F)$ is stable under φ_4, we may write

$$\varphi_4(T_{m-1,-m}(1)) = \sum_{1\le i<j\le m} T_{i,-j}(a_{i,-j}),$$

where $a_{m-1,-m} \neq 0$. By applying φ_4 on $[T_{12}(1), T_{m-1,-m}(1)] = 0$, we have $a_{m-1,-m}b_1 = a_{m-1,-m}b_3 = 0$, which implies that $b_1 = b_3 = 0$. Since $T_{m-2,-(m-1)}(1) \in x(F)$, we may write $\varphi_4(T_{m-2,-(m-1)}) = \sum_{1\le i<j\le m-1} T_{i,-j}(b_{i,-j})$, where $b_{m-2,-(m-1)} \neq 0$. By applying φ_4 on $[T_{m-2,m}(1), T_{m-2,-(m-1)}(1)] = 0$ and $[T_{m-1,m}(1), T_{m-2,-(m-1)}(1)] = 0$, respectively, we get $b_{m-2,-(m-1)}b_4 = b_{m-2,-(m-1)}b_2 = 0$, which implies that $b_4 = b_2 = 0$. So $\widetilde{\varphi_4} = \eta_{t,f}$, as desired.

Step 5 There exist $X_4 = E^{(2m)} + \sum_{2\le i\le m-1} T_{1i}((b_{i,-m}^{(i)})^{-1}b_{1,-m}^{(i)})$ and $X_5 = E^{(2m)} + f(T_{12}(1))T_{2m}(1)$ such that $\mathrm{Int}^{-1}X_5\mathrm{Int}^{-1}X_4\varphi_4(T_{ij}(1)) \equiv T_{ij}(1) \pmod{w(F)}$ for $1 \le i < j \le m$.

Since $w(F)$ is stable under φ_4, we may suppose that

$$\varphi_4(T_{i,-m}(1)) \equiv \sum_{k=1}^{m-1} T_{k,-m}(b_{k,-m}^{(i)}) \pmod{x(F)} \text{ for } 2 \le i \le m-1.$$

By applying φ_4 on $[T_{im}(1), T_{i,-m}(1)] = 0$, we have that $b_{k,-m}^{(i)} = 0$ for $2 \le k \le m-1$ and $k \neq i$. Since $y(F)$ is stable under φ_4 and $T_{i,-m}(1) \notin y(F)$, we have that $b_{i,-m}^{(i)} \neq 0$. Set $X_4 = E^{(2m)} + \sum_{2\le i\le m-1} T_{1i}((b_{i,-m}^{(i)})^{-1}b_{1,-m}^{(i)})$. We see that

$$\mathrm{Int}^{-1}X_4\varphi_4(T_{i,-m}(1)) \equiv T_{i,-m}(b_{i,-m}^{(i)}) \pmod{x(F)} \text{ for } 2 \le i \le m-1.$$

Denote $\mathrm{Int}^{-1}X_4\varphi_4$ by φ_5. We can write

$$\varphi_5(T_{2,-m}(1)) = \sum_{1\le i<j\le m-1} T_{i,-j}(b_{i,-j}) + T_{2,-m}(b_{2,-m}^{(2)}).$$

For $3 \le i < j \le m-1$, by applying φ_5 on $[T_{ij}(1), T_{2,-m}(1)] = 0$ and $[T_{2j}(1), T_{2,-m}(1)] = 0$, respectively, we get $f(T_{ij}(1)) = f(T_{2j}(1)) = 0$ and $b_{k,-l} = 0$ for $1 \le k \le 3, 4 \le l \le m-1$ and $4 \le k < l \le m-1$. By applying φ_5 on $[T_{1i}(1), T_{2,-m}(1)] = 0$, we have $f(T_{1i}(1)) = 0$ for $4 \le i \le m-1$. By applying φ_5 on $[T_{im}(1), T_{i,-m}(1)] = 0$ for $2 \le i \le m-1$, we have $f(T_{im}(1)) = 0$. By applying φ_5 on $[T_{23}(1), T_{m-1,-m}(1)] = 0$ and $[T_{13}(1), T_{m-1,-m}(1)] = 0$, respectively, we have $f(T_{23}(1)) = f(T_{13}(1)) = 0$. By considering the action of φ_5 on $[T_{3m}(1)-$

$T_{3,m-1}(1), T_{2,-m}(1)+T_{2,-(m-1)}(1)] = 0$ and $[T_{1m}(1)-T_{1,m-1}(1), T_{2,-m}(1)+T_{2,-(m-1)}(1)] = 0$, we have $f(T_{1m}(1)) = 0$. Let $X_5 = E^{(2m)} + f(T_{12}(1))T_{2m}(1)$. Then we have

$$\mathrm{Int}^{-1}X_5\varphi_5(T_{ij}(1)) \equiv T_{ij}(1) \pmod{w(F)} \text{ for } 1 \le i < j \le m.$$

Denote $\mathrm{Int}^{-1}X_5\varphi_5$ by φ_6.

Step 6 There exist $h \in F$, $X_6 = E^{(2m)} + W$ with $W \in w(F)$ and $a, b, c \in F$ such that $\rho_{a,b,c}^{-1}\lambda_h^{-1}\mathrm{Int}^{-1}X_6\varphi_6(T_{ij}(1)) \equiv T_{ij}(1) \pmod{T_{1,-2}}$ for $1 \le i < j \le m$.

Suppose that

$$\varphi_6(T_{i,i+1}(1)) = T_{i,i+1}(1) + \sum_{1\le k<l\le m} T_{k,-l}(a_{k,-l}^{(i)}), 1 \le i \le m-1,$$

where $a_{k,-l}^{(i)} \in F$. For $1 \le t \le m-1$ and $t \ne i-1, i+1$, by applying φ_6 on $[T_{i,i+1}(1), T_{t,t+1}(1)] = 0$, we have $a_{t+1,-l}^{(i)} = 0$ for $t+2 \le l \le m$, $l \ne i$ and $a_{k,-(t+1)}^{(i)} = 0$ for $1 \le k \le t-1$, $k \ne i$. By considering the action of φ_6 on $[T_{12}(1), T_{13}(1)] = 0$, $[T_{13}(1), T_{14}(1)] = 0$ and $[T_{12}, T_{14}] = 0$, respectively, we have $a_{2,-3}^{(1)} = 0$ and $a_{3,-4}^{(1)} = 0$. By applying φ_6 on $[T_{23}(1), T_{14}(1)] = 0$, we have $a_{3,-4}^{(2)} = 0$ and $a_{4,-5}^{(2)} = 0$. For $3 \le k \le m-1$, by considering the action of φ_6 on $[T_{2,k+1}(1), T_{k,k+1}(1)] = 0$, we get $a_{1,-(k+1)}^{(k)} = a_{k+1,-(k+2)}^{(k)} = 0$. For $3 \le k \le m-2$, by applying φ_6 on $[T_{k,k+1}(1), T_{k-1,k+2}(1)] = 0$, we obtain $a_{1,-(k+2)}^{(k)} = a_{k+2,-(k+3)}^{(k)} = 0$.

For $j \ne i-1, i+1$, by considering the action of φ_6 on $[T_{i,i+1}(1), T_{j,j+1}(1)] = 0$, we get $a_{i,j+1}^{(i)} = a_{i+1,j}^{(j)}$. Choose

$$X_6 = E^{(2m)} + \sum_{k=3}^{m} T_{1,-k}(a_{1,k-1}^{(k-1)}) + \sum_{2\le i<j\le m} T_{i,-j}(a_{i-1,j}^{(i-1)}),\ h = a_{1,-3}^{(2)}.$$

Then

$$\begin{aligned}
\lambda_h^{-1}\mathrm{Int}^{-1}X_6\varphi_6(T_{12}(1)) &= T_{12}(1) + T_{1,-2}(a_{1,-2}^{(1)}),\\
\lambda_h^{-1}\mathrm{Int}^{-1}X_6\varphi_6(T_{23}(1)) &= T_{23}(1) + T_{1,-2}(a_{1,-2}^{(2)}) + T_{2,-3}(a_{2,-3}^{(2)}) + T_{1,-4}(a_{1,-4}^{(2)}),\\
\lambda_h^{-1}\mathrm{Int}^{-1}X_6\varphi_6(T_{i,i+1}(1)) &= T_{i,i+1}(1) + T_{1,-2}(a_{1,-2}^{(i)}) + T_{i,-(i+1)}(a_{i,-(i+1)}^{(i)}),\ 3 \le i \le m-1.
\end{aligned}$$

Denote $\lambda_h^{-1}\mathrm{Int}^{-1}X_6\varphi_6$ by φ_7.

Now we may assume that

$$\varphi_7(T_{ij}(1)) = T_{ij}(1) + \sum_{1\le k<l\le m} T_{k,-l}(b_{k,-l}^{(ij)}),\ 1 \le i \le m-2,\ i+2 \le j \le m,$$

where $b_{k,-l}^{(ij)} \in F$. For $1 \le t \le m-1$, $t \ne i-1, j$, $2 \le s \le m$, $3 \le p \le m$, $s, p \ne i$, by applying φ_7 on $[T_{t,t+1}(1), T_{ij}(1)] = 0$, $[T_{1s}(1), T_{ij}(1)] = 0$ and $[T_{2p}(1), T_{ij}(1)] = 0$, respectively, we have

$$\begin{aligned}
\varphi_7(T_{13}(1)) &= T_{13}(1) + T_{1,-2}(b_{1,-2}^{(13)}) + T_{1,-3}(b_{1,-3}^{(13)}) + T_{1,-4}(b_{1,-4}^{(13)}) + T_{2,-4}(b_{2,-4}^{(13)}),\\
\varphi_7(T_{14}(1)) &= T_{14}(1) + T_{1,-2}(b_{1,-2}^{(14)}) + T_{1,-4}(b_{1,-4}^{(14)}) + T_{2,-3}(b_{2,-3}^{(14)}),\\
\varphi_7(T_{24}(1)) &= T_{24}(1) + T_{1,-2}(b_{1,-2}^{(24)}) + T_{1,-3}(b_{1,-3}^{(24)}) + T_{2,-3}(b_{2,-3}^{(24)}) + T_{2,-4}(b_{2,-4}^{(24)}),\\
\varphi_7(T_{kl}(1)) &= T_{kl}(1) + T_{1,-2}(b_{1,-2}^{(kl)}) + T_{k,-l}(b_{k,-l}^{(kl)}), k = 1, 2, 5 \le l \le m,\\
\varphi_7(T_{ij}(1)) &= T_{ij}(1) + T_{1,-2}(b_{1,-2}^{(ij)}) + T_{1,-i}(b_{1,-i}^{(ij)}) + T_{i,-j}(b_{i,-j}^{(ij)}), 3 \le i < j \le m, j \ne i+1.
\end{aligned}$$

By applying φ_7 on $[T_{12}(1), T_{13}(1)] = 0$ and $[T_{12}(1), T_{14}(1)] = 0$, respectively, we have $b_{2,-4}^{(13)} = b_{2,-3}^{(14)} = 0$. For $1 \le i < j \le m, 1 \le k < j \le m, k \ne i, 4 \le j \le m$, by applying φ_7 on $[T_{ij}(1), T_{kj}(1)] = 0$, we obtain that $b_{i,-j}^{(ij)} = 0$ and $a_{i,-(i+1)}^{(i)} = 0$. For $3 \le i < j \le m$ and $j \ne i+1$, by applying φ_7 on $[T_{2i}(1) + T_{2,j-1}(1), T_{ij}(1) - T_{j-1,j}(1)] = 0$, we get $b_{1,-i}^{(ij)} = 0$.

By applying φ_7 on $[T_{13}(1), T_{23}(1)] = 0$, $[T_{13}(1), T_{24}(1)] = 0$ and $[T_{23}(1), T_{24}(1)] = 0$, respectively, we get $b_{1,-3}^{(13)} = -a_{2,-3}^{(2)}$, $b_{1,-4}^{(13)} = -b_{2,-3}^{(24)}$ and $b_{1,-3}^{(24)} = a_{1,-4}^{(2)}$. Let $a = b_{1,-3}^{(13)}$, $b = b_{1,-4}^{(13)}$, $c = a_{1,-4}^{(2)}$. Then

$$\rho_{a,b,c}^{-1}\varphi_7(T_{i,i+1}(1)) = T_{i,i+1}(1) + T_{1,-2}(a_{1,-2}^{(i)}) \text{ for } 1 \le i \le m-1,$$
$$\rho_{a,b,c}^{-1}\varphi_7(T_{ij}(1)) = T_{ij}(1) + T_{1,-2}(b_{1,-2}^{(ij)}) \text{ for } 1 \le i < j \le m-1, j \ne i+1.$$

That is to say

$$\rho_{a,b,c}^{-1}\varphi_7(T_{ij}(1)) \equiv T_{ij}(1) \pmod{T_{1,-2}} \text{ for } 1 \le i < j \le m.$$

Denote $\rho_{a,b,c}^{-1}\varphi_7$ by φ_8.

Step 7 There exist some $d, s \in F^*, X_7 = E^{(2m)} + T_{1m}(s^{-1}d)$ and a linear map f from $l_{2m}(F)$ to F such that $\xi_s^{-1} Int X_7 \varphi_8 = \eta_f$.

Now we assume that $\varphi_8(T_{i,-j}(1)) = \sum\limits_{1 \le k < l \le m} T_{k,-l}(c_{k,-l}^{(ij)})$. For $2 \le k \le m$, $k \ne i, j$, $2 \le t \le m-1$, $t+1 \ne i, j$, by applying φ_8 on $[T_{1k}(1), T_{i,-j}(1)] = 0$, $[T_{t,t+1}(1), T_{i,-j}(1)] = 0$ and $[T_{ij}(1), T_{i,-j}(1)] = 0$, respectively, we get

$$\begin{aligned}
\varphi_8(T_{1,-2}(1)) &= T_{1,-2}(c_{1,-2}^{(12)}),\\
\varphi_8(T_{i,-j}(1)) &= T_{i,-j}(c_{i,-j}^{(ij)}) + T_{1,-2}(c_{1,-2}^{(ij)}) \text{ for } 1 \le i \le 2, i < j \le m \text{ and } (i,j) \ne (1,2),\\
\varphi_8(T_{i,-j}(1)) &= T_{i,-j}(c_{i,-j}^{(ij)}) + T_{1,-2}(c_{1,-2}^{(ij)}) + T_{1,-i}(c_{1,-i}^{(ij)}) \text{ for } 3 \le i < j \le m.
\end{aligned}$$

For $3 \le i < j \le m-1$, by applying φ_8 on

$$[T_{i-1,i}(1) - T_{j,j+1}(1), T_{i,-j}(1) + T_{i-1,-(j+1)}(1)] = 0,$$

we obtain $c_{1,-i}^{(ij)} = 0$. For $3 \le i < k \le m-1$, by considering the action of φ_8 on

$$[T_{2i}(1) - T_{2k}(1), T_{i,-m}(1) + T_{k,-m}(1)] = 0,$$

we get that $c_{1,-i}^{(im)} = c_{1,-3}^{(3m)}$ for $4 \le i \le m-1$.

For $3 \le l \ne k \le m$, by applying φ_8 on

$$[T_{2l}(1) - T_{2k}(1), T_{1,-l}(1) + T_{1,-k}(1)] = 0,$$

we get $c_{1,-l}^{(1l)} = c_{1,-k}^{(1k)}$.

For $2 \le i < k < l \le m$, by applying φ_8 on $[T_{1k}(1) - T_{1l}(1), T_{i,-k}(1) + T_{i,-l}(1)] = 0$, we have $c_{i,-k}^{(ik)} = c_{i,-l}^{(il)}$. By applying φ_8 on

$$[T_{13}(1) + T_{23}(1), T_{1,-3}(1) + T_{2,-3}(1)] = 0,$$

and

$$[T_{12}(1)-T_{13}(1),T_{2,-4}(1)+T_{3,-4}(1)]=0,$$

we obtain $c^{(13)}_{1,-3}=c^{(23)}_{2,-3}$ and $c^{(24)}_{2,-4}=c^{(34)}_{3,-4}$. By applying φ_8 on

$$[T_{12}(1)+T_{1j}(1),T_{2,-k}(1)+T_{k,-j}(1)]=0$$

with $4\le k<j\le m$, we get $c^{(2k)}_{2,-k}=c^{(kj)}_{k,-j}$. So all $c^{(ij)}_{i,-j}$ are equal for $1\le i<j\le m$ except $(i,j)=(1,2)$. Let $d=c^{(3m)}_{1,-3}$, $s=c^{(ij)}_{i,-j}$, $X_7=E^{(2m)}+T_{1m}(s^{-1}d)$, then

$$\begin{aligned}
\xi_s^{-1}\mathrm{Int}X_7\varphi_8(T_{1,-2}(1)) &= T_{1,-2}(s^{-1}c^{(12)}_{1,-2}),\\
\xi_s^{-1}\mathrm{Int}X_7\varphi_8(T_{2,-m}(1)) &= T_{2,-m}(s^{-1}c^{(2m)}_{1,-2}-d),\\
\xi_s^{-1}\mathrm{Int}X_7\varphi_8(T_{i,-j}(1)) &= T_{i,-j}(1)+T_{1,-2}(s^{-1}c^{(ij)}_{1,-2})\text{ for }1\le i<j\le m,(i,j)\ne(1,2),(2,m).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\xi_s^{-1}\mathrm{Int}X_7\varphi_8(T_{i,i+1}(1)) &= T_{i,i+1}(1)+T_{1,-2}(s^{-1}a^{(i)}_{1,-2})\text{ for }1\le i\le m-1,\\
\xi_s^{-1}\mathrm{Int}X_7\varphi_8(T_{ij}(1)) &= T_{ij}(1)+T_{1,-2}(s^{-1}b^{(ij)}_{1,-2})\text{ for }1\le i<j\le m,j\ne i+1.
\end{aligned}$$

Let $f(T_{i,i+1}(1))=s^{-1}a^{(i)}_{1,-2}$, $f(T_{1,-2}(1))=s^{-1}c^{(12)}_{1,-2}-1$, $f(T_{2,-m}(1))=s^{-1}c^{(2m)}_{1,-2}-d$, $f(T_{ij}(1))=s^{-1}b^{(ij)}_{1,-2}$, $j\ne i+1$, $f(T_{k,-l}(1))=s^{-1}c^{(kl)}_{1,-2}$, $k,l\ne(1,2),(2,m)$. Then

$$\xi_s^{-1}\mathrm{Int}X_7\varphi_8=\eta_f.$$

Above discussion shows that

$$\varphi=\psi_{c_1}\mathrm{Int}X_1(\mu_\pi)^\delta\xi_g\psi_{c_2}\mathrm{Int}(X_2X_3X_4X_5X_6X_7)\lambda_h\rho_{a,b,c}\xi_s\eta_f.$$

By Lemma 4.1, it is easy to show that φ is of form (5.4). This completes the proof.

References

[1] Frobenius G. Uber die darstellung der endlichen gruppen durch lineare substitutioen[J]. Sitzungsber Deutsch Akad Wiss Berlin, 1897, 994–1015.

[2] Li C K, Tsing N K. Linear preserver problem: a brief introduction and some special techniques[J]. Linear Algebra and its Applications, 1992, 162–164: 217–235.

[3] Li C K, Pierce S. Linear preserver problem[J]. The American Mathematical Monthly, 2001, 108(7): 591–605.

[4] Bresar M, Semrl P. Commutativity preserving linear maps on central simple algebras[J]. Journal of Algebra, 2005, 284(1): 102–110.

[5] Bresar M, Semrl P. On bilinear maps on matrices with applications to commutativity preservers[J]. Journal of Algebra, 2006, 301(2): 803–837.

[6] Choi M D, Jafarian A A, Radjavi H. Linear maps preserving commutativity[J]. Linear Algebra and its Applications, 1987, 87: 227–242.

[7] Cao Youan, Chen Zhonghu, Huang Chong. Commutativity preserving linear maps and Lie automorphisms of strictly triangular matrix space[J]. Linear Algebra Appl., 2002, 350: 41–66.

[8] Marcoux L W, Sourour R. Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras[J]. Linear Algebra and its Applications, 1999, 288(1-3): 89–104.

[9] Omladic M, Radjavi H, Semrl P. Preserving commutativity[J]. Journal of Pure and Applied Algebra, 2001, 156: 309–328.

[10] Petek T. Spectrum and commutativity preserving mappings on triangular matrices[J]. Linear Algebra and its Applications, 2002, 357: 107–122.

[11] Semrl P. Commutativity preserving maps[J]. Linear Algebra Appl., 2008, 429: 1051–1070.

[12] Zhang Jianhua, Yang Aili, Pan Fangfang. Linear maps preserving zero products on nest subalgebras of von Neumann algebras[J]. Linear Algebra Appl., 2006, 412: 348–361.

D_m型正交代数的极大幂零子代数上保零李括积的映射

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摘 要: 令F 表示任意域, $l_{2m}(F)$表示F上D_m型正交李代数的极大幂零子代数. 本文的目的是当$m \geq 5$时, 刻画$l_{2m}(F)$上的每一个双向保零李括积的映射. 利用文献[7]的主要结果和矩阵计算技巧, 本文证明了$l_{2m}(F)$上的一个线性映射φ 是双向保零李括积的当且仅当φ 能够写成内自同构, 图自同构, 广义的对角自同构, 中心映射, 次中心自同构, 极端映射和标量乘法的乘积. 这推广了文献[7]的主要结果.

关键词: 极大幂零子代数; 零李括积; D_m型正交李代数

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