SOME RESULTS FOR CERTAIN SUBCLASS OF MULTIVALENT AND ANALYTIC FUNCTIONS

XIONG Liang-peng, HAN Hong-wei, MA Zhi-yuan
(School of Engineering and Technical, ChengDu University of Technology, Leshan 614007, China)

Abstract: In this paper, we investigate functions of the class $G^{\ast}_{p,c}(a, b, \sigma)$ which are analytic and multivalent in the open unit disk $U = \{z : |z| < 1\}$. By using the method of function theory, we obtain some general results concerning the quasi-Hadamard product and the extreme points and support points of $G^{\ast}_{p,c}(a, b, \sigma)$. Many interesting consequences of the main results extend related works of several earlier authors.

Keywords: analytic functions; multivalent function; quasi-Hadamard product; extreme points; support points

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1 Introduction

Let $A$ denote the functions $f_p(z)$ of the form

$$f_p(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_p > 0; a_{n+p} \geq 0; p \in N^* = \{1, 2, \cdots\}),$$

(1.1)

which are multivalent and analytic in the unit disc $U = \{z \in C : |z| < 1\}$.

Here, we define the general quasi-Hadamard product of the functions $f_{p,i}$ by

$$f_{p,1} * \chi_1, f_{p,2} * \chi_2, f_{p,3} * \cdots * \chi_{s-1}, f_{p,s} = \left(\prod_{i=1}^{s} a_{p,i}\right) z^p - \left(\prod_{i=1}^{s-1} \chi_i\right) \sum_{n=1}^{\infty} \left(\prod_{i=1}^{s} a_{n+p,i}\right) z^{n+p},$$

(1.2)

where $\chi_i$ are any nonnegative real numbers and $f_{p,i}(z) \in A$ are defined as (1.1), $i = 1, 2, \cdots, s$.

A function $f_p(z)$ defined by (1.1) is said to be in the class $G^{\ast}_p(a, b, \sigma)$ if and only if

$$\left|\frac{zf_p(z)}{f_p(z)} - p\right| < \sigma \quad z \in U,$$

(1.3)

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Biography: Xiong Liangpeng(1983–), male, born at Wuhan, Hubei, lecture, major in geometric function theory. E-mail:xlpwxf00163.com.
where \(-1 \leq a < b \leq 1, 0 < \sigma \leq 1\). Moreover, let \(M_p(a, b, \sigma)\) denote the class of functions \(f_p(z)\) such that \(\frac{zf_p'(z)}{p}\) is in the class \(G_p^*(a, b, \sigma)\).

We also have the following special cases on \(G_p^*(a, b, \sigma)\) and \(M_p(a, b, \sigma)\):

(I) For \(a_p \equiv 1\) in (1.1), the classes \(G_p^*(a, b, \sigma) \equiv J_p^*(a, b, \sigma)\) and \(M_p(a, b, \sigma) \equiv C_p(a, b, \sigma)\) were studied by Raina, Nahar [1].

(II) For \(p = 1, a = -1, b = \alpha, \sigma = \beta\), the classes \(G_1^*(-1, \alpha, \beta) \equiv S_0(\alpha, \beta)\) and \(M_1(-1, \alpha, \beta) \equiv C_0(\alpha, \beta)\) introduced by Owa [2] are well known.

Using similar arguments as given by Raina, Nahar [1], we can easily prove the following Lemmas for functions in the classes \(G_p^*(a, b, \sigma)\) and \(M_p(a, b, \sigma)\):

**Lemma 1.1** A function \(f_p(z)\) defined by (1.1) belongs to \(G_p^*(a, b, \sigma)\) if and only if

\[
\sum_{n=1}^{\infty} \left\{ (1 + b\sigma)n + (b - a)p\sigma \right\} a_{n+p} \leq (b - a)p\sigma a_p,
\]  

where \(-1 \leq b < a \leq 1, 0 < \sigma \leq 1, p \in N^* = \{1, 2, \cdots\} \). 

**Lemma 2** A function \(f_p(z)\) defined by (1.1) belongs to \(M_p(a, b, \sigma)\) if and only if

\[
\sum_{n=1}^{\infty} \left\{ (1 + b\sigma)n + (b - a)p\sigma \right\} a_{n+p} \leq (b - a)p\sigma a_p,
\]  

where \(-1 \leq b < a \leq 1, 0 < \sigma \leq 1, p \in N^* = \{1, 2, \cdots\} \). 

Now, we introduce a new general class of analytic functions connected with the classes \(G_p^*(a, b, \sigma)\) and \(M_p(a, b, \sigma)\), which is important in the following discussion.

**Definition 1.1** A function \(f_p(z)\) defined by (1.1) belongs to \(G_{p,c}^*(a, b, \sigma)\) if and only if

\[
\sum_{n=1}^{\infty} \left\{ (1 + b\sigma)n + (b - a)p\sigma \right\} a_{n+p} \leq (b - a)p\sigma a_p,
\]  

where \(-1 \leq b < a \leq 1, 0 < \sigma \leq 1, p \in N^* = \{1, 2, \cdots\} \) and \(c\) is any fixed nonnegative real number.

In fact, for every nonnegative real number \(c\), the class \(G_{p,c}^*(a, b, \sigma)\) is nonempty as the functions of the form

\[
f_p(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(b - a)p\sigma a_p}{c[(1 + b\sigma)n + (b - a)p\sigma]} \lambda_{n+p} z^{n+p},
\]  

where \(a_p > 0, \lambda_{n+p} \geq 0\) and \(\sum_{n=1}^{\infty} \lambda_{n+p} \leq 1\), satisfy inequality (1.6).

We note that

(I) For \(c = 0\), the class \(G_{p,0}^*(a, b, \sigma) \equiv G_p^*(a, b, \sigma)\).

(II) For \(c = 1\), the class \(G_{p,1}^*(a, b, \sigma) \equiv M_p(a, b, \sigma)\).

(III) For \(p = 1, a = -1, b = \alpha, \sigma = \beta\), the class \(G_{1,c}^*(-1, \alpha, \beta) \equiv S_c(\alpha, \beta)\) was studied by Aouf [3].
(IV) For any positive integer $c$, we have the inclusion relation

$$G^*_{p,c}(a,b,\sigma) \subset G^*_{p,c-1}(a,b,\sigma) \subset G^*_{p,c-2}(a,b,\sigma) \subset \cdots \subset G^*_{p,2}(a,b,\sigma) \subset M_p(a,b,\sigma) \subset G^*_{p}(a,b,\sigma).$$

The topology of $\mathcal{A}$ is defined to be the topology of uniform convergence on compact subsets of the unit disk $U$. Suppose that $\mathcal{X}$ is a subset of the space $\mathcal{A}$, then $f \in \mathcal{X}$ is called an extreme point of $\mathcal{X}$ if and only if $f$ can not be expressed as a proper convex combination of two distinct elements of $\mathcal{X}$. The set of all extreme points of $\mathcal{X}$ is denoted by $E\mathcal{X}$.

Furthermore, a function $f$ is called a support point of a compact $\mathcal{F}$ of $\mathcal{A}$ if $f \in \mathcal{F}$ and if there is a continuous linear functional $J$ on $\mathcal{A}$ such that $\Re J$ is non-constant on $\mathcal{F}$ and

$$\Re J(f) = \max\{\Re J(g) : g \in \mathcal{F}\}.$$ 

We shall denote the set of all support points of $\mathcal{F}$ by $\text{supp}\mathcal{F}$.

Throughout this paper we use the notation $H\mathcal{F}$ for the closed convex hull of $\mathcal{F}$.

**Lemma 1.3** (see [4]) Let $\mathcal{A}$ be a locally convex linear topological space and let $\mathcal{F}$ be a compact subset of $\mathcal{A}$, then

(i) If $\mathcal{F}$ is non-empty, then $E\mathcal{F}$ is non-empty.

(ii) $H E\mathcal{F} = H\mathcal{F}$.

(iii) If $H\mathcal{F}$ is compact, then $E H\mathcal{F} \subset \mathcal{F}$.

The main object of the present work is to discuss some interesting results concerning the quasi-Hadamard product of functions belonging to the class $G^*_{p,c}(a,b,\sigma)$, which extends the earlier corresponding studies in [3, 5–10]. Also, we apply this technique in Peng Zhigang [11, 12] to obtain the extreme points and support points of some important classes with $G^*_{p,c}(a,b,\sigma)$.

**2 The Main Theorem**

**Theorem 2.1** Let the functions $f_{p,i}$ defined by (1.1) be in the class $M_p(a,b,\sigma)$ for every $i = 1, 2, 3, \cdots, m$; $m \in N^*$, and let the functions $g_{p,j}$ defined by (1.1) be in the class $G^*_{p}(a,b,\sigma)$ for every $j = 1, 2, \cdots, q, q \in N^*$. If $\prod_{i=1}^{m+q-1} \chi_i = 1$ or for any $i, 0 < \chi_i \leq 1$, then the quasi-Hadamard product $f_{p,1} \ast \chi_1 f_{p,2} \ast \cdots \ast \chi_{m-1} f_{p,m} \ast \chi_m g_{p,1} \ast \chi_{m+1} g_{p,2} \ast \cdots \ast \chi_{m+q-1} g_{p,q}$ belongs to the class $G^*_{p,2m+q-1}(a,b,\sigma) \subset M_p(a,b,\sigma)$.

**Proof** To simplify the notation, we denote by

$$\mathcal{H}_p = f_{p,1} \ast \chi_1 f_{p,2} \ast \cdots \ast \chi_{m-1} f_{p,m} \ast \chi_m g_{p,1} \ast \chi_{m+1} g_{p,2} \ast \cdots \ast \chi_{m+q-1} g_{p,q},$$

the quasi-Hadamard product of the functions $f_{p,1}, f_{p,2}, \cdots, f_{p,m}, g_{p,1}, \cdots, g_{p,q}$.

Clearly,

$$\mathcal{H}_p = \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\} z^p - \left( \prod_{i=1}^{m+q-1} \chi_i \right) \sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{m} a_{n+p,i} \prod_{j=1}^{q} b_{n+p,j} \right\} z^{n+p}. \quad (2.1)$$
To prove the $\mathcal{H}_p \in G_{p,2m+q-1}^*$, we need to show that
\[
\sum_{n=1}^{\infty} \left[ \frac{n+p}{p} \right]^{2m+q-1} \left\{ n(1+b\sigma) + (b-a)p\sigma \right\} \left\{ \prod_{i=1}^{m+q-1} \chi_i \prod_{i=1}^{m} a_{n+p,i} \prod_{j=1}^{q} b_{n+p,j} \right\} 
\leq (b-a)p\sigma \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\}.
\]

As $f_{p,i}(z) \in M_p(a,b,\sigma)$, then for every $i = 1, 2, \cdots, m$, we have
\[
\sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right) \left\{ (1+b\sigma)n + (b-a)p\sigma \right\} a_{n+p,i} \leq (b-a)p\sigma a_{p,i}.
\] (2.2)

Therefore, the condition $a_{n+p,i} \geq 0$ can make sure that
\[
\left( \frac{n+p}{p} \right) \left\{ (1+b\sigma)n + (b-a)p\sigma \right\} a_{n+p,i} \leq (b-a)p\sigma a_{p,i}, i = 1, 2, \cdots, m
\] (2.3)
or
\[
a_{n+p,i} \leq \frac{(b-a)p\sigma}{\left(1+b\sigma\right)n + (b-a)p\sigma} a_{p,i}
\] (2.4)
for every $i = 1, 2, \cdots, m$. Also, since $-1 \leq a < b \leq 1, 0 < \sigma \leq 1$, it implies
\[
\frac{(b-a)p\sigma}{\left(1+b\sigma\right)n + (b-a)p\sigma} \leq \left( \frac{n+p}{p} \right)^{-1},
\] (2.5)
so the right side of the inequality (2.4) is not greater than $\left( \frac{n+p}{n} \right)^{-2} a_{p,i}$, and we obtain
\[
a_{n+p,i} \leq \left( \frac{n+p}{n} \right)^{-2} a_{p,i}
\] (2.6)
for $i = 1, 2, \cdots, m$. Similarly, for $g_{p,j}(z) \in G_p^*(a,b,\sigma)$, from Lemma 1.1 we have
\[
\sum_{n=1}^{\infty} \left\{ (1+b\sigma)n + (b-a)p\sigma \right\} b_{n+p,j} \leq (b-a)p\sigma b_{p,j}
\] (2.7)
for every $j = 1, 2, \cdots, q$. Furthermore, we can obtain
\[
b_{n+p,j} \leq \left( \frac{n+p}{p} \right)^{-1} b_{p,j}
\] (2.8)
for every $j = 1, 2, \cdots, q$.

Using (2.6) for $i = 1, 2, \cdots, m$, (2.8) for $j = 1, 2, \cdots, q-1$, (2.7) for $j = q$ and following
\[
\prod_{i=1}^{m+q-1} \chi_i = 1 \text{ or for any } i, 0 < \chi_i \leq 1, \text{ we have}
\]
\[
\sum_{n=1}^{\infty} \left[ \frac{n+p}{p} \right]^{2m+q-1} \left\{ n(1+b\sigma) + (b-a)p\sigma \right\} \left\{ \prod_{i=1}^{m+q-1} \chi_i \prod_{i=1}^{m} a_{n+p,i} \prod_{j=1}^{q} b_{n+p,j} \right\}.
\]
\[
\leq \sum_{n=1}^{\infty} \left( \frac{n + p}{p} \right)^{2m+q-1} \left\{ (1 + b) + (b - a) \sigma \right\} b_{n+p,q} \left\{ \left( \frac{n + p}{p} \right)^{-2m} \left( \frac{n + p}{p} \right)^{(q-1) \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j}} \right\}
\]

\[
= \sum_{n=1}^{\infty} \left\{ (1 + b) + (b - a) \sigma \right\} b_{n+p,q} \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\} \leq (b - a) \sigma b_{p,q} \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\}
\]

and therefore \( H_p \in \mathcal{G}_{p,2m+q-1}^{*}(a, b, \sigma) \).

Furthermore, since \( \mathcal{G}_{p,2m+q-1}^{*}(a, b, \sigma) \subset \mathcal{G}_{p,2m+q-2}^{*}(a, b, \sigma) \subset \cdots \subset \mathcal{G}_{p,1}^{*}(a, b, \sigma) = \mathcal{M}_{p}(a, b, \sigma) \), which complete the proof of Theorem 2.1.

As \( \mathcal{G}_{p,2m-1}^{*}(a, b, \sigma) \subset \mathcal{G}_{p,2m-2}^{*}(a, b, \sigma) \subset \cdots \subset \mathcal{G}_{p,1}^{*}(a, b, \sigma) = \mathcal{M}_{p}(a, b, \sigma) \), we can obtain the following Corollary 2.1 by setting \( q = 0 \) in Theorem 2.1.

**Corollary 2.1** Let the functions \( f_{p,i} \) defined by (1.1) be in the class \( \mathcal{M}_{p}(a, b, \sigma) \) for every \( i = 1, 2, 3, \ldots, m; m \in \mathbb{N}^* \). If \( \prod_{i=1}^{m-1} x_i = 1 \) or for any \( i, 0 < x_i \leq 1 \), then the quasi-Hadamard product \( f_{p,1} \ast x_1 f_{p,2} \ast \cdots \ast x_{m-1} f_{p,m} \) belongs to the class \( \mathcal{G}_{p,2m-1}^{*}(a, b, \sigma) \subset \mathcal{M}_{p}(a, b, \sigma) \).

As \( \mathcal{G}_{p,q-1}^{*}(a, b, \sigma) \subset \mathcal{G}_{p,q-2}^{*}(a, b, \sigma) \subset \cdots \subset \mathcal{G}_{p,1}^{*}(a, b, \sigma) = \mathcal{M}_{p}(a, b, \sigma) \subset \mathcal{G}_{p}^{*}(a, b, \sigma) \), we can obtain the following Corollary 2.2 by setting \( m = 0 \) in Theorem 2.1.

**Corollary 2.2** Let the functions \( g_{p,j} \) defined by (1.1) be in the class \( \mathcal{G}_{p}^{*}(a, b, \sigma) \) for every \( j = 1, 2, \ldots, q, q \in \mathbb{N}^* \). If \( \prod_{j=1}^{q-1} x_j = 1 \) or for any \( j, 0 < x_j \leq 1 \), then the quasi-Hadamard product \( g_{p,1} \ast x_1 g_{p,2} \ast \cdots \ast x_{q-1} g_{p,q} \) belongs to the class \( \mathcal{G}_{p,q-1}^{*}(a, b, \sigma) \subset \mathcal{M}_{p}(a, b, \sigma) \).

**Remark 2.1** (I) Putting \( p = 1, a = -1, b = a, \sigma = \beta, \chi_i = 1 \) \( i = 1, 2, \ldots, m + q - 1 \) in Theorem 2.1, we obtain the Aouf [3, Theorem 1] and Owa [2, Theorem 8].

(II) Putting \( p = 1, a = -1, b = a, \sigma = \beta, \chi_i = 1 \) \( i = 1, 2, \ldots, m - 1 \) in Corollary 2.1, we obtain the Aouf [3, Corollary 1] and Owa [2, Theorem 7].

(III) Putting \( p = 1, a = -1, b = a, \sigma = \beta, \chi_i = 1 \) \( i = 1, 2, \ldots, q - 1 \) in Corollary 2.2, we obtain the Aouf [3, Corollary 2] and Owa [2, Theorem 6].

(IV) Obviously, \( J_{p}^{*}(a, b, \sigma) \subset \mathcal{G}_{p}^{*}(a, b, \sigma) \) and \( C_{p}(a, b, \sigma) \subset \mathcal{M}_{p}(a, b, \sigma) \), so the corresponding results in Theorem 2.1, Corollaries 2.1, 2.2 with the classes \( J_{p}^{*}(a, b, \sigma) \) and \( C_{p}(a, b, \sigma) \) defined by Raina, Nahar [1] are all right.

**Theorem 2.2** The class \( \mathcal{G}_{p,c}^{*}(a, b, \sigma) \) is compact subset of \( \mathcal{A} \).

**Proof** Montel’s theorem implies that the \( \mathcal{G}_{p,c}(a, b, \sigma) \) contained in \( \mathcal{A} \) is compact if and only if \( \mathcal{G}_{p,c}(a, b, \sigma) \) is closed and locally uniformly bounded (see [4, p.39]). We first assume

\[
f_{p}(z) = a_{p}z^{p} - \sum_{n=1}^{\infty} a_{n+p}z^{n+p} \in \mathcal{G}_{p,c}^{*}(a, b, \sigma),
\]

then (1.6) gives that

\[
a_{n+p} \leq \frac{(b - a)\sigma a_{p}}{\left( \frac{n+p}{p} \right)^{c}(1 + b\sigma)n + (b - a)\sigma}, \quad n = 1, 2, \ldots.
\]
Since \( |z| = r < 1 \), it follows
\[
|f_p(z)| \leq a_p |z|^p + \sum_{n=1}^{\infty} a_{n+p} |z|^{n+p} \leq a_p r^p + \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c(1 + b\sigma)n + (b-a)p\sigma} r^{1+p}
\]
which implies that \( G_{p,c}^*(a, b, \sigma) \) is locally uniformly bounded.

It remains to show that \( G_{p,c}^*(a, b, \sigma) \) is sequentially closed. Suppose that a sequence \( \{f_p^{(k)}(z)\} \) in \( G_{p,c}^*(a, b, \sigma) \) and \( \{f_p^{(k)}(z)\} \to f_p(k \to \infty) \), where
\[
f_p^{(k)}(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}.
\]

Weierstrass' theorem asserts that \( f_p \in A \) (see \([4, \text{p.}38]\)), so we can take
\[
f_p = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p},
\]
moreover, \( a_{n+p} \to a_{n+p} (k \to \infty) \). We next need to consider the \( f_p \in G_{p,c}^*(a, b, \sigma) \). Since \( f_p^{(k)}(z) \in G_{p,c}^*(a, b, \sigma) \), (1.6) implies that
\[
\sum_{n=1}^{M} \frac{(n+p)^c[(1 + b\sigma)n + (b-a)p\sigma]}{(b-a)p\sigma a_p} a_{n+p}^{(k)} \leq \sum_{n=1}^{\infty} \frac{(n+p)^c[(1 + b\sigma)n + (b-a)p\sigma]}{(b-a)p\sigma a_p} a_{n+p} \leq 1
\]
for any \( M \in \mathbb{Z}^+ \). Thus, as \( k \to \infty \), we have
\[
\sum_{n=1}^{\infty} \frac{(n+p)^c[(1 + b\sigma)n + (b-a)p\sigma]}{(b-a)p\sigma a_p} a_{n+p} \leq 1.
\]

Furthermore, taking \( M \to +\infty \), it gives that
\[
\sum_{n=1}^{\infty} \frac{(n+p)^c[(1 + b\sigma)n + (b-a)p\sigma]}{(b-a)p\sigma a_p} a_{n+p} \leq 1.
\]

This completes the proof of Theorem 2.2.

**Theorem 2.3** The extreme points of the class \( G_{p,c}^*(a, b, \sigma) \) are given by

\[
EG_{p,c}^*(a, b, \sigma) = \left\{ a_p z^p, a_p z^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[1 + b\sigma] + (b-a)p\sigma} z^{1+p}, a_p z^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[2(1 + b\sigma) + (b-a)p\sigma]} z^{2+p}, \ldots, a_p z^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c(1 + b\sigma)n + (b-a)p\sigma} z^{n+p}, \ldots \right\},
\]

where \(-1 \leq a < b \leq 1, 0 < \sigma \leq 1, n \in \mathbb{N}^*\).

**Proof** Using similar arguments as given by Xiong et al. [13, Theorem 2.6], we can easily obtain the extreme points on \( G_{p,c}^*(a, b, \sigma) \).
Theorem 2.4 The support points of the class $G^*_{p,c}(a, b, \sigma)$ are given by

$$\text{Supp} G^*_{p,c}(a, b, \sigma) = \left\{ f_p(z) \in G^*_{p,c}(a, b, \sigma) : f_p(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(b-a)p\sigma}{(1+b\sigma)n+(b-a)p\sigma} \phi_{n+p} z^{n+p} \right\},$$

where $-1 \leq a < b \leq 1, 0 < \sigma \leq 1, \phi_{n+p} \geq 0, \sum_{n=1}^{\infty} \phi_{n+p} \leq 1, n \in \mathbb{N}$ and $\phi_{n+p} = 0$ for some $n \geq 1$.

Proof First, let a function

$$f_{p,0}(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(b-a)p\sigma}{(1+b\sigma)n+(b-a)p\sigma} \phi_{n+p} z^{n+p},$$

where $\sum_{n=1}^{\infty} \phi_{n+p} \leq 1, \phi_{n+p} \geq 0, \phi_i = 0$ for some $i \geq 1 + p$. In fact, (1.7) implies that $f_{p,0}(z) \in G^*_{p,c}(a, b, \sigma)$. Now, we need to take

$$b_{n+p} = \begin{cases} 0, & n \geq 1, n + p \neq i, \\ 1, & n \geq 1, n + p = i. \end{cases}$$

Obviously, we have $\lim_{n \to \infty} (b_{n+p})^{\frac{1}{n+p}} < 1$. Furthermore, we define a functional $J$ on $A$ by

$$J(f_p(z)) = \sum_{n=0}^{\infty} (-a_{n+p}) b_{n+p}, f_p(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \subset A, g_p(z) = b_p z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \subset A.$$

It is clearly that the $J$ is a continuous linear functional on $A$ (see [4, p.42]). Moreover, we note that $J(f_{p,0}(z)) = -a_p b_p - \sum_{n=1}^{\infty} \frac{(b-a)p\sigma}{(1+b\sigma)n+(b-a)p\sigma} \phi_{n+p} b_i = -a_p b_p - 0 = -a_p b_p$. However, for any function

$$f_p(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \in G^*_{p,c}(a, b, \sigma), \quad (2.9)$$

we can note that

$$J(f_p(z)) = -a_p b_p - a_i b_i \leq -a_p b_p (i \geq p + 1).$$

So we have

$$\text{Re} J(f_{p,0}) = \max \{ \text{Re} J(f_p(z)) : f_p(z) \in G^*_{p,c}(a, b, \sigma) \}$$

and $\text{Re} J(f_p(z))$ are not constant on $G^*_{p,c}(a, b, \sigma)$. Hence $f_{p,0}$ is a support point of $G^*_{p,c}(a, b, \sigma)$.

Conversely, suppose that $f_{p,0}(z)$ is a support point of $G^*_{p,c}(a, b, \sigma)$, and $J$ is a continuous linear functional on $A$. Note that $\text{Re} J$ is also a continuous linear and is non-constant on $G^*_{p,c}(a, b, \sigma)$, consequently, we have

$$\text{Re} J(f_{p,0}) = \max \{ \text{Re} J(f_p(z)) : f_p(z) \in G^*_{p,c}(a, b, \sigma) \}. $$
Let \( \mathcal{M} = \text{Re} J(f_{p,0}) \)

and

\[ G_J = \{ f_p(z) \in G_{p,c}^*(a, b, \sigma) : \text{Re} J(f_p(z)) = \mathcal{M} \}. \]

On the one hand, suppose that

\[ \text{Re} J(f_{p,1}) = \text{Re} J(f_{p,2}) = \mathcal{M}, \]

where \( f_{p,1} \in G_J, f_{p,2} \in G_J, 0 < t < 1. \) Then

\[ \text{Re} J(tf_{p,1} + (1-t)f_{p,2}) = t\text{Re} J(f_{p,1}) + (1-t)\text{Re} J(f_{p,2}) = t\mathcal{M} + (1-t)\mathcal{M} = \mathcal{M} \]

and so \( tf_{p,1} + (1-t)f_{p,2} \in G_J \), which gives the convexity of \( G_J \).

On the other hand, suppose that \( \text{Re} J(f_p^{(k)}(z)) = \mathcal{M} \) and \( f_p^{(k)}(z) \to f_p(z) \), where \( f_p^{(k)}(z) \in G_J \). Then \( \text{Re} J(f_p^{(k)}(z)) \to \text{Re} J(f_p(z)) \) and so \( \text{Re} J(f_p(z)) = \mathcal{M} \), which implies that the \( G_J \) is closed. Furthermore, Theorem 2.2 makes sure that the class \( G_J \subset G_{p,c}^*(a, b, \sigma) \) is locally uniformly bounded. Therefore, the class \( G_J \) is a convex compact subset of \( G_{p,c}^*(a, b, \sigma) \). Thus, \( E_G J \) is not empty (see [Lemma 1.3]). Now, suppose that \( g_{p,0}(z) \in E_G J \) and \( g_{p,0}(z) = tg_{p,1}(z) + (1-t)g_{p,2}(z) \), where \( 0 < t < 1, g_{p,1}(z) \in G_{p,c}^*(a, b, \sigma), g_{p,2}(z) \in G_{p,c}^*(a, b, \sigma) \). Then since

\[ \text{Re} J(g_{p,1}) \leq \mathcal{M}, \text{Re} J(g_{p,2}) \leq \mathcal{M}, t\text{Re} J(g_{p,1}) + (1-t)\text{Re} J(g_{p,2}) = \text{Re} J(g_{p,0}) = \mathcal{M}, \]

it follows that

\[ \text{Re} J(g_{p,1}) = \text{Re} J(g_{p,2}) = \mathcal{M}, \]

which implies \( g_{p,1} \in G_J, g_{p,2} \in G_J \). Again, because \( g_{p,0} \in E_G J \), so \( g_{p,1} = g_{p,2} = g_{p,0} \). Thus \( g_{p,0} \in E_G J \subset G_{p,c}(a, b, \sigma) \). Suppose

\[ E_G J \setminus \{ a_p z^p \} = \left\{ a_p z^p - \frac{(b-a)p\sigma a_p}{(n+p)^c[(1+b\sigma)n + (b-a)p\sigma]} z^{n+p} : n \in Z_1 \right\}, \]

where \( Z_1 \) is a subset of \( Z_0 = \{1, 2, \cdots \} \). We assert that \( Z_1 \) is a proper subset of \( Z_0 \). In fact, if it is not the case, then

\[ E_G J \setminus \{ a_p z^p \} = \left\{ a_p z^p - \frac{(b-a)p\sigma a_p}{(n+p)^c[(1+b\sigma)n + (b-a)p\sigma]} z^{n+p} : n \in Z_0 \right\}. \]

Since \( E_G J \subset G_J \), it follows that

\[ \text{Re} J(a_p z^p) - \frac{(b-a)p\sigma a_p}{(n+p)^c[(1+b\sigma)n + (b-a)p\sigma]} z^{n+p} = \mathcal{M} \quad (2.10) \]

for all \( n \in Z_0 \). Hence,

\[ \text{Re} J(a_p z^p) - \frac{(b-a)p\sigma a_p}{(n+p)^c[(1+b\sigma)n + (b-a)p\sigma]} \text{Re} J(z^{n+p}) = \mathcal{M} \quad (2.11) \]
for all $n \in Z_0$. Let $n \to +\infty$. Since $z^{n+p} \to 0$ in the metric of $A$ and $J$ is a continuous linear functional on $A$, it follows that $\text{Re}J(z^{n+p}) \to 0$. Thus, By (2.10) and (2.11) we have $\text{Re}J(a_p z^p) = M$ and we also find that $\text{Re}J(z^{n+p}) = 0$ for all $n \in Z_0$. Furthermore, for any $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \in G^*_p(a, b, \sigma)$, since $J$ is continuous on $A$ and $\text{Re}J(z^{n+p}) = 0$ for $n \in Z_0$, it follows that

$$\text{Re}J(f_p) = \text{Re}J(a_p z^p) - \sum_{n=1}^{\infty} a_{n+p} \text{Re}J(z^{n+p}) = \text{Re}J(a_p z^p) = M,$$

which contradicts the fact that $\text{Re}J$ is not constant on $G^*_p(a, b, \sigma)$. This shows that there is an integer $i(i \geq 1)$ not belonging to $Z_1$. In other words,

$$a_p z^p = \frac{(b-a)p\sigma a_p}{(1 + b\sigma)i + (b-a)ps} z^{i+p}$$

is not belonging to $E G_J$. Because $G_J$ is a convex compact set, so $G_J = HE G_J$ (see [Lemma 1.3]). Following Theorem 2.3, since $f_{p,0}(z) \in G_J$, it gives that

$$f_{p,0}(z) = \phi_1 a_p z^p + \sum_{n=1}^{\infty} \phi_{n+p} f_{n+p}(z), \quad (2.12)$$

where $\phi_1 \geq 0, \phi_{n+p} > 0$ and $\phi_1 + \sum_{n=1}^{\infty} \phi_{n+p} = 1, f_{n+p}(z) \in E G_J$.

Because

$$a_p z^p = \frac{(b-a)p\sigma a_p}{(1 + b\sigma)i + (b-a)ps} z^{i+p}$$

is not belonging to $E G_J$. So

$$f_{p,0}(z) = \phi_1 a_p z^p - \sum_{n=1, n \neq i}^{\infty} \phi_{n+p} \left[ a_p z^p - \frac{(b-a)p\sigma a_p}{(1 + b\sigma)n + (b-a)ps} z^{n+p} \right]$$

$$= a_p z^p - \sum_{n=1, n \neq i}^{\infty} \phi_{n+p} \frac{(b-a)p\sigma a_p}{(1 + b\sigma)n + (b-a)ps} z^{n+p}.$$
Corollary 2.4 The support points of the class $M_p(a, b, \sigma)$ are given by

$$\text{Supp} M_p(a, b, \sigma) = \left\{ f(z) \in M_p(a, b, \sigma) : f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(b-a)p^2 \sigma a_p}{(n+p)(1+b\sigma)n + (b-a)p\sigma} \phi_{n+p} z^{n+p} \right\},$$

where $-1 \leq a < b \leq 1, 0 < \sigma \leq 1, \phi_{n+p} \geq 0, \sum_{n=1}^{\infty} \phi_{n+p} \leq 1, n \in \mathbb{N}^*$ and $\phi_{n+p} = 0$ for some $n \geq 1$.

Remark 2.2 (I) Putting $p = 1, a = -1, b = \alpha, \sigma = \beta$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $S_c(\alpha, \beta)$ defined by Aouf [3].

(II) Putting $a_p \equiv 1, c = 0$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $J_p^*(a, b, \sigma)$ defined by Raina, Nahar [1].

(III) Putting $a_p \equiv 1, c = 1$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $C_p(a, b, \sigma)$ defined by Raina, Nahar [1].

(IV) Putting $p = 1, a = -1, b = \alpha, \sigma = \beta, c = 0$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $S_0(\alpha, \beta)$ defined by Owa [2].

(VI) Putting $p = 1, a = -1, b = \alpha, \sigma = \beta, c = 1$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $C_0(\alpha, \beta)$ defined by Owa [2].

References


多叶解析函数族子类的一些结果

熊良鹏 , 韩红伟 , 马致远
(成都理工大学工程技术学院, 四川 乐山 614007)

摘要: 本文研究了在单位开圆盘 $U = \{ z : |z| < 1 \}$ 内多叶解析的函数族 $G^{*}_{p,c}(a, b, \sigma)$ 的性质。利用函数论的方法, 获得了 $G^{*}_{p,c}(a, b, \sigma)$ 族相关的准哈达玛乘积的一般化结果及 $G^{*}_{p,c}(a, b, \sigma)$ 的极值点与支撑点。推广了文献相应的研究工作。

关键词: 解析函数; 多叶函数; 准哈达玛乘积; 极值点; 支撑点