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REMARKS ON RICCI SOLITONS IN TRANS-SASAKIAN MANIFOLDS

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Abstract: In this article we mainly study Ricci solitons in trans-Sasakian manifold of type (α, β). By the calculation of Ricci tensor, we obtain that 3-dimensional compact trans-Sasakian manifold equipping with Ricci solitons (g, ξ, λ) is homothetic to a Sasakian manifold and a trans-Sasadkian manifold admitting a gradient Ricci soliton is an Einstein manifold in case of α, β are constants.

Keywords: Ricci soliton; gradient Ricci soliton; trans-Sasakian manifold; Sasakian manifold; Einstein manifold

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1 Introduction

Let (M, ϕ, η, ξ) be a $(2n+1)$-dimensional almost contact manifold. Then the product $\overline{M} = M \times \mathbb{R}$ is a almost Hermitian manifold with almost complex structure J and product metric G being Hermitian metric. In [10], Gray and Harvella gave sixteen different structures of the almost Hermitian manifold $(\overline{M}, J, G)$. Using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$, the trans-Sasakian structure $(\phi, \eta, \xi, \alpha, \beta)$ on M, was defined (see [15]) that is the generalization of Sasakian and Kenmotsu structure on a contact metric manifold (see [1, 12]), where α, β are smooth functions on M. In general, we denote $(M, \phi, \eta, \xi, \alpha, \beta)$ by a trans-Sasakian manifold of type (α, β). Note that trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifolds respectively.

Recall that a Ricci soliton is the generalization of Einstein metric and defined on a Riemannian manifold (M, g) by

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.1}$$

where V is a smooth vector field , λ a constant on M. It is called gradient Ricci soliton if $V = \nabla f$ for some smooth function f on M. The Ricci soliton became important not only for studying topology of manifold but in study of string theory. Compact Ricci solitons are the fixed point of Ricci flow

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}$$

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projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. More details about Ricci soliton can refer to [2, 4].

Recently in [3], Calin and Carasmareanu started to study Ricci solitons in f-Koenmotsu manifolds. Later Nagaraja and Premalatha [13] also considered Ricci soliton (g, V, λ) in f-Koenmotsu manifolds and Ricci soliton in 3-dimensional trans-Sasakian manifolds when V is a conformal killing vector field, and gave the conditions for Ricci solitons to be shrinking, steady and expanding. Otherwise, De [9] studied Ricci solitons on normal almost contact metric manifolds.

Concerning the Ricci solitons in contact manifolds, Sharama [16] began to study the Ricci solitons in K-contact manifolds, where the contact structure ξ is a killing vector field, i.e., $\mathcal{L}_\xi g = 0$, which is not in general in a trans-Sasakian manifold. Recently, He and Zhu [11] proved that a Sasakian manifold satisfying the gradient Ricci soliton equation is necessarily Einstein. Also, Cho [5, 6] considered contact Ricci solitons and transversal Ricci solitons in 3-contact manifolds, and proved that a compact contact Ricci soliton is Sasakian-Einstein and a 3-contact manifold admitting a transversal Ricci soliton is either Sasakian or locally isometric to one of the following Lie group with a left invariant metric: $SU(2)$, $SL(2, \mathbb{R})$, $E(2)$, respectively.

Motivated by the above work, in this paper, we study the Ricci soliton in a 3-dimensional trans-Sasakian manifold $(M, \phi, \eta, \xi, \alpha, \beta)$ of type (α, β) in case of $V = \xi$ in Ricci soliton equation (1.1) and the gradient Ricci solitons in trans-Sasakian manifolds.

2 Preliminaries

An almost contact manifold (M, ϕ, ξ, η) is a $(2n+1)$-dimensional Riemannian manifold M equipped with an almost contact structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ a unit vector field, η a one-form dual to ξ satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta \circ \phi = 0,\ \phi \circ \xi = 0. \tag{2.1}$$

It is well-known that there exists a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ g(X, \xi) = \eta(X), \tag{2.3}$$

where $X, Y \in \mathfrak{X}(M)$. If there are two smooth functions α, β on (M, ϕ, ξ, η) such that

$$(\nabla_X \phi)Y = -\alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

then M is called a trans-Sasakian manifold of type (α, β), denote by $(M, \phi, \xi, \eta, \alpha, \beta)$, where ∇ is the Levi-Civita connection with respect to metric g. It is clear that a trans-Sasakian manifold of type $(1, 0)$ is a Sasakian manifold and a trans-Sasakian manifold of type $(0, 1)$ is a Kenmotsu manifold. A trans-Sasakian manifold of type $(0, 0)$ is called cosymplectic manifold.

Using (2.4), it follows that for any $X, Y \in \mathfrak{X}(M)$

$$\nabla_X \xi = -\alpha\phi(X) + \beta(X - \eta(X)\xi),\ (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.5}$$

Then it is easy to get the divergence $\mathrm{div}\xi = \mathrm{tr}(X \to \nabla_X \xi) = 2n\beta$ and $\nabla_\xi \xi = 0$.

Let Ric be the Ricci tensor on a Riemmaian manifold (M, g), then the Ricci operator $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by $Ric(X, Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$. It is well known that for any vector field $X, Y \in \mathfrak{X}(M)$, the following results were hold [7, Theorem 3.2, Proposition 3.4]:

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi(X) - \eta(X)\phi(Y)) \\ &\quad +(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.6}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.7}$$

$$\mathrm{Ric}(X,\xi) = (2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\phi X)\alpha. \tag{2.8}$$

Lemma 2.1 For any Riemannain manifold (M, g) and a local orthogonal frame $\{e_j\}$ on M, $j = 1, \cdots, \mathrm{dim}M$, the gradient of scalar curvature r satisfies

$$\frac{1}{2}\nabla r = \sum_j (\nabla Q)(e_j, e_j),$$

where $(\nabla Q)(X, Y) = \nabla_X Q(Y) - Q(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$.

Proof For any $X \in \mathfrak{X}(M)$,

$$\begin{aligned} X(r) &= \sum_j \nabla_X \mathrm{Ric}(e_j, e_j) = \sum_j \nabla_X g(Qe_j, e_j) \\ &= \sum_j \Big\{ g(\nabla_X(Qe_j), e_j) + g(Qe_j, \nabla_X e_j) \Big\} \\ &= \sum_j g((\nabla Q)(e_j, X), e_j) = 2\sum_j g((\nabla Q)(e_j, e_j), X). \end{aligned}$$

Note that the last equation is held because of the second Bianchi identity.

3 Ricci Solitons in 3-Dimensional Trans-Sasakian Mianifolds

In this section we consider Ricci soliton (g, ξ, λ) in 3-dimensional trans-Sasakian manifolds $(M, \phi, \xi, \eta, \alpha, \beta)$, i.e., there exists some constant λ satisfies

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_\xi g = \lambda g. \tag{3.1}$$

The next lemma play important role in proving our results.

Lemma 3.1 For any $(2n + 1)$-dimensional manifold with trans-Sasakian structure $(\phi, \xi, \eta, \alpha, \beta)$, we have

$$\frac{1}{2}\xi r = 2n\beta^2,$$

where r is the scalar curvature.

Proof In term of (3.1) for any vector field X,

$$Q(X) = \lambda X + \beta\phi^2 X. \tag{3.2}$$

We compute the differentiation of (3.2) with respect to any vector field Y,

$$\begin{aligned}(\nabla_Y Q)X &= \nabla_Y(Q(X)) - Q(\nabla_Y X)\\ &= \nabla_Y(\lambda X + \beta\phi^2 X) - \lambda\nabla_Y X - \beta\phi^2(\nabla_Y X)\\ &= Y(\beta)\phi^2 X - \alpha\beta g(X,\phi Y)\xi + \beta^2 g(\phi X,\phi Y)\xi\\ &\quad -\alpha\beta\eta(X)\phi(Y) - \beta^2\eta(X)\phi^2(Y). \end{aligned}\tag{3.3}$$

Since there is a canonical splitting of tangent bundle $\ker\eta \oplus \mathrm{span}\xi$ as the case of a contact structure, we can choose an orthogonal frame $\{e_1, \cdots, e_{2n+1}\}$ such that $e_{j+n} = \phi e_j$, $e_{2n+1} = \xi$, $j = 1, \cdots, n$. It reduces from Lemma 2.1 and (3.3) that

$$\begin{aligned}\frac{1}{2}\xi r &= \frac{1}{2}g(\nabla r,\xi) = \sum_{j=1}^{2n+1} g((\nabla Q)(e_j,e_j),\xi) = \sum_{j=1}^{2n+1} g((\nabla_{e_j} Q)e_j,\xi)\\ &= \beta^2\sum_{j=1}^{2n} g(\phi e_j,\phi e_j) = 2n\beta^2.\end{aligned}$$

For the 3-dimensional trans-Sasakian manifolds, the Ricci tensor Ric may express as follows (see [7]):

$$\begin{aligned}\mathrm{Ric}(X,Y) &= (\frac{1}{2}r + \xi\beta - (\alpha^2-\beta^2))g(X,Y)\\ &\quad -(\frac{1}{2}r + \xi\beta - 3(\alpha^2-\beta^2))\eta(X)\eta(Y)\\ &\quad -(Y\beta + \phi(Y)\alpha)\eta(X) - (X\beta + \phi(X)\alpha)\eta(Y),\end{aligned}\tag{3.4}$$

where r is the scalar curvature.

Thus

$$\mathrm{Ric}(\phi X,\phi Y) = (\frac{1}{2}r + \xi\beta - (\alpha^2-\beta^2))g(\phi X,\phi Y), \tag{3.5}$$

$$\mathrm{Ric}(\xi,\xi) = 2(\alpha^2-\beta^2) - 2\xi\beta. \tag{3.6}$$

By the first equation of (2.5), a straightforward calculation implies that

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X\xi, Y) + g(X,\nabla_Y\xi) = 2\beta g(\phi X,\phi Y).$$

Therefore

$$(\mathcal{L}_\xi g)(\phi X,\phi Y) = 2\beta g(\phi^2 X,\phi^2 Y) = 2\beta g(\phi X,\phi Y). \tag{3.7}$$

Applying (3.7), (3.5) in Ricci soliton equation (3.1), we have

$$\frac{1}{2}r + \xi\beta - (\alpha^2-\beta^2) + \beta = \lambda. \tag{3.8}$$

Obviously, since $\phi\xi = 0$,

$$(\mathcal{L}_\xi g)(\xi, \xi) = 0. \tag{3.9}$$

On the other hand, it implies from equations(3.6), (3.9) and Ricci soliton equation (3.1) that

$$-2\xi\beta + 2(\alpha^2 - \beta^2) = \lambda. \tag{3.10}$$

Then from (3.8) and (3.10) we obtain

$$r + 2\beta = 3\lambda. \tag{3.11}$$

Differentiating (3.11) w.r.t. ξ and together with Lemma 3.1 when $n = 1$, we get

$$\xi\beta = -2\beta^2. \tag{3.12}$$

It implies immediately from (3.12) and (3.11) that $\lambda = 2(\alpha^2 + \beta^2)$, then we have the following result.

Proposition 3.2 A Ricci soliton (g, λ, ξ) in a 3-dimensional trans-Sasakian manifold is shrinking.

Moreover, we get from equation (3.12) the following.

Theorem 3.3 If $(M, \phi, \xi, \eta, \alpha, \beta)$ is a 3-dimensional compact and connected trans-Sasakian manifold admitting Ricci soliton (g, λ, ξ), then M is homothetic to a Sasakian manifold .

Proof Using (3.12) and div$\xi = 2\beta$, we get $\beta = 0$ and α is a non-zero constant. It deduces that for any $X, Y \in \mathfrak{X}(M)$,

$$\alpha^{-2}(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi) = g(Y, \xi)X - g(X, Y)\xi,$$

and $(\mathcal{L}_\xi g)(X, Y) = 0$, i.e., ξ is a killing vector field. Thus it completes the proof of theorem by [14, Theorem 1.1]. The detail of proof can be seen in [8, Theorem 3.1].

Corollary 3.4 A 3-dimensional compact and connected trans-Sasakian manifold M of type(α, β) admitting Ricci soliton (g, λ, ξ) is an Einstein manifold.

Proof From the proof of Theorem 3.3, we know $\beta = 0$. Thus the scalar curvature $r = 3\lambda$ is constant via (3.11). Moreover, Sharama [16] proved that a compact Ricci soliton of constant scalar curvature is Einstein, then we obtain immediately the result.

4 Gradient Ricci Solitons in Trans-Sasakian Manifolds

In this section we consider gradient Ricci solitons in trans-Sasakian manifolds. We assume that $(M, \phi, \xi, \alpha, \beta)$ is a $(2n + 1)$-dimensional trans-Sasakian manifold.

First, we note that the following conclusion has been proved by taking Lie derivative of $\mathcal{L}_V g$ with respect to ξ.

Lemma 4.1 [11] For any manifold with a almost contact metric structure (ϕ, ξ, η, g),

$$\mathcal{L}_\xi(\mathcal{L}_V g)(Y, \xi) = R(V, \xi, \xi, Y) + g(\nabla_\xi \nabla_\xi V, Y) + \nabla_Y g(\nabla_\xi V, \xi)$$

for any vector field Y.

Using (2.7), equation (2.6) implies

$$\begin{aligned} R(X,\xi,\xi,Y) &= g(R(X,\xi)\xi,Y) = g\Big((\alpha^2-\beta^2)(X-\eta(X)\xi)+(\xi\beta)\phi^2X,Y\Big) \\ &= -(\alpha^2-\beta^2-\xi\beta)g(\phi X,\phi Y). \end{aligned} \tag{4.1}$$

When α,β =constant, it implies immediately from (2.8) that

$$\mathrm{Ric}(X,\xi)=2n(\alpha^2-\beta^2)\eta(X).$$

Then

$$\begin{aligned} (\mathcal{L}_\xi\mathrm{Ric})(Y,\xi) &= \nabla_\xi(\mathrm{Ric}(\xi,Y))-\mathrm{Ric}([\xi,Y],\xi) \\ &= \nabla_\xi(2n(\alpha^2-\beta^2)\eta(Y)))-\mathrm{Ric}(\nabla_\xi Y-\nabla_Y\xi,\xi) \\ &= 2n(\alpha^2-\beta^2)g(\nabla_\xi Y,\xi)-\mathrm{Ric}(\nabla_\xi Y,\xi) \\ &= 2n(\alpha^2-\beta^2)\eta(\nabla_\xi Y)-\mathrm{Ric}(\nabla_\xi Y,\xi)=0, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &(\mathcal{L}_\xi g)(Y,\xi)=g(\nabla_Y\xi,\xi)=0, \\ &R(V,\xi,\xi,Y)=-(\alpha^2-\beta^2)g(\phi V,\phi Y)=-(\alpha^2-\beta^2)g(V,Y). \end{aligned}$$

On the other hand,

$$\begin{aligned} 2(\lambda-2n(\alpha^2-\beta^2))g(X,\xi) &= 2(\lambda g(X,\xi)-\mathrm{Ric}(X,\xi))=(\mathcal{L}_Vg)(X,\xi) \\ &= g(\nabla_XV,\xi)+g(\nabla_\xi V,X). \end{aligned}$$

Replacing X by ξ in above equation, we get

$$\lambda-2n(\alpha^2-\beta^2)=g(\nabla_\xi V,\xi).$$

This implies $\nabla_Y g(\nabla_\xi V,\xi)=0$ since α,β,λ are constant. Therefore, from Lemma 4.1, taking the Lie derivative $\mathcal{L}_\xi$ to the Ricci soliton equation (1.1) yields

$$-(\alpha^2-\beta^2)g(V,Y)+g(\nabla_\xi\nabla_\xi V,Y)=0. \tag{4.3}$$

In case of where $V=\nabla f$ for some smooth function f, since for any $X\in\mathfrak{X}(M)$ Ricci soliton equation (1.1) yields $\nabla_X\nabla f+QX=\lambda X$,

$$\nabla_\xi\nabla_\xi\nabla f=\nabla_\xi(\lambda\xi-Q\xi)=-\nabla_\xi(2n(\alpha^2-\beta^2)\xi)=0.$$

Using (4.3), therefore we have

$$(\alpha^2+\beta^2)g(V,Y)=0. \tag{4.4}$$

Next we consider the following cases:

(i) If $\alpha=0$ then $\beta\neq0$ since $\alpha^2\neq\beta^2$. So we have $g(V,Y)=0$ via (4.4), i.e., $\nabla f=0$ for any $Y\perp\xi$. It follows that f =constant.

(ii) If $\alpha\neq0$ then $g(V,Y)=0$ by (4.4), i.e., f =constant.

Summarizing the above discussion, we obtain the following conclusions:

Theorem 4.2 Any trans-Sasakian manifold $(M,\phi,\xi,\eta,\alpha,\beta)$ admitting a gradient Ricci soliton is an Einstein manifold provided α and β are constants.

Remark 4.3 In fact, our result can be regarded as the generalization of [11, Theorem 1.1].

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关于带有Ricci 孤子的trans-Sasakian 流形的注记

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摘 要: 本文主要研究带有Ricci 孤子的(α, β)型trans-Sasakian流形, 证明了带有Ricci 孤子(g, ξ, λ)的3-维紧致trans-Sasakian流形是一个Sasakian流形. 此外, 如果α, β 是常数, 得到带有梯度Ricci 孤子的trans-Sasakian流形是Einstein流形.

关键词: Ricci 孤子; 梯度Ricci 孤子; trans-Sasakian 流形; Sasakian 流形; Einstein 流形

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