REMARKS ON RICCI SOLITONS IN TRANS-SASAKIAN MANIFOLDS

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Abstract: In this article we mainly study Ricci solitons in trans-Sasakian manifold of type $(\alpha, \beta)$. By the calculation of Ricci tensor, we obtain that 3-dimensional compact trans-Sasakian manifold equipping with Ricci solitons $(g, \xi, \lambda)$ is homothetic to a Sasakian manifold and a trans-Sasadkian manifold admitting a gradient Ricci soliton is an Einstein manifold in case of $\alpha, \beta$ are constants.

Keywords: Ricci soliton; gradient Ricci soliton; trans-Sasakian manifold; Sasakian manifold; Einstein manifold

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1 Introduction

Let $(M, \phi, \eta, \xi)$ be a $(2n + 1)$-dimensional almost contact manifold. Then the product $\overline{M} = M \times \mathbb{R}$ is a almost Hermitian manifold with almost complex structure $J$ and product metric $G$ being Hermitian metric. In [10], Gray and Harvella gave sixteen different structures of the almost Hermitian manifold $(\overline{M}, J, G)$. Using the structure in the class $W_4$ on $(\overline{M}, J, G)$, the trans-Sasakian structure $(\phi, \eta, \xi, \alpha, \beta)$ on $M$, was defined (see [15]) that is the generalization of Sasakian and Kenmotsu structure on a contact metric manifold (see [1, 12]), where $\alpha, \beta$ are smooth functions on $M$. In general, we denote $(M, \phi, \eta, \xi, \alpha, \beta)$ by a trans-Sasakian manifold of type $(\alpha, \beta)$. Note that trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds respectively.

Recall that a Ricci soliton is the generalization of Einstein metric and defined on a Riemannian manifold $(M, g)$ by

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = \lambda g,$$

where $V$ is a smooth vector field, $\lambda$ a constant on $M$. It is called gradient Ricci soliton if $V = \nabla f$ for some smooth function $f$ on $M$. The Ricci soliton became important not only for studying topology of manifold but in study of string theory. Compact Ricci solitons are the fixed point of Ricci flow

$$\frac{\partial}{\partial t} g = -2\text{Ric}$$

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projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively. More details about Ricci soliton can refer to [2, 4].

Recently in [3], Calin and Carasmareanu started to study Ricci solitons in $f$-Koenmotsu manifolds. Later Nagaraja and Premalatha [13] also considered Ricci soliton $(g, V, \lambda)$ in $f$-Koenmotsu manifolds and Ricci soliton in 3-dimensional trans-Sasakian manifolds when $V$ is a conformal killing vector field, and gave the conditions for Ricci solitons to be shrinking, steady and expanding. Otherwise, De [9] studied Ricci solitons on normal almost contact metric manifolds.

Concerning the Ricci solitons in contact manifolds, Sharama [16] began to study the Ricci solitons in $K$-contact manifolds, where the contact structure $\xi$ is a killing vector field, i.e., $\mathcal{L}_\xi g = 0$, which is not in general in a trans-Sasakian manifold. Recently, He and Zhu [11] proved that a Sasakian manifold satisfying the gradient Ricci soliton equation is necessarily Einstein. Also, Cho [5, 6] considered contact Ricci solitons and transversal Ricci solitons in 3-contact manifolds, and proved that a compact contact Ricci soliton is Sasakian-Einstein and a 3-contact manifold admitting a transversal Ricci soliton is either Sasakian or locally isometric to one of the following Lie group with a left invariant metric: $SU(2)$, $SL(2, \mathbb{R})$, $E(2)$, respectively.

Motivated by the above work, in this paper, we study the Ricci soliton in a 3-dimensional trans-Sasakian manifold $(M, \phi, \eta, \xi, \alpha, \beta)$ in case of $V = \xi$ in Ricci soliton equation (1.1) and the gradient Ricci solitons in trans-Sasakian manifolds.

2 Preliminaries

An almost contact manifold $(M, \phi, \xi, \eta)$ is a $(2n + 1)$-dimensional Riemannian manifold $M$ equipped with an almost contact structure $(\phi, \xi, \eta)$, where $\phi$ is a $(1, 1)$-tensor field, $\xi$ a unit vector field, $\eta$ a one-form dual to $\xi$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \eta \circ \phi = 0, \phi \circ \xi = 0.$$  \hspace{1cm} (2.1)

It is well-known that there exists a Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$  \hspace{1cm} (2.2)

$$g(\phi X, Y) = -g(X, \phi Y), g(X, \xi) = \eta(X),$$  \hspace{1cm} (2.3)

where $X, Y \in \mathfrak{X}(M)$. If there are two smooth functions $\alpha, \beta$ on $(M, \phi, \xi, \eta)$ such that

$$(\nabla_X \phi)Y = -\alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$  \hspace{1cm} (2.4)

then $M$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$, denote by $(M, \phi, \xi, \eta, \alpha, \beta)$, where $\nabla$ is the Levi-Civita connection with respect to metric $g$. It is clear that a trans-Sasakian manifold of type $(1, 0)$ is a Sasakian manifold and a trans-Sasakian manifold of type $(0, 1)$ is a Kenmotsu manifold. A trans-Sasakian manifold of type $(0, 0)$ is called cosymplectic manifold.
Using (2.4), it follows that for any $X, Y \in \mathfrak{X}(M)$

$$\nabla_X \xi = - \alpha \phi(X) + \beta (X - \eta(X) \xi), \quad (\nabla_X \eta)Y = - \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

Then it is easy to get the divergence $\text{div} \xi = \text{tr}(X \rightarrow \nabla_X \xi) = 2n\beta$ and $\nabla \xi \xi = 0$.

Let $\text{Ric}$ be the Ricci tensor on a Riemannian manifold $(M, g)$, then the Ricci operator $Q : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is defined by $\text{Ric}(X, Y) = g(QX, Y), X, Y \in \mathfrak{X}(M)$. It is well known that for any vector field $X, Y \in \mathfrak{X}(M)$, the following results were hold [7, Theorem 3.2, Proposition 3.4]:

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi(X) - \eta(X)\phi(Y)) + (Y \alpha)\phi X - (X \alpha)\phi Y + (Y \beta)\phi^2 X - (X \beta)\phi^2 Y,$$

$$2\alpha\beta + \xi \alpha = 0,$$

$$\text{Ric}(X, \xi) = (2n(\alpha^2 - \beta^2) - \xi \beta)\eta(X) - (2n - 1)X \beta - \phi X \alpha.$$  \hfill (2.6)

**Lemma 2.1** For any Riemannian manifold $(M, g)$ and a local orthogonal frame $\{e_j\}$ on $M$, $j = 1, \cdots, \dim M$, the gradient of scalar curvature $r$ satisfies

$$\frac{1}{2} \nabla r = \sum_j (\nabla Q)(e_j, e_j),$$

where $(\nabla Q)(X, Y) = \nabla_X Q(Y) - Q(\nabla_X Y), X, Y \in \mathfrak{X}(M)$.

**Proof** For any $X \in \mathfrak{X}(M)$,

$$X(r) = \sum_j \nabla_X \text{Ric}(e_j, e_j) = \sum_j \nabla_X g(Qe_j, e_j)$$

$$= \sum_j \left\{ g(\nabla_X (Qe_j), e_j) + g(Qe_j, \nabla_X e_j) \right\}$$

$$= \sum_j g((\nabla Q)(e_j, X), e_j) = 2 \sum_j g((\nabla Q)(e_j, e_j), X).$$

Note that the last equation is held because of the second Bianchi identity.

### 3 Ricci Solitons in 3-Dimensional Trans-Sasakian Manifolds

In this section we consider Ricci soliton $(g, \xi, \lambda)$ in 3-dimensional trans-Sasakian manifolds $(M, \phi, \xi, \eta, \alpha, \beta)$, i.e., there exists some constant $\lambda$ satisfies

$$\text{Ric} + \frac{1}{2} \mathcal{L}_\xi g = \lambda g.$$  \hfill (3.1)

The next lemma play important role in proving our results.

**Lemma 3.1** For any $(2n + 1)$-dimensional manifold with trans-Sasakian structure $(\phi, \xi, \eta, \alpha, \beta)$, we have

$$\frac{1}{2} \xi r = 2n\beta^2,$$

where $r$ is the scalar curvature.
Proof In term of (3.1) for any vector field $X$,

$$Q(X) = \lambda X + \beta \phi^2 X.$$  \hspace{1cm} (3.2)

We compute the differentiation of (3.2) with respect to any vector field $Y$,

$$(\nabla_Y Q)X = \nabla_Y (Q(X)) - Q(\nabla_Y X)
= \nabla_Y (\lambda X + \beta \phi^2 X) - \lambda \nabla_Y X - \beta \phi^2 (\nabla_Y X)
= Y(\beta)\phi^2 X - \alpha \beta g(X, \phi Y)\xi + \beta^2 g(\phi X, \phi Y)\xi
- \alpha \beta \eta(X)\phi(Y) - \beta^2 \eta(X)\phi^2(Y).$$ \hspace{1cm} (3.3)

Since there is a canonical splitting of tangent bundle $\text{ker} \eta \oplus \text{span}\xi$ as the case of a contact structure, we can choose an orthogonal frame \{e_1, \cdots, e_{2n+1}\} such that $e_{j+n} = \phi e_j$, $e_{2n+1} = \xi$, $j = 1, \cdots, n$. It reduces from Lemma 2.1 and (3.3) that

$$\frac{1}{2}\xi r = \frac{1}{2}g(\nabla r, \xi) = \sum_{j=1}^{2n+1} g((\nabla Q)(e_j, e_j), \xi) = \sum_{j=1}^{2n+1} g((\nabla e_j Q)e_j, \xi)
= \beta^2 \sum_{j=1}^{2n} g(\phi e_j, \phi e_j) = 2n\beta^2.$$  \hspace{1cm} (3.4)

For the 3-dimensional trans-Sasakian manifolds, the Ricci tensor $\text{Ric}$ may express as follows (see [7]):

$$\text{Ric}(X, Y) = \left(\frac{1}{2}r + \xi \beta - (\alpha^2 - \beta^2)\right)g(X, Y)
- \left(\frac{1}{2}r + \xi \beta - 3(\alpha^2 - \beta^2)\right)\eta(X)\eta(Y)
- (Y \beta + \phi(Y)\alpha)\eta(X) - (X \beta + \phi(X)\alpha)\eta(Y),$$ \hspace{1cm} (3.5)

where $r$ is the scalar curvature. Thus

$$\text{Ric}(\phi X, \phi Y) = \left(\frac{1}{2}r + \xi \beta - (\alpha^2 - \beta^2)\right)g(\phi X, \phi Y),$$ \hspace{1cm} (3.6)

By the first equation of (2.5), a straightforward calculation implies that

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2\beta g(\phi X, \phi Y).$$

Therefore

$$(\mathcal{L}_\xi g)(\phi X, \phi Y) = 2\beta g(\phi^2 X, \phi^2 Y) = 2\beta g(\phi X, \phi Y).$$ \hspace{1cm} (3.7)

Applying (3.7), (3.5) in Ricci soliton equation (3.1), we have

$$\frac{1}{2}r + \xi \beta - (\alpha^2 - \beta^2) + \beta = \lambda.$$ \hspace{1cm} (3.8)
Obviously, since $\phi \xi = 0$,

$$\mathcal{L}_\xi g(\xi, \xi) = 0. \quad (3.9)$$

On the other hand, it implies from equations (3.6), (3.9) and Ricci soliton equation (3.1) that

$$-2\xi \beta + 2(\alpha^2 - \beta^2) = \lambda. \quad (3.10)$$

Then from (3.8) and (3.10) we obtain

$$r + 2\beta = 3\lambda. \quad (3.11)$$

Differentiating (3.11) w.r.t. $\xi$ and together with Lemma 3.1 when $n = 1$, we get

$$\xi \beta = -2\beta^2. \quad (3.12)$$

It implies immediately from (3.12) and (3.11) that $\lambda = 2(\alpha^2 + \beta^2)$, then we have the following result.

**Proposition 3.2** A Ricci soliton $(g, \lambda, \xi)$ in a 3-dimensional trans-Sasakian manifold is shrinking.

Moreover, we get from equation (3.12) the following.

**Theorem 3.3** If $(M, \phi, \xi, \eta, \alpha, \beta)$ is a 3-dimensional compact and connected trans-Sasakian manifold admitting Ricci soliton $(g, \lambda, \xi)$, then $M$ is homothetic to a Sasakian manifold.

**Proof** Using (3.12) and $\text{div} \xi = 2\beta$, we get $\beta = 0$ and $\alpha$ is a non-zero constant. It deduces that for any $X, Y \in \mathfrak{X}(M)$,

$$\alpha^{-2}(\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi) = g(Y, \xi)X - g(X, Y)\xi,$$

and $\mathcal{L}_\xi g(X, Y) = 0$, i.e., $\xi$ is a killing vector field. Thus it completes the proof of theorem by [14, Theorem 1.1]. The detail of proof can be seen in [8, Theorem 3.1].

**Corollary 3.4** A 3-dimensional compact and connected trans-Sasakian manifold $M$ of type $(\alpha, \beta)$ admitting Ricci soliton $(g, \lambda, \xi)$ is an Einstein manifold.

**Proof** From the proof of Theorem 3.3, we know $\beta = 0$. Thus the scalar curvature $r = 3\lambda$ is constant via (3.11). Moreover, Sharama [16] proved that a compact Ricci soliton of constant scalar curvature is Einstein, then we obtain immediately the result.

### 4 Gradient Ricci Solitons in Trans-Sasakian Manifolds

In this section we consider gradient Ricci solitons in trans-Sasakian manifolds. We assume that $(M, \phi, \xi, \alpha, \beta)$ is a $(2n + 1)$-dimensional trans-Sasakian manifold.

First, we note that the following conclusion has been proved by taking Lie derivative of $\mathcal{L}_V g$ with respect to $\xi$.

**Lemma 4.1** [11] For any manifold with a almost contact metric structure $(\phi, \xi, \eta, g)$,

$$\mathcal{L}_\xi (\mathcal{L}_V g)(Y, \xi) = R(V, \xi, \xi, Y) + g(\nabla_\xi \nabla_\xi V, Y) + \nabla_Y g(\nabla_\xi V, \xi)$$

for any vector field $Y$. 
Using (2.7), equation (2.6) implies
\[ R(X, \xi, \xi, Y) = g(R(X, \xi)\xi, Y) = g\left((\alpha^2 - \beta^2)(X - \eta(X)\xi) + (\xi\beta)\phi^2 X, Y\right) \]
\[ = -(\alpha^2 - \beta^2 - \xi\beta)g(\phi X, \phi Y). \]  
(4.1)

When \(\alpha, \beta =\)constant, it implies immediately from (2.8) that
\[ \text{Ric}(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X). \]

Then
\[ (\mathcal{L}_\xi\text{Ric})(Y, \xi) = \nabla_\xi(\text{Ric}(\xi, Y)) - \text{Ric}(\xi, [Y, \xi], \xi) \]
\[ = \nabla_\xi(2n(\alpha^2 - \beta^2)\eta(Y))) - \text{Ric}(\nabla_\xi Y - \nabla_Y \xi, \xi) \]
\[ = 2n(\alpha^2 - \beta^2)g(\nabla_\xi Y, \xi) - \text{Ric}(\nabla_\xi Y, \xi) \]
\[ = 2n(\alpha^2 - \beta^2)\eta(\nabla_\xi Y) - \text{Ric}(\nabla_\xi Y, \xi) = 0, \]  
(4.2)

and
\[ (\mathcal{L}_\xi g)(Y, \xi) = g(\nabla_Y \xi, \xi) = 0, \]
\[ R(V, \xi, \xi, Y) = -(\alpha^2 - \beta^2)g(\phi V, \phi Y) = -(\alpha^2 - \beta^2)g(V, Y). \]

On the other hand,
\[ 2(\lambda - 2n(\alpha^2 - \beta^2))g(X, \xi) = 2(\lambda g(X, \xi) - \text{Ric}(X, \xi)) = (\mathcal{L}_V g)(X, \xi) \]
\[ = g(\nabla_X V, \xi) + g(\nabla_\xi V, X). \]

Replacing \(X\) by \(\xi\) in above equation, we get
\[ \lambda - 2n(\alpha^2 - \beta^2) = g(\nabla_\xi V, \xi). \]

This implies \(\nabla_Y g(\nabla_\xi V, \xi) = 0\) since \(\alpha, \beta, \lambda\) are constant. Therefore, from Lemma 4.1, taking the Lie derivative \(\mathcal{L}_\xi\) to the Ricci soliton equation (1.1) yields
\[ -(\alpha^2 - \beta^2)g(V, Y) + g(\nabla_\xi \nabla_\xi V, Y) = 0. \]  
(4.3)

In case of where \(V = \nabla f\) for some smooth function \(f\), since for any \(X \in \mathfrak{X}(M)\) Ricci soliton equation (1.1) yields \(\nabla_X \nabla f + QX = \lambda X, \)
\[ \nabla_\xi \nabla_\xi \nabla f = \nabla_\xi (\lambda \xi - Q\xi) = -\nabla_\xi (2n(\alpha^2 - \beta^2)\xi) = 0. \]

Using (4.3), therefore we have
\[ (\alpha^2 + \beta^2)g(V, Y) = 0. \]  
(4.4)

Next we consider the following cases:
(i) If \(\alpha = 0\) then \(\beta \neq 0\) since \(\alpha^2 \neq \beta^2\). So we have \(g(V, Y) = 0\) via (4.4), i.e., \(\nabla f = 0\) for any \(Y \perp \xi\). It follows that \(f =\)constant.
(ii) If \(\alpha \neq 0\) then \(g(V, Y) = 0\) by (4.4), i.e., \(f =\)constant.

Summarizing the above discussion, we obtain the following conclusions:

**Theorem 4.2** Any trans-Sasakian manifold \((M, \phi, \xi, \eta, \alpha, \beta)\) admitting a gradient Ricci soliton is an Einstein manifold provided \(\alpha\) and \(\beta\) are constants.

**Remark 4.3** In fact, our result can be regarded as the generalization of [11, Theorem 1.1].
References


关于带有Ricci 孤子的trans-Sasakian 流形的注记

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摘要: 本文主要研究带有Ricci 孤子的(α, β)型trans-Sasakian流形, 证明了带有Ricci 孤子(g, ξ, λ)的3维紧致trans-Sasakian流形是一个Sasakian流形。此外, 如果α, β是常数, 得到带有梯度Ricci 孤子的trans-Sasakian流形是Einstein流形。

关键词: Ricci 孤子; 梯度Ricci 孤子; trans-Sasakian 流形; Sasakian 流形; Einstein 流形