ON DUALLY FLAT AND CONFORMALLY FLAT 
(\(\alpha, \beta\))-METRICS

CHENG Xin-yue, ZHANG Ting, YUAN Min-gao
(School of Math. and Statistics, Chongqing University of Technology, Chongqing 400054, China)

Abstract: In this paper, from the relation between the sprays of two dually flat and conformally flat \((\alpha, \beta)\)-metrics, we obtain that locally dually flat and conformally flat Randers metrics are Minkowskian. Further, we extend the result to the non-Randers type and show that the locally dually flat and conformally flat \((\alpha, \beta)\)-metrics of non-Randers type must be Minkowskian under an extra condition.

Keywords: \((\alpha, \beta)\)-metric; dually flat Finsler metric; conformally flat Finsler metric, Minkowski metric

2010 MR Subject Classification: 53B40; 53C60

Document code: A Article ID: 0255-7797(2014)03-0417-06

1 Introduction

The notion of dually flat metrics was first introduced by Amari and Nagaoka when they studied the information geometry on Riemannian space [1, 2]. Later on, the notion of locally dually flat Finsler metrics was introduced by Shen [3]. A Finsler metric \(F = F(x, y)\) on an \(n\)-dimensional manifold \(M\) is called the locally dually flat Finsler metric if at every point there is a coordinate system \((x^i)\) in which the geodesic coefficients are in the following form

\[G^i = -\frac{1}{2}g^{ij}H_{ij},\]

where \(H = H(x, y)\) is a local scalar function on the tangent bundle \(TM\) of \(M\) and satisfies \(H(x, \lambda y) = \lambda^p H(x, y)\) for all \(\lambda > 0\). Such a coordinate system is called an adapted coordinate system. It is shown that a Finsler metric on an open subset \(U \subseteq \mathbb{R}^n\) is dually flat if and only if it satisfies the following PDE

\[(F^2)_{x^k y^l} y^k - 2(F^2)_{x^l} = 0.\]

In this case, \(H = -\frac{1}{4}(F^2)_{x^k} y^k\). Recently, Shen, Zhou and the second author studied locally dually flat Randers metrics \(F = \alpha + \beta\) and classified locally dually flat Randers metrics \(F = \alpha + \beta\) with isotropic \(S\)-curvature [4]. Later, Xia characterized locally dually flat \((\alpha, \beta)\)-metrics on an \(n\)-dimensional manifold \(M(n \geq 3)\) [5].

The study on conformal properties has a long history. Two Finsler metrics \(F\) and \(\tilde{F}\) on a manifold \(M\) are said to be conformally related if there is a scalar function \(\sigma(x)\) on \(M\) such that \(F = e^{\sigma(x)} \tilde{F}\). A Finsler metric which is conformally related to a Minkowski metric is called conformally flat Finsler metric. In 1989, Ichijyo and Hashiguchi defined a
conformally invariant Finsler connection in a Finsler space with \((\alpha, \beta)\)-metric and gave the condition for a Randers space to be conformally flat based on their connection (see [6]). Later, S. Kikuchi found a conformally invariant Finsler connection and gave a necessary and sufficient condition for a Finsler metric to be conformally flat by a system of partial differential equations under an extra condition (see [7]). By using Kikuchi’s conformally invariant Finsler connection, Hojo, Matsumoto and Okubo studied conformally Berwald Finsler spaces and its applications to \((\alpha, \beta)\)-metrics (see [8]). Recently, Kang proved that any conformally flat Randers metric of scalar flag curvature is projectively flat and classified completely conformally flat Randers metrics of scalar flag curvature (see [9]). On the other hand, Bacso and the second author studied the global conformal transformations on a Finsler space \((M, F)\). They obtain the relations between some important geometric quantities of \(F\) and their correspondences respectively, including Riemann curvatures, Ricci curvatures and S-curvatures (see [10, 11]). The Weyl theorem states that the projective and conformal properties of a Finsler metric determine the metric properties uniquely. Thus the conformal properties of a Finsler metric deserve extra attention.

In this paper, we study and classify locally dually flat and conformally flat \((\alpha, \beta)\)-metrics. Firstly, we can prove the following theorem.

**Theorem 1.1** Let \(F = \alpha + \beta\) be a locally dually flat Randers metric on an \(n\)-dimensional manifold \(M\) \((n \geq 3)\). Assume that \(F\) is conformally flat. Then it must be Minkowskian.

Further, following Xia’s main result on locally dually flat \((\alpha, \beta)\)-metrics in [5], we study and characterize locally dually flat and conformally flat \((\alpha, \beta)\)-metrics of non-Randers type. We get the following theorem.

**Theorem 1.2** Let \(F = \alpha \phi(s), s = \frac{\beta}{\alpha}\), be an \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold \(M\) \((n \geq 3)\). Suppose that \(\phi\) satisfies one of the following conditions:

(i) \(\phi(s)\) is a polynomial of \(s\) with \(\phi'(0) = 0\);

(ii) \(\phi(s)\) is an analytic function with \(\phi'(0) = \phi''(0) = 0\);

(iii) \(\phi'(0) \neq 0, s(k_2 - k_3 s^2)(\phi \phi' - s \phi'' - s\phi''') - (\phi'^2 + \phi\phi''') + k_1(\phi - s\phi') \neq 0\),

where \(k_1, k_2\) and \(k_3\) are constants. Then, if \(F\) is locally dually flat with \(\alpha\) conformally flat, \(F\) must be Minkowskian.

### 2 Preliminary

Let \(M\) be an \(n\)-dimensional \(C^\infty\) manifold and \(TM\) denotes the tangent bundle of \(M\). A Finsler metric on \(M\) is a function \(F : TM \to (0, \infty)\) with the following properties:

(a) \(F\) is \(C^\infty\) on \(TM\setminus\{0\}\);

(b) At any point \(x \in M, F_x(y) := F(x, y)\) is a Minkowski norm on \(T_xM\),

we call the pair \((M, F)\) an \(n\)-dimensional Finsler manifold.

Let \((M, F)\) be a Finsler manifold and \(g_{i\bar{j}}(x, y) := \frac{1}{2}[F^2(x, y)]_{\bar{i}y_j}\). For any non-zero vector \(y = y^i \frac{\partial}{\partial x^i} \in T_xM, F\) induces an inner product \(g_y\) on \(T_xM\) as \(g_y(u, v) := g_{i\bar{j}}(x, y)u^i\bar{v}_j\), where \(u = u^i \frac{\partial}{\partial x^i} \in T_xM, v = v^j \frac{\partial}{\partial x^j} \in T_xM\).
The geodesic $\sigma = \sigma(t)$ of a Finsler metric $F$ is characterized by the following system of 2nd order ordinary differential equations

$$\frac{d^2 \sigma^i(t)}{dt^2} + 2G^i(\sigma(t), \frac{d}{dt}\sigma(t)) = 0,$$

where $G^i := \frac{1}{2} g^{ij}(\{F^2\} - [F^2]_x y^k - [F^2]_y)$, where $(g^{ij}) = (g_{ij})^{-1}$. $G^i$ are called the geodesic coefficients of $F$.

By the definition, an $(\alpha, \beta)$-metric is a Finsler metric expressed in the following form

$$F = \alpha \phi(s), \quad s = \beta \alpha,$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_0$, $x \in M$. It is proved (see [12]) that $F = \alpha \phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi = \phi(s)$ is a $C^\infty$ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

In particular, when $\phi = 1 + s$, the metric $F = \alpha \phi(\beta/\alpha)$ is just the Randers metric $F = \alpha + \beta$. Let $G^i$ and $G^\alpha_i$ denote the geodesic coefficients of $F$ and $\alpha$, respectively. Denote

$$r_{ij} := (b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}),$$

$$s^i_j := a^i_{ij}, \quad s_i := b^i s_{ji}, \quad s_0 := s_i y^i, \quad r_{00} := r_{ij} y^i y^j,$$

where $(a^i_j) := (a_{ij})^{-1}$ and $b_{ij}$ denote the covariant derivative of $\beta$ with respect to $\alpha$. Then we have

**Lemma 2.1** (see [12]) The geodesic coefficients of $G^i$ are related to $G^\alpha_i$ by

$$G^i = G^\alpha_i + \alpha Q s^i_0 + \{ -2Q \alpha s_0 + r_{00} \} \{ \Psi b^i + \Theta \alpha^{-1} y^i \}, \quad (2.1)$$

where $s^i_0 := s^i_j y^j$ and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \quad \Psi := \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']^{\prime}}.$$

In order to prove our theorems, we need some lemmas about locally dually flat $(\alpha, \beta)$-metrics. Shen, Zhou and the second author first characterized locally dually flat Randers metrics and obtained the following lemma.

**Lemma 2.2** (see [4]) Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$. Then $F$ is locally dually flat if and only if in an adapted coordinate system, $\beta$ and $\alpha$ satisfy

$$r_{00} = \frac{2}{3} \theta_\beta - \frac{5}{3} \tau/\beta^2 + [\tau + \frac{2}{3}(\tau b^2 - b_m \theta_m)]\alpha^2, \quad (2.2)$$

$$s_{k0} = -\frac{1}{3}(\theta b_k - \beta \theta_k), \quad (2.3)$$

$$G^\alpha_m = \frac{1}{3}(2\theta + \tau \beta)y^m - \frac{1}{3}(\tau b^m - \theta^m)\alpha^2, \quad (2.4)$$
where \( \tau = \tau(x) \) is a scalar function and \( \theta = \theta_k y^k \) is a 1-form on \( M \) and \( \theta^m := a^{mk} \theta_k \).

Later, Xia characterized locally dually flat \((\alpha, \beta)\)-metrics.

**Lemma 2.3** (see [5]) Let \( F = \alpha \phi(\beta/\alpha) \) be an \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( M \) \((n \geq 3)\). Suppose \( F \) is not Riemannian and \( \phi \) satisfies one of the following:

(i) \( \phi(s) \) is a polynomial of \( s \) with \( \phi'(0) = 0; \)

(ii) \( \phi(s) \) is an analytic function with \( \phi'(0) = \phi''(0) = 0; \)

(iii) \( \phi'(0) \neq 0, \) \( s(k_2 - k_3 s^2)(\phi \phi'' - s \phi''') - (\phi'^2 + \phi''') + k_1 \phi(\phi - s \phi') \neq 0, \)

where \( k_1, k_2 \) and \( k_3 \) are constants. Then \( F \) is locally dually flat on \( M \) if and only if \( \alpha \) and \( \beta \) satisfy

\[
\begin{align*}
\sigma_{i0} &= \frac{1}{3} (\beta \theta_i - \theta b_i), \\
\sigma_{r0} &= \frac{2}{3} \theta \beta - (\theta b^l x^l) \alpha^2, \\
G_{\alpha}^l &= \frac{1}{3} (2 \theta y^l + \theta^l \alpha^2),
\end{align*}
\]

where \( \theta := \theta_i(x) y^i \) is a 1-form on \( M \) and \( \theta^l := a^{lk} \theta_k \).

### 3 Proof of Theorems

Now we are in the position to prove the theorems. First, we prove Theorem 1.1.

**Proof of Theorem 1.1** Let \( F = \alpha \phi(\beta/\alpha) \) and \( \tilde{F} = \tilde{\alpha} \phi(\tilde{\beta}/\tilde{\alpha}) \) be two \((\alpha, \beta)\)-metrics. If \( F \) and \( \tilde{F} \) are conformally related, that is \( F = e^{\sigma(x)} \tilde{F} \), then we have the following relations:

\[
\begin{align*}
\tilde{\alpha} &= e^{-\sigma(x)} \alpha, \\
\tilde{\beta} &= e^{-\sigma(x)} \beta, \tilde{a}_{ij} = e^{-2\sigma(x)} a_{ij}, \\
\tilde{b}_i &= e^{-\sigma(x)} b_i,
\end{align*}
\]

where \( \sigma := \frac{\partial \tau}{\partial x^i} \), \( \sigma^i := a^{ij} \sigma_j \), and “\( \parallel \)” denotes the covariant derivative with respect to \( \tilde{\alpha} \).

Let \( F = \alpha + \beta \) and \( \tilde{F} = \tilde{\alpha} + \tilde{\beta} \) be two Randers metrics and \( F = e^{\sigma(x)} \tilde{F} \). Then the above relations still hold. Assume \( F \) is conformally flat, then \( \tilde{F} \) is Minkowskian. In this case, \( \tilde{b}_{ij} = 0 \) and (3.1), (3.2), (3.3) are reduced to:

\[
\begin{align*}
b_{ij} &= b_i \sigma^r a_{ij} - b_j \sigma_i, \\
r_{ij} &= b_i \sigma^r a_{ij} - \frac{1}{2} \sigma_j b_j - \frac{1}{2} \sigma_i b_i, \\
s_{ij} &= \frac{1}{2} \sigma_j b_i - \frac{1}{2} \sigma_i b_j.
\end{align*}
\]

For any Finsler metric \( F \), the geodesic coefficients \( G^i \) can be expressed as:

\[
G^i = \frac{1}{4} g^{il} \{(F^2)_{x^i x^l} y^k - (F^2)_{x^i x^k}\}.
\]

\(420\) Journal of Mathematics Vol. 34
In particular, for \( \bar{\alpha} \) and \( \alpha \), by (3.7), their geodesic coefficients \( G^i_{\bar{\alpha}} \) and \( G^i_{\alpha} \) have the relation

\[
G^i_{\bar{\alpha}} = G^i_{\alpha} - \sigma_0 y^i + \frac{1}{2} \alpha^2 \sigma^i, \tag{3.8}
\]

where \( \sigma_0 := \sigma_k y^k \) and \( \sigma^i := a^i_l \sigma_l \).

If \( F \) is locally dually flat, then Lemma 2.2 holds for \( F \). Note that \( \alpha \) is also conformally flat since \( F \) is conformally flat, then \( \bar{\alpha} \) is Euclidean and \( G^i_{\bar{\alpha}} = 0 \). Combining (2.4) and (3.8) yields

\[
\left\{ \frac{1}{3} (2 \theta + \tau \beta) - \sigma_0 \right\} y^i = \left\{ \frac{1}{3} (\tau b^i - \theta^i) - \frac{1}{2} \sigma^i \right\} \alpha^2. \tag{3.13}
\]

For the dimension of manifold \( M \) satisfies \( n \geq 3 \) and \( \alpha^2 \) is not divisible in this circumstances, we immediately have \( \sigma^i = \frac{\tau}{2} (\tau b^i - \theta^i) \), \( \sigma_0 = \frac{1}{3} (2 \theta + \tau \beta) \). Comparing the above two equations, one easily has

\[
\theta_i = \frac{1}{4} \tau b_i. \tag{3.9}
\]

Combining (2.2), (3.5) and (3.9) we get

\[
\left( \frac{3}{2} \tau - \sigma_0 \right) \beta = (t + \tau + \frac{1}{2} \tau b^2) \alpha^2, \tag{3.10}
\]

where \( t := b_i \sigma^i \).

When \( n \geq 3 \), \( \alpha^2 \) is indivisible, then from (3.10) we have

\[
\sigma_i = \frac{3}{2} \tau b_i, \tag{3.11}
\]

\[
t + \tau + \frac{1}{2} \tau b^2 = 0. \tag{3.12}
\]

Plugging (3.11) into (3.12) yields \( \tau (1 + 2b^2) = 0 \). Considering that \( 1 + 2b^2 \neq 0 \), one has \( \tau = 0 \). Then \( \sigma_i = 0 \), i.e., \( \sigma \) is a constant. In this case, \( F \) is Minkowskian.

In the end, we are going to prove Theorem 1.2.

**Proof of Theorem 1.2** Assume that \( F = \alpha \phi (\beta / \alpha) \) is an \((\alpha, \beta)\) -metric satisfying the conditions in Theorem 1.2, \( \alpha = e^{\sigma(x)} \bar{\alpha} \) and \( \alpha \) is conformally flat. Then \( \bar{\alpha} \) is Euclidean and (2.5), (2.6), (2.7) in Lemma 2.3 hold. By (2.7) and (3.8) we have

\[
\left( \frac{2}{3} \theta - \sigma_0 \right) y^i = \left( - \frac{1}{2} \sigma^i - \frac{1}{3} \theta^i \right) \alpha^2. \tag{3.13}
\]

Then by (3.13) and the fact that \( \alpha^2 \) is indivisible when \( n \geq 3 \) again, naturally we get

\[
\theta_i = \frac{3}{2} \sigma_i, \tag{3.14}
\]

\[
\theta^i = - \frac{3}{2} \sigma^i. \tag{3.15}
\]

We use \( a_{ij} \) to lower the index of (3.15) and obtain

\[
\theta_i = - \frac{3}{2} \sigma_i. \tag{3.16}
\]

Comparing (3.14) with (3.16), instantly we conclude \( \sigma_i = 0 \) and \( \theta_i = 0 \). Then \( \sigma \) is a constant and obviously \( \alpha \) is Euclidean. According to (2.5) and (2.6), we get \( s_{ij} = 0 \) and \( r_{ij} = 0 \), which implies that \( \beta \) is parallel with respect to \( \alpha \). Therefore, \( F \) is Minkowskian.
References


对偶平坦和共形平坦的\((\alpha, \beta)\)-度量

程新跃, 张 岷, 袁敏高

(重庆理工大学数学与统计学院, 重庆 400054)

摘要: 本文主要研究了对偶平坦和共形平坦的\((\alpha, \beta)\)-度量, 利用对偶平坦和共形平坦与其测地线的关系, 得到了局部对偶平坦和共形平坦的Randers度量是Minkowskian度量的结论。进一步, 推广到非Randers型的情形, 我们证明了局部对偶平坦和共形平坦的非Randers型的\((\alpha, \beta)\)-度量在附加的条件下一定是Minkowskian度量。

关键词: \((\alpha, \beta)\)度量; 对偶平坦的Finsler度量; 共形平坦的Finsler度量; Minkowskian度量

MR(2010)主题分类号: 53B40; 53C60 中图分类号: O186.1