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WEAK SOLUTIONS FOR A FOURTH ORDER ELLIPTIC PROBLEM

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Abstract: In this work we shall study the existence of weak solutions for a fourth order elliptic problem. By virtue of mountain pass theorem and fountain theorem, combining with variational method, several existence theorems for weak solutions are obtained. The results obtained here improve some existing results in the literature.

Keywords: elliptic problem; mountain pass theorem; fountain theorem; weak solution

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1 Introduction

In this paper, we shall investigate the existence of weak solutions for the fourth order elliptic problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, the parameter c is less than the first eigenvalue of $(-\Delta, H_0^1)$ and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

It is well-known that fourth order problems were studied by many authors. For example, $u^{(4)}(t) = f(t, u(t))$ subject to boundary value conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ can be used to model the deflection of elastic beams simply supported at the endpoints [1–3]. In [4], Lazer and Mckenna pointed out that fourth order elliptic problems furnish a model to study traveling waves in suspension bridges. Since then, more general nonlinear fourth order elliptic boundary value problems were studied, we refer the interested reader to [5–13].

Meanwhile, as is known to all, fountain and dual fountain theorems by Bartsch and Willem [14, 15] are effective tools for studying the existence of infinitely many large energy solutions and small energy solutions, for instance, see [16–19] and the references therein.

In [5], Yang and Zhang consider the existence of positive, negative and sign-changing solutions for (1.1). They present their results on invariant sets of the gradient flows of the corresponding variational functionals.

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Chen and Tang [16] investigated the fractional boundary value problem of the following form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}u'(t)+\dfrac{1}{2}\,{}_tD_T^{-\beta}u'(t)\right)+\nabla F(t,u(t))=0 \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0. \end{cases}$$

In their paper, they adopted fountain and dual fountain theorems to obtain the existence of infinitely many solutions under some adequate conditions. It is no doubt that the results in the literature are significantly improved.

In this paper, we first adopt mountain pass theorem to obtain the existence of nontrivial solutions for (1.1) under some appropriate conditions imposed on f, i.e., sublinearity at 0 and superlinearity at ∞ with respect to u. Secondly, we only consider the nonlinearity f is asymptotically linear at infinity and we utilize fountain theorem to obtain the existence of infinitely many solutions for (1.1). Note that, this condition at infinity is indeed (f_1) of [5]. In [5], the authors assume that the nonlinearity f satisfies conditions $(f_1),(f_4),(f_6)$ and (f_7) (see [5, P130]) to obtain the existence of weak solutions for (1.1). However, in this paper, we only need conditions (f_1) and (f_7) to establish the existence of infinitely many solutions for (1.1). Therefore, our results here improve and extend the corresponding ones in [5].

2 Preliminaries and Main Results

Let $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. We will always assume the parameter c is less than the first eigenvalue of $(-\Delta,H_0^1)$. Denote by $0<\lambda_1<\lambda_2\le\cdots\le\lambda_j\le\dots$ the eigenvalues of $(-\Delta,H_0^1)$ and $\mu_k(c)=\lambda_k(\lambda_k-c)$ the eigenvalues of $(\Delta^2+c\Delta,H_0^1\cap H^2)$. We also denote by φ_j the eigenfunction associated with λ_j and consequently with μ_j, and $\varphi_1>0$ for $x\in\Omega$. We define a space $X:=H_0^1\cap H^2$. Clearly, X is a Hilbert space with the inner product

$$(u,v)=\int_\Omega(\Delta u\Delta v-c\nabla u\nabla v)\mathrm{d}x.$$

We denote by $\|u\|_2$ the norm in $L^2(\Omega)$ and $\|u\|$ the norm in X which is given by $\|u\|^2=(u,u)$. Furthermore, we have the Poincaré inequality $\|u\|^2\ge\mu_1\|u\|_2^2$.

We define a functional on X as follows

$$J(u)=\frac{1}{2}\int_\Omega(|\Delta u|^2-c|\nabla u|^2)\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x=\frac{1}{2}\|u\|^2-\int_\Omega F(x,u)\mathrm{d}x, \tag{2.1}$$

where $F(x,u)=\int_0^u f(x,t)\mathrm{d}t$. Clearly, the existence of weak solutions for (1.1) is equivalent to the existence of critical points of the functional J. By simple computation, we have

$$(J'(u),v)=\int_\Omega(\Delta u\Delta v-c\nabla u\nabla v)\mathrm{d}x-\int_\Omega f(x,u)v\mathrm{d}x=(u,v)-(Au,v),\ \forall u,v\in X, \tag{2.2}$$

where $(Au,v)=\int_\Omega f(x,u)v\mathrm{d}x,\ \forall u,v\in X$. Obviously, $J\in C^1(X,\mathbb{R})$ (see line 7 from below in [13, P. 798]).

As we have mentioned, we utilize the critical point theory to prove our main results. Let us recall two definitions about (PS) condition and $(\mathrm{PS})_c$ condition that will be used below. One can refer to [20–23] for more details.

Definition 2.1 Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$. We say J satisfies (PS) condition if for every sequence $\{u_n\} \subset X$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, there exists a subsequence of $\{u_n\}$ which is convergent in X.

Definition 2.2 Let X be a real Banach space, $J \in C^1(X,\mathbb{R})$ and $d \in \mathbb{R}$. We say J satisfies $(\mathrm{PS})_c$ condition if the existence of a sequence $\{u_n\} \subset X$ such that $J(u_n) \to d$ and $J'(u_n) \to 0$ as $n \to \infty$ lead to d is a critical value of J.

Lemma 2.1 (Mountain Pass Theorem) Let X be a Banach space and $J \in C^1(X,\mathbb{R})$ be a functional satisfying (PS) condition. If $e \in X$ and $0 < r < \|e\|$ are such that

$$a := \max\{J(0), J(e)\} < \inf_{\|u\|=r} J(u) =: b,$$

then $c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t))$ is a critical value of J with $c \geq b$, where Γ is the set of paths joining the points 0 and e, i.e., $\Gamma := \{\gamma \in C([0,1],X) : \gamma(0) = 0,\ \gamma(1) = e\}$.

For $j,k \in \mathbb{N}$, $X = \overline{\mathrm{span}}\{\varphi_j : j = 1,2\ldots\}$. Denote $X_j := \mathrm{span}\{\varphi_j\}$, $Y_k := \bigoplus_{j=1}^{k} X_j$ and $Z_k := \overline{\bigoplus_{j=k+1}^{\infty} X_j}$. Clearly, $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for all $j \in \mathbb{N}$. We also find $X = Y_k \bigoplus Z_k$.

Lemma 2.2 (see [14]) Let X be defined above. Suppose that

(A1) $J \in C^1(X,\mathbb{R})$ is an even functional.

If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(A2) $a_k := \max\limits_{u\in Y_k,\|u\|=\rho_k} J(u) \leq 0$;

(A3) $b_k := \inf\limits_{u\in Z_k,\|u\|=r_k} J(u) \to \infty$ as $k \to \infty$;

(A4) J satisfies $(\mathrm{PS})_c$ condition for all $c > 0$,

then J has an unbounded sequence of critical values.

Theorem 2.1 If f satisfies the following two conditions:

(H1) $f(x,0) = 0$;

(H2) $\lim\limits_{t\to0} \frac{f(x,t)}{t} = \xi$, $\lim\limits_{|t|\to\infty} \frac{f(x,t)}{t} = \eta$, uniformly a.e. in $x \in \Omega$, where

$$0 \leq \xi < \mu_1 = \lambda_1(\lambda_1 - c) < \eta < +\infty,$$

then (1.1) has at least a weak solution.

Lemma 2.3 Suppose (H1), (H2) hold, then J satisfies (PS) condition.

Proof Since Ω is bounded and (H2) holds, then if $\{u_k\}$ is bounded in X, by using the Sobolev embedding theorem and the standard procedures, we can get a subsequence converges strongly in X. So we need only to show that $\{u_k\}$ is bounded in X.

Assume that $\{u_k\} \subset X$ is a (PS) sequence, i.e.,

$$J(u_k) \to d, \quad J'(u_k) \to 0, \quad \text{as } k \to \infty. \tag{2.3}$$

From (H2) we know that

$$|f(x,t)t| \le C(1+|t|^2). \tag{2.4}$$

(2.3) implies that for all $\varphi \in X$,

$$(J'(u_k),\varphi) = \int_\Omega (\Delta u_k \Delta\varphi - c\nabla u_k \nabla\varphi)\mathrm{d}x - \int_\Omega f(x,u_k)\varphi \mathrm{d}x \to 0. \tag{2.5}$$

Setting $\varphi = u_k$ and using (2.4), we have

$$\|u_k\|^2 = \int_\Omega f(x,u_k)u_k \mathrm{d}x + (J'(u_k),u_k) \le C|\Omega| + C\|u_k\|_2^2 + o(1)\|u_k\|. \tag{2.6}$$

We claim that $\|u_k\|_2$ is bounded. Assume, by contradiction, that passing to a subsequence, $\|u_k\|_2^2 \to \infty$ as $k \to \infty$. We put $\omega_k := \frac{u_k}{\|u_k\|_2}$, then $\|\omega_k\|_2 = 1$. Moreover, from (2.6) we know

$$\|\omega_k\|^2 \le o(1) + C + \frac{o(1)}{\|u_k\|_2}\frac{\|u_k\|}{\|u_k\|_2} \le o(1) + C + o(1)\|\omega_k\|. \tag{2.7}$$

Hence, $\|\omega_k\|$ is bounded. Passing to a subsequence, we may assume that there exists $\omega \in X$ and $\|\omega\|_2 = 1$ such that

$$\omega_k \rightharpoonup w, \text{ weakly in } X, k\to\infty, \quad \omega_k \to w, \text{ strongly in } L^2(\Omega), k \to \infty.$$

From (2.5) we derive

$$\int_\Omega (\Delta\omega\Delta\varphi - c\nabla\omega\nabla\varphi)\mathrm{d}x - \int_\Omega \eta\omega\varphi \mathrm{d}x = 0, \ \forall \varphi \in X. \tag{2.8}$$

Then $\omega \in X$ is a weak solution of the equation

$$\Delta^2\omega + c\Delta\omega = \eta\omega.$$

Taking $\varphi(x) = \varphi_1(x)$, from (2.8) we have

$$\int_\Omega (\Delta\omega\Delta\varphi_1 - c\nabla\omega\nabla\varphi_1)\mathrm{d}x - \int_\Omega \eta\omega\varphi_1 \mathrm{d}x = 0. \tag{2.9}$$

On the other hand, since $\varphi_1(x) > 0$ is the eigenfunction of $\lambda_1(\lambda_1 - c)$, we have also

$$\int_\Omega (\Delta\omega\Delta\varphi_1 - c\nabla\omega\nabla\varphi_1)\mathrm{d}x - \int_\Omega \lambda_1(\lambda_1-c)\omega\varphi_1 \mathrm{d}x = 0. \tag{2.10}$$

Together (2.9) with (2.10), note that $\|\omega\|_2 = 1$, we know that $\lambda_1(\lambda_1 - c) = \eta$, which contradicts $\lambda_1(\lambda_1 - c) < \eta$. Hence $\|u_k\|_2$ is bounded. Then, from (2.6) we know that $\{u_k\}$ is bounded in X. This completes the proof.

Proof of Theorem 2.1 (H2) implies that, for any $\varepsilon > 0$, there exists $C_1 > 0$, such that

$$F(x,t) \ge \frac{1}{2}(\eta - \varepsilon)t^2 - C_1, \forall x \in \Omega, t \ne 0. \tag{2.11}$$

Taking $\varepsilon > 0$ such that $\eta - \varepsilon > \mu_1$, $\phi = \varphi_1$, from (2.11) we obtain

$$\begin{aligned} J(s\phi) &\le \frac{1}{2}\int_\Omega (|\Delta(s\phi)|^2 - c|\nabla(s\phi)|^2)\mathrm{d}x - \frac{1}{2}(\eta-\varepsilon)\int_\Omega s^2\phi^2\mathrm{d}x + C_1|\Omega| \\ &\le \frac{s^2}{2}\|\phi\|^2 - \frac{s^2}{2}(\eta-\varepsilon)\|\phi\|_2^2 + C_1|\Omega| \\ &\le \frac{s^2}{2}\left(1 - \frac{\eta-\varepsilon}{\mu_1}\right)\|\phi\|^2 + C_1|\Omega|. \end{aligned}$$

Therefore, by $1 - \frac{\eta-\varepsilon}{\mu_1} < 0$ implies

$$\lim_{s\to\infty} J(s\phi) \to -\infty. \tag{2.12}$$

From (H2), we can find α, such that $2 < \alpha < 2^*$, where $2^* = \begin{cases} \frac{2N}{N-2}, N > 2, \\ +\infty, N \le 2. \end{cases}$ (H1), (H2) imply that for all given $\varepsilon > 0$, there exists $C_0 > 0$, such that

$$F(x,t) \le \frac{1}{2}(\xi+\varepsilon)|t|^2 + C_0|t|^\alpha. \tag{2.13}$$

(2.13), the Poincaré inequality and the Sobolev embedding theorem enable us to obtain

$$J(u) \ge \frac{1}{2}\|u\|^2 - \frac{\xi+\varepsilon}{2}\int_\Omega |u|^2\mathrm{d}x - C_0\int_\Omega |u|^\alpha\mathrm{d}x \ge \left(\frac{1}{2} - \frac{\xi+\varepsilon}{2\mu_1}\right)\|u\|^2 - C_s\|u\|^\alpha, \tag{2.14}$$

where C_s is a constant. In (2.14), by taking $\varepsilon > 0$ such that $\xi + \varepsilon < \mu_1$, and choosing $\|u\| = \rho > 0$ small enough, we obtain $J(u) \ge R > 0$, if $\|u\| = \rho$.

From (2.12), we know that there exists $e \in X$, $\|e\| > \rho$, such that $J(e) < 0$. Define

$$\Gamma := \{\gamma : [0,1] \to X | \gamma \text{ is continuous and } \gamma(0) = 0, \gamma(1) = e\},$$

and $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$. From Lemma 2.1 it follows that

$$J(0) = 0, J(e) < 0, \text{ and } J(u)|_{\partial B_\rho} \ge R > 0.$$

Moreover, J satisfies (PS) condition by Lemma 2.3. By the mountain pass theorem, we know c is a critical value of J and there is at least one nontrivial critical point in X corresponding to this value. This completes the proof.

Theorem 2.2 If f satisfies the following two conditions:

(H3) $\mu_k < \liminf\limits_{|t|\to\infty} \frac{f(x,t)}{t} \le \limsup\limits_{|t|\to\infty} \frac{f(x,t)}{t} < \mu_{k+1}$, uniformly in Ω;

(H4) $f(t,-u) + f(t,u) = 0$,

then (1.1) has infinitely many weak solutions.

Lemma 2.4 (see [5, Lemma 4.1]) Let f satisfy (H3). Then J satisfies (PS) condition.

Proof Take $\mu_k < b_1 \le b_2 < \mu_{k+1}$ and $M > 0$ such that for $|u| \ge M$, $b_1 \le \frac{f(x,u)}{u} \le b_2$ by (H3). Now let $\{u_k\}$ be a (PS) sequence for $J(u)$. Writing $u_k = v_k + w_k$ with $v_k \in Y_k$ and $w_k \in Z_k$. Considering the inner product of $J'(u)$ and $v_k - w_k$, we find

$$\begin{aligned}
&o(1)\cdot\|u_k\| = (J'(u_k), v_k - w_k) = (u_k, v_k - w_k) - \int_\Omega f(x,u_k)(v_k - w_k)\mathrm{d}x \\
=&(v_k + w_k, v_k - w_k) - \int_{|u_k|\ge M} \frac{f(x,u_k)}{u_k}(v_k^2 - w_k^2)\mathrm{d}x - \int_{|u_k|<M} f(x,u_k)(v_k - w_k)\mathrm{d}x \\
&\le \|v_k\|^2 - \|w_k\|^2 - b_1\int_{|u_k|\ge M} v_k^2\mathrm{d}x + b_2\int_{|u_k|\ge M} w_k^2\mathrm{d}x - \int_{|u_k|<M} f(x,u_k)(v_k - w_k)\mathrm{d}x \\
=&\|v_k\|^2 - \|w_k\|^2 - b_1\int_\Omega v_k^2\mathrm{d}x + b_1\int_{|u_k|<M} v_k^2\mathrm{d}x + b_2\int_\Omega w_k^2\mathrm{d}x - b_2\int_{|u_k|<M} w_k^2\mathrm{d}x \\
&- \int_{|u_k|<M} f(x,u_k)(v_k - w_k)\mathrm{d}x \\
\le&\|v_k\|^2 - \|w_k\|^2 - \frac{b_1}{\mu_k}\|v_k\|^2 + \frac{b_2}{\mu_{k+1}}\|w_k\|^2 + b_1\int_{|u_k|<M} v_k^2\mathrm{d}x - b_2\int_{|u_k|<M} w_k^2\mathrm{d}x \\
&- \int_{|u_k|<M} f(x,u_k)(v_k - w_k)\mathrm{d}x.
\end{aligned} \tag{2.15}$$

By (2.15) and Hölder inequality, we obtain

$$\begin{aligned}
&o(1)\cdot\|u_k\| \le \left(1 - \frac{b_1}{\mu_k}\right)\|v_k\|^2 + \left(\frac{b_2}{\mu_{k+1}} - 1\right)\|w_k\|^2 + \frac{b_1 b_2}{b_2 - b_1}\int_{|u_k|<M} u_k^2\mathrm{d}x \\
&+ \left(\int_{|u_k|<M} |f(x,u_k)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{|u_k|<M} |v_k - w_k|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(1 - \frac{b_1}{\mu_k}\right)\|v_k\|^2 + \left(\frac{b_2}{\mu_{k+1}} - 1\right)\|w_k\|^2 + \frac{b_1 b_2}{b_2 - b_1}M^2|\Omega| + C\left(\int_{|u_k|<M} |u_k|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(1 - \frac{b_1}{\mu_k}\right)\|v_k\|^2 + \left(\frac{b_2}{\mu_{k+1}} - 1\right)\|w_k\|^2 + \frac{b_1 b_2}{b_2 - b_1}M^2|\Omega| + CM\sqrt{|\Omega|} \\
&\le -a\|u_k\|^2 + \frac{b_1 b_2}{b_2 - b_1}M^2|\Omega| + CM\sqrt{|\Omega|}.
\end{aligned}$$

So, $\{u_k\}$ is bounded, where $a = \min\left\{\frac{b_1}{\mu_k} - 1, 1 - \frac{b_2}{\mu_{k+1}}\right\} > 0$. A standard argument shows that $J(u)$ satisfies e (PS) condition. This completes the proof.

Proof of Theorem 2.2 (H4) and Lemma 2.4 enable us to obtain that (A1) and (A4) in Lemma 2.2 are satisfied.

By $\liminf\limits_{|t|\to\infty} \frac{f(x,t)}{t} > \mu_k$, there exist $M_1 > 0$ and $\varepsilon > 0$ such that $f(x,t) \ge (\mu_k + \varepsilon)t$, for all $|t| \ge M_1$ and $x \in \Omega$. We know $f(x,t) - (\mu_k + \varepsilon)t$ is continuous and bounded on $x \in \Omega$ and $|t| \le M_1$, and thus there exists $C > 0$ such that $-C \le f(x,t) - (\mu_k + \varepsilon)t \le C$. Therefore,

$$f(x,t) \ge (\mu_k + \varepsilon)t - C,\ \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.16}$$

By the definition of $F(x,u)$, we see

$$F(x,u)=\int_0^u f(x,t)\mathrm{d}t\geq\int_0^u[(\mu_k+\varepsilon)t-C]\mathrm{d}t\geq\frac{u^2}{2}(\mu_k+\varepsilon)-Cu,\ \forall(x,t)\in\Omega\times\mathbb{R}. \tag{2.17}$$

For any $u\in Y_k$, and it is easy to verify that $\|\cdot\|_2$ is a norm of Y_k. Since all the norms of a finite dimensional normed space are equivalent, so there exists positive constant C_1 such that $\|u\|_2\leq C_1\|u\|$. In view of (2.17), by Hölder inequality, we obtain

$$\begin{aligned}J(u)&=\frac{1}{2}\|u\|^2-\int_\Omega F(x,u)\mathrm{d}x\leq\frac{1}{2}\|u\|^2-\int_\Omega\frac{u^2}{2}(\mu_k+\varepsilon)\mathrm{d}x+C|\Omega|^{\frac{1}{2}}\|u\|_2\\&\leq\frac{1}{2}\|u\|^2-\frac{\mu_k+\varepsilon}{2\mu_k}\|u\|^2+CC_1|\Omega|^{\frac{1}{2}}\|u\|=\frac{1}{2}\|u\|^2\left(1-\frac{\mu_k+\varepsilon}{\mu_k}\right)+CC_1|\Omega|^{\frac{1}{2}}\|u\|.\end{aligned} \tag{2.18}$$

Since $1-\frac{\mu_k+\varepsilon}{\mu_k}<0$, then there exists positive constants d_k such that

$$J(u)\leq 0,\ \text{for each } u\in Y_k \text{ and } \|u\|\geq d_k. \tag{2.19}$$

On the other hand, by $\limsup\limits_{|t|\to\infty}\frac{f(x,t)}{t}<\mu_{k+1}$, there exist $M_2>0$ and $\varepsilon\in(0,\mu_{k+1})$ such that

$$f(x,t)\leq(\mu_{k+1}-\varepsilon)t,\ |t|\geq M_2$$

and $x\in\Omega$. For the reason that $f(x,t)-(\mu_{k+1}-\varepsilon)t$ is continuous and bounded on $|t|\leq M_2$ and $x\in\Omega$, then there is a $C>0$ such that $f(x,t)-(\mu_{k+1}-\varepsilon)t\leq C,\ |t|\leq M_2$ and $x\in\Omega$. Consequently,

$$f(x,t)\leq(\mu_{k+1}-\varepsilon)t+C,\ \forall(x,t)\in\Omega\times\mathbb{R}. \tag{2.20}$$

Therefore, we have

$$F(x,u)\leq\frac{u^2}{2}(\mu_{k+1}-\varepsilon)+Cu,\ \forall(x,t)\in\Omega\times\mathbb{R}. \tag{2.21}$$

For any $u\in Z_k$, let $\beta_k:=\sup\limits_{u\in Z_k,\|u\|=1}\|u\|_2$. Since X is compactly embedded into $L^2(\Omega)$, there holds (see [21, Lemma 3.8]), $\beta_k\to0,$ as $k\to\infty$. By (2.21) and Hölder inequality, we arrive at

$$\begin{aligned}J(u)=&\frac{1}{2}\|u\|^2-\int_\Omega F(x,u)\mathrm{d}x\geq\frac{1}{2}\|u\|^2-\int_\Omega\frac{u^2}{2}(\mu_{k+1}-\varepsilon)\mathrm{d}x-\int_\Omega Cu\mathrm{d}x\\ \geq&\frac{1}{2}\|u\|^2-\frac{\mu_{k+1}-\varepsilon}{2\mu_{k+1}}\|u\|^2-C|\Omega|^{\frac{1}{2}}\left(\int_\Omega|u|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ \geq&\frac{1}{2}\|u\|^2\left(1-\frac{\mu_{k+1}-\varepsilon}{\mu_{k+1}}\right)-C|\Omega|^{\frac{1}{2}}\beta_k\|u\|.\end{aligned} \tag{2.22}$$

Choosing $r_k:=1/\beta_k$, we easily $r_k\to\infty$ as $k\to\infty$, then

$$J(u)\geq\frac{1}{2}\left(1-\frac{\mu_{k+1}-\varepsilon}{\mu_{k+1}}\right)r_k^2-C|\Omega|^{\frac{1}{2}}\to\infty,\ \text{as } k\to\infty.$$

Hence, $b_k := \inf_{u\in Z_k,\|u\|=r_k} J(u) \to \infty$ as $k \to \infty$. Combining this and (2.19), we can take $\rho_k := \max\{d_k, r_k + 1\}$, and thus $a_k := \max_{u\in Y_k,\|u\|=\rho_k} J(u) \le 0$.

Up until now, we have proved the functional J satisfies all the conditions of Lemma 2.2, then J has an unbounded sequence of critical values. Equivalently, (1.1) has infinitely many weak solutions. This completes the proof.

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四阶椭圆方程的弱解

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摘要: 本文研究了一个四阶椭圆方程解的存在性问题. 利用山路定理和喷泉定理, 结合变分方法, 获得了该问题弱解的几个存在性定理，推广了现有的一些结果.

关键词: 椭圆方程; 山路定理; 喷泉定理; 弱解

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