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A NOTE ON CANTOR EXPANSIONS OF REAL NUMBERS

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Abstract: This paper is mainly concerned with Cantor expansions of real numbers. By construction of Moran sets, Hausdorff dimensions of certain sets related to the number of distinct digits in the expansion are determined. It is a complement to the statistical properties studied by Erdös and Renyi.

Keywords: Cantor expansion; digit; Hausdorff dimension

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1 Introduction

Let $\{q_n\}$ be a sequence of positive integers satisfying the condition that $q_n \geq 2$ ($n = 1, 2, \cdots$), then every real number $x \in [0, 1]$ can be represented as a Cantor series

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{q_1 q_2 \cdots q_n},$$

where $\varepsilon_n(x) \in \{0, 1, \cdots, q_n - 1\}$ is called the n-th digit in the Cantor expansion of x.

Note that in the special case of $q_n = q$ for a fixed integer $q \geq 2$, the Cantor series corresponds to the q-ary expansion of x.

In Galambos' book [2], there is a good account of the Cantor expansion and many other representations of real numbers by infinite series, from different viewpoints such as probability theory as well as metric number theory.

In [1], several interesting statistical properties of the digit sequence $\{\varepsilon_n(x)\}$ were investigated by Erdös and Renyi. For instance, let $d_n(x)$ denote the number of distinct numbers in the first n digits $\{\varepsilon_1(x), \varepsilon_2(x), \cdots, \varepsilon_n(x)\}$ of the Cantor series, it was proved that

$$\lim_{n\to\infty} \frac{d_n(x)}{n} = 1$$

for almost all x (with respect to the Lebesgue measure), under the condition that

$$\sum_{n=1}^{\infty} \frac{1}{q_n} < \infty.$$

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In this note, we are concerned with the following set

$$E_\delta = \{x \in [0,1] : \lim_{n\to\infty} \frac{d_n(x)}{n^\delta} = c\},$$

where $0 < \delta < 1$ and $c > 0$. By the theorem of Erdös and Renyi, the set E_δ has Lebesgue measure zero. We shall prove the following

Theorem 1 The set E_δ has Hausdorff dimension δ provided that

$$\lim_{n\to\infty} q_n = \infty \text{ and } \lim_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} = 1.$$

Let us remark that if $\{q_n\}$ increases rapidly, then the set E_δ always has Hausdorff dimension 1.

2 Proof of the Theorem

In order to avoid complicated calculation, we only present a proof for the special case that $q_n = n+1$ and $c = 1$.

Let us begin with some notations. For each $n \geq 1$, define

$$I(a_1, a_2, \cdots, a_n) = \{x \in [0,1] : \varepsilon_1(x) = a_1, \varepsilon_2(x) = a_2, \cdots, \varepsilon_n(x) = a_n\},$$

which is an interval of length $|I(a_1, a_2, \cdots, a_n)| = \frac{1}{(n+1)!}$. We call it a a rank-$n$ basic interval.

First we bound the Hausdorff dimension $\dim_H(E_\delta)$ from above. For each $n \geq 2$, the set E_δ can be covered by nearly

$$N(E_\delta, n) := \binom{n}{[n^\delta]} [n^\delta]! [n^\delta]^{n-[n^\delta]}$$

rank-n basic intervals. It follows from Stirling's formula that

$$\dim_H(E_\delta) \leq \lim_{n\to\infty} \frac{\log N(E_\delta, n)}{\log(n+1)!} = \delta.$$

Actually, the right hand side is an upper bound for the box-counting dimension of the set $\dim_H(E_\delta)$. For more details about the Hausdorf and box-counting dimension, we refer to Falconer's book [3].

To bound the Hausdorff dimension $\dim_H(E_\delta)$ from below, we construct a subset of E_δ as follows. Let F_δ be the set of $x \in [0,1]$ subject to the restriction that

$$\varepsilon_n(x) \in \{1, 2, \cdots, [n^\delta]\}, \quad n = 1, 2, \cdots.$$

The set F_δ is a homogeneous Moran set (see [4]) which has Hausdorff dimension

$$\dim_H(F_\delta) = \lim_{n\to\infty} \frac{\log\left(\prod_{k=1}^n [k^\delta]\right)}{\log(n+1)!} = \delta.$$

It is easy to see that $F_\delta \subset E_\delta$. The proof is finished now.

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关于实数Cantor级数展开的一个注记

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摘 要: 本文主要研究实数的Cantor级数展开式. 通过构造Moran集的方法, 确定了由Cantor级数中不同字符个数的渐近值所定义的一类集合的Hausdorff维数. 本文结果可视为Erdös 和Renyi关于Cantor级数统计性质研究的补充.

关键词: Cantor 展开; 字符; Hausdorff 维数

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