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THE PROBABILITY OF RUIN WITH FINITE HORIZON IN A DISCRETE-TIME MULTI-RISK MODEL

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Abstract: This paper investigates the probability of ruin for the discrete-time multi-risk model. By using the classical method of large deviation, we obtain the ruin probability within finite horizon, which extends the corresponding results of the discrete-time unit-risk model.

Keywords: ruin probability; large deviation; discrete-time multi-risk model

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1 Introduction

Recently, a lot of papers were published on the issue of the ruin probability, we refer the reader to [1–6] for more details. We say that a random variable ξ (or its distribution function F) is heavy-tailed if it has no finite exponential moments. In ruin probability theory, heavy-tailed distributions are often used to model large claims. They play a key role in some fields such as insurance, financial mathematics and queueing theory. In this paper, we use an important subclass of heavy-tailed distribution, which is called $\mathcal{C}$. A distribution function F belongs to $\mathcal{C}$ if

$$\lim_{y\downarrow 1}\liminf_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=1 \quad \text{or, equivalently,} \quad \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=1.$$

Set

$$\gamma(y):=\liminf_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)} \text{ and } \gamma_F:=\inf\Big\{-\frac{\log\gamma(y)}{\log y}:y>1\Big\}.$$

In [7], γ_F is called the upper Matuszewska index of the nonnegative and nondecreasing function $f(x)=(\bar{F}(x))^{-1}$, $x>0$. Without any danger of confusion, we simply call γ_F the

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upper Matuszewska index of the distribution F. See Chapter 2.1 of [7] and [8] for more details of the Matuszewska index.

The ruin probability in a discrete-time unit-risk model was investigated in [9–11] among other. In these papers, they always assumed that the company provides only one kind of insurance contract. In reality this assumption is not correct, for example, an unexpected claim even might induce more than one type of claim in an umbrella insurance policy. A typical example is motor insurance where an accident could cause claims for vehicle damages and the third party injuries. So the ruin probability of multi-risk models is more valuable. In present paper, we consider a discrete-time multi-risk model with a random interest rate and assume that the insurance company has k classes of insurance contract. Other basic assumptions of this model are as follows:

A1 For $s = 1, 2, \cdots, k$, the sth type related net incomes $\xi_{si}, i = 1, 2, \cdots, n$, constitute a sequence of dependent random variables with common distribution F_s concentrated on $(-\infty, +\infty)$, where the net incomes ξ_{si}, called the insurance risk, is understood as the total incoming premium minus the total claim amount within year n. And $\{\xi_{si}, i = 1, 2, \cdots, n\}$ satisfy the assumption condition: $\exists\ x_0 > 0$ and $c > 0$ such that

$$P(|\xi_{si}| > x_i|\xi_{sj} = x_j \text{ with } j \in J) \le cP(\xi_{si} > x_i)$$

for all $1 \le i \le n$, $\emptyset \ne J \subset \{1, 2, \cdots, n\}\backslash\{i\}$, $x_i > x_0$, and $x_j > x_0$ with $j \in J$. Remark that this condition is assumption B of [1], we denote this assumption condition by assumption A which will be used in Proposition 2.1.

A2 For $j = 1, 2, \cdots, n$, η_j, called the financial risk, is the discount factor from year j to year $j-1$. They constitute a sequence of independent identical distributed random variables, satisfy that there exist some constants $a < a \le b < \infty$ such that $P(a \le \eta_i \le b) = 1$ for all $1 \le i \le n$.

A3 $\{\xi_{si}, i = 1, 2, \cdots, n\}_{s=1}^{k}$ and $\{\eta_j, j = 1, 2, \cdots, n\}$ are mutually independent.

Let the initial capital of the insurance company be $x \ge 0$. We tacitly assume that the sth type related net incomes ξ_{si} is calculated at the beginning of year i, and the discounted value of the surplus of the company accumulated till the beginning of year m can be characterized by S_{sm}, $m = 1, 2, \cdots$, which satisfy the recurrence equation below:

$$S_{s0} = x, \quad S_{sm} = x - \sum_{i=1}^{m} \xi_{si} \prod_{j=1}^{i-1} Y_j.$$

Hence the ruin probability in the considered multi-risk model with initial capital $x \ge 0$ is defined by $\varphi^k(x, n) = P\Big(\max\limits_{1\le m\le n} \sum\limits_{s=1}^{k} \sum\limits_{i=1}^{m} \xi_{si} \prod_{j=1}^{i-1} \eta_j\Big)$.

2 Main Result

In this section, several propositions used in Section 3 are provided, and our main result is presented.

Proposition 2.1 Suppose that $\{\xi_i, 1 \leq i \leq n\}$ are n real-valued random variables with $F_i \in \mathcal{C}$ for all $1 \leq i \leq n$, and $F_i * F_j \in \mathcal{C}$ for all $1 \leq i \neq j \leq n$, and satisfy that $\exists\ x_0 > 0$ and $c > 0$ such that

$$P(|\xi_i| > x_i | \xi_j = x_j \ \text{ with } \ j \in J) \leq cP(\xi_i > x_i)$$

for all $1 \leq i \leq n$, $\emptyset \neq J \subset \{1, 2, \cdots, n\} \backslash \{i\}$, $x_i > x_0$, and $x_j > x_0$ with $j \in J$. If there exist some constants $0 < a \leq b < \infty$ such that $P(a \leq \eta_i \leq b) = 1$ for all $1 \leq i \leq n$, then the following relation

$$P\Big(\max_{1 \leq m \leq n} \sum_{k=1}^{m} \xi_k \prod_{j=1}^{k} \eta_j > x\Big) \sim P\Big(\sum_{k=1}^{n} \xi_k \prod_{j=1}^{k} \eta_j > x\Big) \sim \sum_{k=1}^{n} P\Big(\xi_k \prod_{j=1}^{k} \eta_j > x\Big),$$

as $x \to \infty$. Here $F_i * F_j \in \mathcal{C}$ denotes the convolution of distribution F_i and F_j.

This proposition is the conclusion of Theorem 2 of [1] when distribution function belongs to $\mathcal{C}$, and plays a key role in the proof of our result. Repeated using Corollary 2.5 of [8], we can get the following proposition.

Proposition 2.2 If $\xi \in \mathcal{C}$, there exist some constants $0 < a \leq b < \infty$ such that $P(a \leq \eta_i \leq b) = 1$ for all $1 \leq i \leq k$, then $\xi \prod_{i=1}^{k} \eta_k \in \mathcal{C}$.

From Theorem 3.3 of [8], we can obtain Proposition 2.3, that is

Proposition 2.3 If $\xi \in \mathcal{C}$, $E\eta^p < \infty$ for some $p > \gamma_F$, and ξ and η are mutually independent, then $P(\xi\eta > x) \asymp P(\xi > x)$.

The following exhibits one such situation in a discrete-time multi-risk model and is the main result of this paper.

Theorem 2.1 Suppose that assumptions A1–A3 hold, and $F_s \in \mathcal{C}$ and $\gamma_{F_s} > 0$ for $1 \leq s \leq k$. If there exists some positive constant $p > \max\limits_{1 \leq s \leq k} \gamma_{F_s}$ such that $E\eta_i^p < \infty$ for all $1 \leq i \leq n$, then

$$\varphi^k(x, n) \sim \sum_{s=1}^{k} \sum_{i=1}^{n} P\Big(\xi_{si} \prod_{j=1}^{i-1} \eta_j > x\Big) \quad \text{as} \quad x \to \infty \tag{2.1}$$

holds for any $n \geq 1$.

3 Proof

For notational convenience, throughout this section, all limit relations are for $x \to \infty$ unless stated otherwise. In order to prove Theorem 2.1, we need to divide the proof into the following two theorems.

Theorem 3.1 Under the conditions of Theorem 2.1, we have

$$\varphi^k(x, n) \lesssim \sum_{s=1}^{k} \sum_{i=1}^{n} P\Big(\xi_{si} \prod_{j=1}^{i-1} \eta_j > x\Big). \tag{3.1}$$

Proof We give the proof of Theorem 3.1 by induction approach. It is trivial that relation (3.1) holds for $k = 1$ from Proposition 2.1. When $k = 2$, for any fixed $0 < \varepsilon < \frac{1}{2}$, we have

$$
\begin{aligned}
\varphi^2(x,n) &= P\Big(\max_{1\le m\le n}\sum_{s=1}^{2}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big)\\
&\le P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j + \max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > x\Big)\\
&\le P\Big(\Big\{\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big\}\bigcup\Big\{\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big\}\\
&\qquad \bigcup\Big\{\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x, \max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x\Big\}\Big)\\
&\le P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big) + P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big)\\
&\quad + P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x, \max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x\Big)\\
&= P_1 + P_2 - P_3. \qquad (3.2)
\end{aligned}
$$

First, we deal with P_1. By Proposition 2.1 and Proposition 2.2, for any $0<\delta<1$, we have

$$
P_1 \le (1+\delta)\sum_{i=1}^{n}P\Big(\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big) \le (1+\delta)^2\sum_{i=1}^{n}P\Big(\xi_{1i}\prod_{j=1}^{i-1}\eta_j > x\Big). \qquad (3.3)
$$

Similarly to the proof of relation (3.3), we have

$$
P_2 \le (1+\delta)^2\sum_{i=1}^{n}P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > x\Big). \qquad (3.4)
$$

Now we deal with P_3. For any $0<\delta<1$ and any fixed $0<\varepsilon<\frac{1}{2}$,

$$
\begin{aligned}
&P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x \mid \eta_j, 1\le j\le n\Big)\\
\le\ & P\Big(\sum_{i=1}^{n}\xi_{1i}^{+}\prod_{j=1}^{i-1}\eta_j > \varepsilon x \mid \eta_j, 1\le j\le n\Big) \le P\Big(\sum_{i=1}^{n}\xi_{1i}^{+}b^{i-1} > \varepsilon x\Big)\\
\le\ & \sum_{i=1}^{n}P\Big(\xi_{1i}^{+}b^{i-1} > \frac{\varepsilon x}{n}\Big) = \sum_{i=1}^{n}P\Big(\xi_{1i}b^{i-1} > \frac{\varepsilon x}{n}\Big) \le n\delta, \qquad (3.5)
\end{aligned}
$$

where the last step is obtained by $\lim\limits_{x\to\infty}P\Big(\xi_{1i}b^{i-1} > \frac{\varepsilon x}{n}\Big) = 0$ for $1\le i\le n$. Since $\xi_{2i}\prod\limits_{j=1}^{i-1}\in\mathcal{C}$ for $1\le i\le n$, there exists some $M_1>0$, for any fixed $0<\varepsilon<\frac{1}{2}$ such that

$$
P\Big(\xi_{2i}\prod_{j=1}^{i-1} > \varepsilon x\Big) \le M_1 P\Big(\xi_{2i}\prod_{j=1}^{i-1} > x\Big). \qquad (3.6)
$$

So, from relation (3.5), Proposition 2.1 and relation (3.6), we get

$$
\begin{aligned}
& \frac{P_3}{\sum_{s=1}^{2}\sum_{i=1}^{n} P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big)} \\
= \ & \frac{E\Big[\prod_{s=1}^{2} P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > \varepsilon x \mid \eta_j, 1\le j\le n\Big)\Big]}{\sum_{s=1}^{2}\sum_{i=1}^{n} P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big)} \\
= \ & \frac{E\Big\{P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x \mid \eta_j, 1\le j\le n\Big)E\Big[P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x \mid \eta_j, 1\le j\le n\Big)\Big]\Big\}}{\sum_{s=1}^{2}\sum_{i=1}^{n} P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big)} \\
\le \ & n\delta\cdot\frac{P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x\Big)}{\sum_{i=1}^{n} P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > x\Big)} \\
\le \ & n\delta(1+\delta)M_1. \qquad (3.7)
\end{aligned}
$$

Combining relation (3.3), (3.4) and (3.7) to relation (3.2), by the arbitrariness of $0<\delta<1$, relation (3.1) holds for $k=2$.

Next suppose that relation (3.1) holds for $k-1$, we want to prove it right for k. By the similar method from relation (3.2) to (3.7), for any fixed $0<\varepsilon<\frac{1}{2}$ and any $0<\delta<1$,

$$
\begin{aligned}
\varphi^k(x,n) &= P\Big(\max_{1\le m\le n}\sum_{s=1}^{k}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big) \\
&\le P\Big(\Big\{\max_{1\le m\le n}\sum_{s=1}^{k-1}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big\}\bigcup\Big\{\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{ki}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big\} \\
&\qquad \bigcup\Big\{\max_{1\le m\le n}\sum_{s=1}^{k-1}\sum_{i=1}^{m}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > \varepsilon x, \max_{1\le m\le n}\sum_{i=1}^{m}\xi_{ki}\prod_{j=1}^{i-1}\eta_j > \varepsilon x\Big\}\Big) \\
&\le P\Big(\max_{1\le m\le n}\sum_{s=1}^{k-1}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big) + P\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\xi_{ki}\prod_{j=1}^{i-1}\eta_j > (1-\varepsilon)x\Big) \\
&\quad +P\Big(\max_{1\le m\le n}\sum_{s=1}^{k-1}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > \varepsilon x, \max_{1\le m\le n}\sum_{i=1}^{m}\xi_{ki}\prod_{j=1}^{i-1}\eta_j > \varepsilon x\Big) \\
&\le (1+\delta)^2\sum_{s=1}^{k-1}\sum_{i=1}^{n} P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big) + (1+\delta)^2\sum_{i=1}^{n} P\Big(\xi_{ki}\prod_{j=1}^{i-1}\eta_j > x\Big) \\
&\quad +n\delta(1+\delta)M_1\sum_{s=1}^{k}\sum_{i=1}^{n} P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big).
\end{aligned}
$$

The arbitrariness of $0<\delta<1$ gives result (3.1). This ends the proof of Theorem 3.1.

Theorem 3.2 Under the conditions of Theorem 2.1, we have

$$\varphi^k(x,n) \gtrsim \sum_{s=1}^{k}\sum_{i=1}^{n} P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big). \tag{3.8}$$

Proof We prove relation (3.8) by induction in k. For the case of $k=1$, relation (3.8) holds from Proposition 2.1. From Theorem 3.1, it suffices to prove that relation (3.8) holds for $k=2$.

When $k=2$, for any fixed $0<\varepsilon<1$ and $x>0$, we have

$$\begin{aligned}
\varphi^2(x,n) &= P\Big(\max_{1\le m\le n}\sum_{s=1}^{2}\sum_{i=1}^{m}\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big)\\
&\ge P\Big(\sum_{s=1}^{2}\sum_{i=1}^{n}\xi_{si}\prod_{j=1}^{i-1}\eta_j > x\Big)\\
&\ge P\Big(\Big\{\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x, \sum_{i=1}^{n}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > -\varepsilon x\Big\}\\
&\quad \bigcup\Big\{\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > -\varepsilon x, \sum_{i=1}^{n}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x\Big\}\Big)\\
&= P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x, \sum_{i=1}^{n}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > -\varepsilon x\Big)\\
&\quad +P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > -\varepsilon x, \sum_{i=1}^{n}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x\Big)\\
&\quad -P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x, \sum_{i=1}^{n}\xi_{2i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x\Big)\\
&= Q_1+Q_2-Q_3.
\end{aligned} \tag{3.9}$$

For Q_1, we have

$$\begin{aligned}
Q_1 &\ge P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x, \bigcap_{i=1}^{n}\Big\{\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \frac{-\varepsilon x}{n}\Big\}\Big)\\
&= E\Big[P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x, \bigcap_{i=1}^{n}\Big\{\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \frac{-\varepsilon x}{n}\Big\} \mid \eta_j, 1\le j\le n\Big)\Big]\\
&= E\Big[P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x \mid \eta_j, 1\le j\le n\Big)\prod_{i=1}^{n}P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \frac{-\varepsilon x}{n} \mid \eta_j, 1\le j\le n\Big)\Big].
\end{aligned}$$

For $1 \le j \le n$, in view of $\eta_j \in [a, b]$, for any fixed $0 < \varepsilon < 1$, we have

$$\begin{aligned} & P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \frac{-\varepsilon x}{n} \mid \eta_j, 1 \le j \le n\Big) \\ = \;& P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \frac{-\varepsilon x}{n}, \xi_{2i} \ge 0 \mid \eta_j, 1 \le j \le n\Big) + P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > \frac{-\varepsilon x}{n}, \xi_{2i} < 0 \mid \eta_j, 1 \le j \le n\Big) \\ \ge \;& P\Big(\xi_{2i}a^{i-1} > \frac{-\varepsilon x}{n}, \xi_{2i} \ge 0\Big) + P\Big(\xi_{2i}b^{i-1} > \frac{-\varepsilon x}{n}, \xi_{2i} < 0\Big) \\ = \;& P\Big(\xi_{2i} > \frac{-\varepsilon x}{nb^{i-1}}\Big). \end{aligned} \tag{3.10}$$

Since $\lim\limits_{x\to\infty} P\Big(\xi_{2i} > \frac{-\varepsilon x}{n}\Big) = 1$ for $1 \le i \le n$, then for any small $0 < \delta < 1$ and any fixed $0 < \varepsilon < 1$, we have

$$P\Big(\xi_{2i} > \frac{-\varepsilon x}{nb^{i-1}}\Big) > 1 - \delta. \tag{3.11}$$

Thus, combing relation (3.10), relation (3.11), Proposition 2.1 and Proposition 2.2, for any small $0 < \delta < 1$ and any fixed $0 < \varepsilon < 1$,

$$\begin{aligned} Q_1 \;&\ge\; (1-\delta)^n P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x\Big) \\ &\ge\; (1-\delta)^{n+1}\sum_{i=1}^{n} P\Big(\xi_{1i}\prod_{j=1}^{i-1}\eta_j > (1+\varepsilon)x\Big) \\ &\ge\; (1-\delta)^{n+2}\sum_{i=1}^{n} P\Big(\xi_{1i}\prod_{j=1}^{i-1}\eta_j > x\Big). \end{aligned} \tag{3.12}$$

Symmetrically,

$$Q_2 \ge (1-\delta)^{n+2}\sum_{i=1}^{n} P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > x\Big). \tag{3.13}$$

Next we turn to Q_3. In view of $\xi_{1i} \in \mathcal{C}$, and η_j is bounded for $1 \le i \le n$, by Proposition 2.3, there is some $M_2 > 0$ such that

$$P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > x\Big) \ge M_2 P(\xi_{2i} > x).$$

Thus, we have

$$\sum_{i=1}^{n} P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j > x\Big) \ge nM_2 P(\xi_{21} > x). \tag{3.14}$$

Because of $\xi_{2i} \in \mathcal{C}$ for $1 \le i \le n$, there exists some $M_3 > 0$ such that

$$P(\xi_{2i}b > \frac{(1+\varepsilon)x}{n}) \le M_3 P(\xi_{2i}b > x). \tag{3.15}$$

So, by relation (3.14) and (3.15), we have

$$
\begin{aligned}
&\frac{P\Big(\sum\limits_{i=1}^{n}\xi_{2i}\prod\limits_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)}{\sum\limits_{i=1}^{n}P\Big(\xi_{2i}\prod\limits_{j=1}^{i-1}\eta_j>x\Big)}\\
\le\;&\frac{P\Big(\sum\limits_{i=1}^{n}\xi_{2i}\prod\limits_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)}{nM_2P(\xi_{21}>x)}\\
\le\;&\frac{P\Big(\sum\limits_{i=1}^{n}\xi_{2i}^{+}\prod\limits_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)}{nM_2P(\xi_{21}>x)}\\
\le\;&\frac{P\Big(\sum\limits_{i=1}^{n}\xi_{2i}^{+}b^{i-1}>(1+\varepsilon)x\Big)}{nM_2P(\xi_{21}>x)}\\
\le\;&\sum_{i=1}^{n}\frac{P\Big(\xi_{2i}b^{i-1}>\frac{(1+\varepsilon)x}{n}\Big)}{nM_2P(\xi_{21}>x)}\\
\le\;&\frac{M_3}{M_2} \qquad (3.16)
\end{aligned}
$$

Then, using Fatou's lemma, and relation (3.16), we obtain

$$
\begin{aligned}
&\limsup_{x\to\infty}\frac{Q_3}{\sum_{s=1}^{2}\sum_{i=1}^{n}P\Big(\xi_{si}\prod_{j=1}^{i-1}\eta_j>x\Big)}\\
\le\;&\limsup_{x\to\infty}\frac{E\Big\{\prod\limits_{s=1}^{2}P\Big(\sum\limits_{i=1}^{n}\xi_{si}\prod\limits_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)\Big\}}{\sum\limits_{i=1}^{n}P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j>x\Big)}\\
\le\;&E\Big\{\limsup_{x\to\infty}\frac{\prod\limits_{s=1}^{2}P\Big(\sum\limits_{i=1}^{n}\xi_{si}\prod\limits_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)}{\sum\limits_{i=1}^{n}P\Big(\xi_{2i}\prod_{j=1}^{i-1}\eta_j>x\Big)}\Big\}\\
\le\;&(1+\delta)\frac{M_3}{M_2}E\Big\{\limsup_{x\to\infty}P\Big(\sum_{i=1}^{n}\xi_{1i}\prod_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)\Big\}\\
=\;&0, \qquad (3.17)
\end{aligned}
$$

where in the fourth equality, we use the following true

$$
\limsup_{x\to\infty}P\Big(\xi_{1i}\prod_{j=1}^{i-1}\eta_j>(1+\varepsilon)x\mid\eta_j,1\le j\le n\Big)\le\limsup_{x\to\infty}P\Big(\xi_{1i}b^{i-1}>(1+\varepsilon)x\Big)=0.
$$

Combing relation (3.12), relation (3.13) and relation (3.17), the arbitrariness of $0 < \delta < 1$ gives result (3.8) for $k = 2$. So Theorem 3.2 holds.

From Theorem 3.1 and Theorem 3.2, we know that Theorem 2.1 holds.

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离散时多元风险模型的破产概率

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摘要: 本文研究了离散时多元风险模型的破产概率问题. 利用经典大偏差的方法, 获得了有限水平的破产概率, 推广了离散时一元风险模型的相应结论.

关键词: 破产概率; 大偏差; 离散时多元风险模型

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