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BOUNDARY VALUE PROBLEM FOR FIRST-ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATION WITH DELAY

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Abstract: In this paper, the properties of solution of boundary value problem for first-order impulsive integro-differential equation with delay are discussed. Using the iterative analysis method, the existence and uniqueness of solution and the sufficient condition for uniform stability of trivial solution are obtained, which extend the previous results on integro-differential equation in periodic boundary value problem.

Keywords: iterative analysis; boundary value problem; existence; stability

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1 Introduction

Impulsive differential equation is mathematical model to simulate process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, etc. During the last two decades, impulsive differential equations were studied by many researchers [1–5]. Some classical tools were used to study such a problem in the literatures. These classical technique include the coincidence degree theory of Mawhin [6], the method of upper and lower solutions [7] and some fixed point theorems [8]. In [9, 10], the author used iterative analysis method to obtain the existence of solution of functional differential equation. In [11–15], by the iterative analysis method, authors got the existence of periodic solution or anti-periodic solution of equation without delay. However, there is few paper on two points boundary value problem for impulsive integro-differential equation with delay. In this paper we employ the iterative analysis method to obtain the existence, uniqueness and stability of integro-differential equation for boundary value problem.

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We consider the following boundary value problem for first-order impulsive integro-differential equation with delay

$$u'(t)+\alpha(t)u(t)=f(t,u(t-\tau),Pu(t)),\qquad t\in J^{'},\tag{1.1}$$

$$\triangle u(t_j)=I_j(u(t_j)),\quad j=1,\cdots,m,\tag{1.2}$$

$$u(0)-u(T)=M,\tag{1.3}$$

$$u(t)=\varphi(t),\ \ t\in[-\tau,0],\tag{1.4}$$

where $J=[0,T]$, $M\in R$, $J^+=[-\tau,T]$, $T>0$, $\tau>0$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, $J^{'}=J-\{t_1,\cdots,t_m\}$, $\alpha\in C[J,R^+]$, $\varphi\in C[[-\tau,0],R]$, $f:J\times R\times R\to R$ is continuous at every point $(t,u,Pu)\in J^{'}\times R\times R$, $I_j\in C[R,R]$, $\triangle u(t_j)=u(t_j^+)-u(t_j^-)$, $t_{j+1}-t_j=\tau$ for all $j=1,\cdots,m$, $Pu(t)=\displaystyle\int_0^t K(t,s)u(s)ds$, $K\in C[D,R^+]$, where $D=\{(t,s)\in R^2:0\le s\le t\le T\}$.

(i) If $M=0$, (1.1)–(1.4) is periodic boundary value problem;

(ii) If $M\neq 0$, (1.1)–(1.4) is two points boundary value problem.

This paper is organized as follows. In Section 2, some notations and preliminaries are introduced. In Section 3, we prove the existence, uniqueness and stability of solution of first-order impulsive integro-differential equation with delay by using the iterative analysis method.

2 Preliminaries

Let $J^-=J^+-\{t_1,\cdots,t_m\}$, $PC(J^+,R)=\{u:J^+\to R;\ u(t)$ is continuous everywhere except for some t_j at which $u(t_j^+)$ and $u(t_j^-)$ exist and $u(t_j^-)=u(t_j)$, $j=1,\cdots,m\}$. $PC^{'}(J^+,R)=\{u\in PC(J^+,R);\ u'(t)$ is continuous on J^-, where $u'(0^+)$, $u'(T^-)$, $u'(t_j^+)$ and $u'(t_j^-)$ exist, $j=1,\cdots,m\}$.

And let $E=\{u\in PC(J^+,R):u(t)=\varphi(t),t\in[-\tau,0]\}$ with norm

$$\parallel u\parallel_E=\sup\{\mid u(t)\mid: t\in J^+\},$$

it is easy to see that E is a Banach space.

Let $\parallel\Psi\parallel=\sup\{\mid\varphi(t)\mid:t\in[-\tau,0]\}$. Let $E_0=\{PC(J^+,R)\bigcap PC^{'}(J^+,R)\}$, a function $u\in E_0$ is called a solution of problem (1.1)–(1.4) if it satisfies (1.1)–(1.4).

The following are the basic hypotheses:

(H_1) $f(t,0,0)=0$ and there exists $L(t)>0, L(t)\in L^1[0,T]$ such that

$$\mid f(t,x_1,x_2)-f(t,y_1,y_2)\mid\le L(t)(\mid x_1-y_1\mid+\mid x_2-y_2\mid);$$

(H_2) $I_j(0)=0$ and there exists $q_j>0$ such that $\mid I_j(u_1)-I_j(u_2)\mid\le q_j\mid u_1-u_2\mid$.

We denote

$$K^*=\sup_{t\in J}\left\{\int_0^t\mid K(t,s)\mid ds\right\},\ \ D=\int_0^T L(s)ds,\ B=\sum_{j=1}^m q_j,$$

$$A=\frac{D}{1-e^{-a(T)}},\ \ C=\frac{B+D(1+K^*)}{1-e^{-a(T)}}.$$

(H$_3$) $0 < C < 1$.

To obtain the main theorem of this paper, we need the following lemma.

Lemma 2.1 The solution of the boundary value problem (1.1)–(1.4) can be presented as

$$u(t) = \int_0^T g_\alpha(t,s) f(s, u(s-\tau), Pu(s)) ds + M g_\alpha(t,0) + \sum_{j=1}^{m} g_\alpha(t,t_j) I_j(u(t_j)), \quad t \in J, \tag{2.1}$$

where

$$g_\alpha(t,s) = \frac{1}{1-e^{-a(T)}} \begin{cases} e^{-[a(t)-a(s)]}, & 0 \le s \le t \le T, \\ e^{-[a(T)+a(t)-a(s)]}, & 0 \le t < s \le T \end{cases}$$

with

$$a(t) = \int_0^t \alpha(s) ds, \qquad t \in J.$$

Proof Set $y(t) = e^{a(t)} u(t)$ for $t \in J$. Then $y(t)$ satisfies the impulsive boundary value problem

$$\begin{aligned} y'(t) &= e^{a(t)} f^*(t, y(t-\tau), Py(t)), \\ y(0) &= e^{-a(T)} y(T) + M, \\ y(t_j^+) &= y(t_j^-) + I_j^*(y(t_j)), \end{aligned}$$

where $f^*(t, y(t-\tau), Py(t)) = f(t, u(t-\tau), Pu(t))$ and $I_j^*(y(t_j)) = e^{a(t_j)} I_j(e^{-a(t_j)} y(t_j))$.

For $t \in [0, t_1]$, there is no impulsive effect in this interval, and we obtain

$$y(t) = y(0) + \int_0^t e^{a(s)} f^*(s, y(s-\tau), Py(s)) ds$$

and

$$y(t_1^-) = y(0) + \int_0^{t_1} e^{a(s)} f^*(s, y(s-\tau), Py(s)) ds. \tag{2.2}$$

Considering Cauchy problem (1.1) and (2.2) on $(t_1, t_2]$, we have

$$\begin{aligned} y(t) &= y(t_1^-) + \int_{t_1}^t e^{a(s)} f^*(s, y(s-\tau), Py(s)) ds + I_1^*(y(t_1)) \\ &= y(0) + \int_0^t e^{a(s)} f^*(s, y(s-\tau), Py(s)) ds + I_1^*(y(t_1)). \end{aligned}$$

The procedure can be repeated on $(t_2, t_3]$, $(t_3, t_4]$, $\cdots$, $(t_m, T]$ and we attain

$$y(t) = y(0) + \int_0^t e^{a(s)} f^*(s, y(s-\tau), Py(s)) ds + \sum_{j: t_j \in (0,t)} I_j^*(y(t_j)), \quad t \in J. \tag{2.3}$$

Using the expression above for $t = T$, we get

$$y(0) = \frac{1}{e^{a(T)}-1} \int_0^T e^{a(s)} f^*(s, y(s-\tau), Py(s)) ds + \frac{e^{a(T)}}{e^{a(T)}-1} M + \frac{1}{e^{a(T)}-1} \sum_{j=1}^{m} I_j^*(y(t_j)).$$

Substituting this value into (2.3), we obtain that for every $t\in J$,

$$\begin{aligned} u(t) &= \int_0^t \left[\frac{e^{-(a(t)-a(s))}}{e^{a(T)}-1}+e^{-(a(t)-a(s))}\right] f(s,u(s-\tau),Pu(s))ds \\ &\quad + \int_t^T \frac{e^{-(a(t)-a(s))}}{e^{a(T)}-1} f(s,u(s-\tau),Pu(s))ds \\ &\quad + \sum_{j=1}^m \frac{e^{-(a(t)-a(t_j))}}{e^{a(T)}-1} I_j(u(t_j)) + \sum_{j:t_j\in(0,t)} e^{-(a(t)-a(t_j))} I_j(u(t_j)) + \frac{e^{-a(t)}}{1-e^{-a(T)}}M \\ &= \int_0^T g_\alpha(t,s) f(s,u(s-\tau),Pu(s))ds + Mg_\alpha(t,0) + \sum_{j=1}^m g_\alpha(t,t_j) I_j(u(t_j)). \end{aligned}$$

The proof of Lemma 2.1 is completed.

Remark 1 (i) If $M=0$, the solution of the periodic boundary value problem (1.1)–(1.4) can be presented as

$$u(t) = \int_0^T g_\alpha(t,s) f(s,u(s-\tau),Pu(s))ds + \sum_{j=1}^m g_\alpha(t,t_j) I_j(u(t_j)), \quad t\in J. \tag{2.4}$$

(ii) If $M\neq 0$, the solution of the two points boundary value problem (1.1)–(1.4) can be presented as

$$u(t) = \int_0^T g_\alpha(t,s) f(s,u(s-\tau),Pu(s))ds + Mg_\alpha(t,0) + \sum_{j=1}^m g_\alpha(t,t_j) I_j(u(t_j)), \ t\in J. \tag{2.5}$$

3 Main Results

Theorem 3.1 Suppose that hypotheses (H_1)–(H_3) hold, the boundary value problem (1.1)–(1.4) has a unique solution $u(t)$ on $[-\tau,T]$ and

$$\| u(t) \|_E \le \frac{2AD \| \Psi \| + A \mid M \mid}{D(1-C)}. \tag{3.1}$$

Proof We define the iteration

$$u^{(k)}(t) = \begin{cases} \int_0^T g_\alpha(t,s) f(s,u^{(k-1)}(s-\tau),Pu^{(k-1)}(s))ds + Mg_\alpha(t,0) & \\ \qquad\qquad + \sum\limits_{j=1}^m g_\alpha(t,t_j) I_j(u^{(k-1)}(t_j)), & t\in J, \\ \| \Psi \|, & t\in[-\tau,0] \end{cases}$$

and

$$u^{(0)}(t) = \begin{cases} \int_0^T g_\alpha(t,s) f(s,u(0),Pu(0))ds + Mg_\alpha(t,0), & t\in J, \\ \| \Psi \|, & t\in[-\tau,0]. \end{cases}$$

Applying inductive method, we obtain that the following inequality holds

$$\begin{aligned}
\| u^{(1)} - u^{(0)} \|_E &= \sup_{t\in[-\tau,T]} | u^{(1)}(t) - u^{(0)}(t) | \\
&\le \frac{1}{1-e^{-a(T)}} \int_{-\tau}^{T-\tau} L(s+\tau)(\| u^{(0)} \|_E (1+K^*) + \| \Psi \|) ds \\
&\quad + \frac{B}{1-e^{-a(T)}} \| u^{(0)} \|_E \\
&\le A(1+C) \| \Psi \| + \frac{A \mid M \mid}{D} C, \\
\| u^{(2)} - u^{(1)} \|_E &\le \frac{1}{1-e^{-a(T)}} \int_{-\tau}^{T-\tau} L(s+\tau) \| u^{(1)} - u^{(0)} \|_E (1+K^*) ds \\
&\quad + \frac{B}{1-e^{-a(T)}} \| u^{(1)} - u^{(0)} \|_E \\
&\le A(1+C)C \| \Psi \| + \frac{A \mid M \mid}{D} C^2.
\end{aligned}$$

Again, using induction, we can derive

$$\| u^{(j+1)} - u^{(j)} \|_E \le A(1+C)C^j \| \Psi \| + \frac{A \mid M \mid}{D} C^{j+1}, \qquad j = 0, 1, \cdots .$$

Furthermore,

$$| u^{(n+1)}(t) | \ \le \sum_{j=0}^{n} | u^{(j+1)}(t) - u^{(j)}(t) | + | u^{(0)}(t) |$$

and

$$\begin{aligned}
\| u^{(n+1)} \|_E &= \sup_{t\in[-\tau,T]} | u^{(n+1)}(t) | \\
&\le \sum_{j=0}^{n} A(1+C)C^j \| \Psi \| + \frac{A \mid M \mid}{D} C^{j+1} + A \| \Psi \| + \frac{A \mid M \mid}{D} \\
&\le \frac{2AD \| \Psi \| + A \mid M \mid}{D(1-C)}.
\end{aligned}$$

For $\forall p \in N$, $m + p \ge m$, we have

$$\begin{aligned}
| u^{(m+p)}(t) - u^{(m)}(t) | &\le \sum_{j=m+1}^{m+p} | u^{(j)}(t) - u^{(j-1)}(t) | \\
&\le A(1+C) \cdot \frac{C^m}{1-C} \cdot \| \Psi \| + \frac{A \mid M \mid}{D} \cdot \frac{C^{1+m}}{1-C}.
\end{aligned}$$

Therefore, the sequence $\{u^{(k)}(t)\}$ is uniformly convergent on $[-\tau, T]$, let $\lim_{k\to\infty} u^{(k)}(t) = u(t)$. Obviously, $u(t)$ is a solution of boundary value problem (1.1)–(1.4), which satisfies inequality (3.1).

Next, we prove the uniqueness. Suppose that $v(t)$ is another solution of boundary value problem (1.1)–(1.4), it has

$$\begin{aligned}\| u-v \|_E &= \sup_{t\in[-\tau,T]} | u(t)-v(t) | \\ &\le \frac{1}{1-e^{-a(T)}}\int_{-\tau}^{T-\tau} L(s+\tau)\cdot \| u-v \|_E (1+K^*)ds + \frac{B}{1-e^{-a(T)}}\cdot \| u-v \|_E \\ &\le C \| u-v \|_E .\end{aligned}$$

From (H_3), $0 < C < 1$, it has $\| u-v \|_E = 0$. Certainly, the uniqueness of solution holds.

The proof of Theorem 3.1 is completed.

Remark 2 (i) If $M = 0$ and the hypotheses (H_1)–(H_3) hold, the periodic boundary value problem (1.1)–(1.4) has a unique solution $u(t)$ on $[-\tau, T]$, and

$$\| u(t) \|_E \le \frac{2A \| \Psi \|}{1-C}. \tag{3.2}$$

(ii) If $M \ne 0$ and the hypotheses (H_1)–(H_3) hold, the two points boundary value problem (1.1)–(1.4) has a unique solution $u(t)$ on $[-\tau, T]$, and

$$\| u(t) \|_E \le \frac{2AD \| \Psi \| + A | M |}{D(1-C)}. \tag{3.3}$$

Definition 3.2 The trivial solution of (1.1)–(1.4) is said to be stable if for any $t_0 > 0$ and $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\| \Psi \| < \delta$ implies that $| u(t, t_0, \varphi(t_0)) | < \varepsilon$ for $t \ge t_0$. The trivial solution of (1.1)–(1.4) is said to be uniformly stable if δ is independent of t_0.

Theorem 3.3 If $M = 0$ and hypotheses (H_1)–(H_3) hold, the trivial solution of the system (1.1)–(1.4) is uniformly stable.

Proof Suppose that the trivial solution of the system (1.1)–(1.4) is not stable, we have that $\exists\ \varepsilon_1 > 0$, $\forall\ \delta(\varepsilon_1, t_0) > 0$, $\exists\ t_1 \ge t_0$,

$$| u(t_1, t_0, \varphi(t_0)) | \ge \varepsilon_1 \quad \text{as} \quad \| \Psi \| < \delta(\varepsilon_1, t_0). \tag{3.4}$$

From Theorem 3.1, we have $\| u(t) \|_E \le \frac{2A\|\Psi\|}{1-C}$.

Letting $\delta(\varepsilon_1, t_0) = \frac{1-C}{2A}\varepsilon_1$, by simple calculation, we have

$$\| u(t) \|_E < \varepsilon_1 \quad \text{as} \quad \| \Psi \| < \delta(\varepsilon_1, t_0),$$

which is contradicted with (3.4), the trivial solution of the system (1.1)–(1.4) is stable. In addition, $\delta(\varepsilon_1, t_0) = \frac{1-C}{2A}\varepsilon_1$ is independent of t_0, thus the trivial solution is uniformly stable.

It completes the proof.

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一阶脉冲时滞积分微分方程边值问题

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摘要: 本文研究了一类一阶脉冲时滞积分微分方程边值问题解的性质. 利用迭代分析方法, 得到了该类边值问题解的存在性、唯一性和平凡解一致稳定的充分条件, 推广了已有积分微分方程周期边值问题解的结论.

关键词: 迭代分析方法; 边值问题; 存在性; 稳定性

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