

UPPER BOUND ON 2-STRONG EDGE CHROMATIC NUMBERS OF GRAPHS

TIAN Jing-jing¹, NIE Yu-feng¹, WANG Li-gong¹, CHANG Jian²

(1. Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710129, China)

(2. Institute of Mathematics Science, Inner Mongolia Normal University, Huhhot 010022, China)

Abstract: In this paper, we study the upper bound for the the r -strong edge chromatic number of regular graph. By probability method, we prove that if $3 \leq \Delta \leq 730$, then $\chi'_s(G, 2) \leq 2\Delta + 1$ by the general local lemma, which extends some corresponding results in [11, 12].

Keywords: 2-strong edge coloring; 2-strong edge chromatic number; the general local lemma

2010 MR Subject Classification: 05C15

Document code: A **Article ID:** 0255-7797(2014)02-0259-06

1 Introduction

Graph coloring is one of the chief topics in graph research. The graph coloring was applied to chemistry, biology, VLSI, etc., see [1, 2], It is well known to compute the chromatic number of graphs is NP-Hard in graphs theory. In the past, the people had some results about it by combination methods, see [3, 4, 5, 6, 7]. In 1974, Edrös proved $r(k, k) \geq 2^{\frac{k}{2}}$ by probability. At ICM2002, Noga Alon had a report about the method and challenge of Discrete mathematics. The viewpoint that the problem of graph coloring studied by probability drew the field of graph attention. For instance, some conclusions was gotten by probability methods. In [8], Alon proved that $\alpha'(G) \leq \Delta + 2$ for any graph whose girth is at least $2000\Delta \log \Delta$, where Δ is maximum degree of G . In [9], Rahul Muthu et al. improved the result of [8]. In [10], Hamed Hatami proved that if $\Delta > 10^{20}$, then $\chi_{as}(G) < \Delta + 300$, where $\Delta(G)$ is maximum degree of G . In 2006, Zhang Zhongfu and Akbaria presented the concept of the r -strong edge coloring independently, see [11, 12]. When $r = 2$, the r -strong edge-chromatic number is denoted by $\chi'_s(G, 2)$. Let Δ be the maximum degree of G . In this paper, we study the upper bound for the the r -strong edge chromatic number of regular graph by probability method, prove that if $3 \leq \Delta \leq 730$, then $\chi'_s(G, 2) \leq 2\Delta + 1$ by the general local lemma. All the graphs $G = G(V, E)$ discussed in this paper are finite, undirected, simple and connected.

* **Received date:** 2013-01-25

Accepted date: 2013-09-02

Foundation item: Supported by National Natural Science Foundation of China (11071169).

Biography: Tian Jingjing (1979 -), female, born at Hanzhong, Shaanxi, lecturer, Ph.D., major in graph theory.

2 Main Results

Definition 2.1 A proper edge coloring of a graph G is a map $f : E(G) \rightarrow C$, where C is a set of colors such that no two edges with the same colors are incident with the same vertex, see [3].

Definition 2.2 For a graph $G(V, E)$, if a proper k -edge coloring f is satisfied with $S(u) \neq S(v)$ for $uv \in E(G)$, where $S(u) = \{f(uv) | uv \in E\}$, then f is called k -adjacent Vertex-distinguishing edge coloring of G , is abbreviated k -ASEC, and

$$\chi'_{as}(G) = \min\{k | k\text{-ASEC of } G\}$$

is called the adjacent Vertex-distinguishing edge chromatic number of G , see [4].

Definition 2.3 For any $u, v \in V(G)$, $d(u, v)$ denotes the distance between u and v and $N(v)$ denotes the set of all vertices adjacent to vertex v . A proper edge coloring of a graph G is called an r -strong edge coloring if for any two distinct vertices $u, v \in V(G)$ with $d(u, v) \leq r$, we have $S(u) \neq S(v)$. The r -strong edge coloring number $\chi'_s(G, r)$ is the minimum number of colors required for an r -strong edge coloring of the graph G , see [11, 12].

Obviously, $\chi'_s(G, 1) = \chi'_{as}$ when $r = 1$, $\chi'_s(G, r) = \chi'_s$ when $r \geq \text{Diam}(G)$, where $\text{diam}(G)$ is the diameter of the graph.

Definition 2.4 Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, 2, \dots, n\}$ is called a dependency digraph for the events A_1, A_2, \dots, A_n if for each $i, 1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \in E\}$, see [13, 14].

Lemma 2.2 (the general local lemma) Consider a set $\varepsilon = \{A_1, A_2, \dots, A_n\}$ of (typically bad) events such that each A_i is mutually independent of $\varepsilon - (D_i \cup A_i)$ for some $D_i \subseteq \varepsilon$. If there exist $x_1, x_2, \dots, x_n \in [0, 1]$ such that for each $1 \leq i \leq n$,

$$\Pr(A_i) \leq x_i \prod_{A_j \in D_i} (1 - x_j),$$

then the probability that none of the events in ε occurs is at least

$$\prod_{i=1}^n (1 - x_i) > 0,$$

see [13, 14]

Theorem 2.3 If $G(V, E)$ has $2 \leq \Delta \leq 730$, then $\chi'_s(G, 2) \leq 2\Delta + 1$.

Proof By Vizing theorem, it is possible to color all edges of G by $\Delta + 1$ colors properly, so we have proper edge coloring f_0 . And then each of edges in G is recolored randomly and independently with an equal probability $\frac{1}{16(\Delta)^{\frac{3}{2}}}$ to one of Δ new colors, name this edge coloring of G as g . We will use Lemma 1 to show that a positive probability, g is 2-strong edge coloring. In order to show that, the following conditions should be satisfied:

- [1] The coloring is proper-no pair of adjacent edges are colored with the same color;
- [2] The coloring is adjacent-vertex-distinguishing-no pair of adjacent vertices meet the same color set;
- [3] No pair of vertices whose distance is 2 meet the same color set.

Step 1 The following bad events are defined in order to satisfy above:

- (1) For each pair of adjacent edges e, f , let $A_{e,f}$ be the event that both e and f are colored with the same color;
- (2) For each edge $e = uv$, such that $\deg(u) = \deg(w)$, let B_e be the set of all edges which are connected u or w , then E_{B_e} be the event that the edges which are adjacent to u and w are colored properly, and $S(u) = S(w)$;
- (3) For each path whose length is 2, $P_{uv} = uewfv$, such that $\deg(u) = \deg(v)$, let P_{uv} be the set of all edges which are incident with u or v , then $E_{P_{uv}}$ is the event that the edges which are incident with u and v are colored properly, and $S(u) = S(v)$.

It remains to show that with positive probability none of these events happen, then G has a 2 -strong edge-coloring. To prove this we apply the local lemma. Let us construct dependency graph H whose nodes are all the events of two nodes E_X and E_Y (where each of X and Y is either a pair of incident edges, or the set of all edges that are adjacent to an edge together with that edge itself, or the set of all edges incident to two vertices u and v which distance is 2) are adjacent if and only if X and Y contain at least one common edge. Since the occurrence of each event E_X depends only on the edges of X , H is dependency graph for our events. In order to apply the general local lemma, we need estimates for the probability of each event and the number of nodes of each type in H which are adjacent to any give node. These estimates are given in the two steps below.

Step 2 Estimate the probability of each event: If $A_{e,f}$ occurs, then e, f in $A_{e,f}$ are recolored with the same color. So

$$\begin{aligned} \Pr(A_{e,f}) &= \binom{\Delta}{1} \left(\frac{1}{16\Delta^{\frac{3}{2}}}\right)^2 = \frac{1}{16^2\Delta^2}, \\ \Pr(E_{P_{uv}}) &= \left[\binom{\Delta}{1} \left(\frac{1}{16\Delta^{\frac{3}{2}}}\right)^\Delta + \binom{\Delta}{2} \left(\frac{1}{2 \cdot 16\Delta^{\frac{3}{2}}}\right)^\Delta + \dots + \binom{\Delta}{\Delta-1} \left(\frac{1}{(\Delta-1)16\Delta^{\frac{3}{2}}}\right)^\Delta\right]^2 \\ &\leq \left[\binom{\Delta}{1} + \binom{\Delta}{2} + \dots + \binom{\Delta}{\Delta-1}\right]^2 \left[\left(\frac{1}{16\Delta^{\frac{3}{2}}}\right)^\Delta\right]^2 \\ &\leq \left[2^\Delta \left(\frac{1}{16\Delta^{\frac{3}{2}}}\right)^\Delta\right]^2 = \frac{1}{(64\Delta^3)^\Delta}. \end{aligned}$$

Let $e = uv$, $\deg(u) = \deg(v) = \Delta$, if E_{B_e} occurs, then edges in B_e are colored properly and $S(u) = S(v)$, namely, $S(u) - \{f(e)\} = S(v) - \{f(e)\} = S$. Assume S is a fixed set which has i member of the new colors and $\Delta - i$ members of the old colors. With probability

$(1 - \frac{1}{8}\Delta)$ preserves its previous color. This event happens with the probability of

$$\begin{aligned}
 & [i!(1 - \frac{1}{8}\Delta)^{\Delta-i}(\frac{1}{16\Delta^{\frac{3}{2}}})^i]^2 \leq [i^i \exp(\frac{8}{\Delta}(i - \Delta))(\frac{1}{16\Delta^{\frac{3}{2}}})^i]^2 \\
 = & i^{2i} \exp(\frac{16}{\Delta}(i)) \exp(\frac{16}{\Delta}(-\Delta))(\frac{1}{16\Delta^{\frac{3}{2}}})^{2i} = (\frac{i^2 \exp(\frac{16}{\Delta})}{16^2\Delta^2})^i \exp(-16),
 \end{aligned}$$

$$\begin{aligned}
 Pr(E_{B_e}) & \leq \sum_{i=0}^{\Delta} \binom{\Delta}{i} \binom{\Delta}{i} (\frac{i^2 \exp(\frac{16}{\Delta})}{16^2\Delta^2})^i \exp(-16) \\
 & \leq \sum_{i=0}^{\Delta} (\frac{e\Delta}{i} \frac{e\Delta}{i} \frac{i^2 \exp(\frac{16}{\Delta})}{16^2\Delta^2})^i \exp(-16) \leq \exp(-16).
 \end{aligned}$$

Step 3 Estimate the dependency events number, in the following table:

	$A_{e,f}$	E_{B_e}	$E_{P_{uv}}$
$A_{e,f}$	$4\Delta - 5$	$3\Delta - 2$	$\Delta(\Delta - 1) + 2(\Delta - 1)^2$
E_{B_e}	$(2\Delta - 1)(2\Delta - 2)$	$(2\Delta - 2)\Delta + 1$	$(2\Delta - 1)\Delta(\Delta - 1)$
$E_{P_{uv}}$	$2\Delta(2\Delta - 1) + 1 + 2(\Delta - 2)$	$2\Delta^2 + \Delta - 2$	$2(\Delta - 1)^3 + \Delta(\Delta - 1)$

For the dependency events numbers in the table, we have some explanations. For each event $A_{e,f}$ of Type $\langle 1 \rangle$, the corresponding vertex of $A_{e,f}$ in H is adjacent to at most $4\Delta - 5$ events of type $\langle 1 \rangle$. This is because that each edge e and f is adjacent to at most $2\Delta - 2$ (except for e itself) edge, there are two edge e and f in event of type $\langle 1 \rangle$, e, f are adjacent to at most $2 \cdot (2\Delta - 2) = 4\Delta - 4$, the edge adjacent to e (or f) contains f (or e), the set $\{e, f\}$ is compute twice, so the result is $2 \cdot (2\Delta - 2) - 1 = 4\Delta - 5$. For each event $A_{e,f}$ of type $\langle 2 \rangle$, the corresponding vertex of $A_{e,f}$ in H is adjacent to at most $3\Delta - 2$ events of type $\langle 2 \rangle$. This is because that each vertex is adjacent to at most Δ vertices, there are three vertices in type $\langle 2 \rangle$, except the common vertex u of e and f , another terminate vertex u_1 of e and another terminate vertex u_2 of f are compute twice, so the result is $3 \cdot \Delta - 2 = 3\Delta - 2$. The others can be explained similarly.

Step 4 Find the real constant $x_i(0 \leq x_i \leq 1)$ for applying Lemma 3. Let $\frac{2}{16^2\Delta^2}, \frac{1}{8\Delta^2}, \frac{2}{(64\Delta^3)^\Delta}$ be the constants associated with events of types (1), (2), (3).

Step 5 Conclude that with positive probability no events of type (1), (2), (3), provided that

$$\frac{1}{16^2\Delta^2} \leq \frac{2}{16^2\Delta^2} (1 - \frac{2}{16^2\Delta^2})^{4\Delta-5} (1 - \frac{2}{(64\Delta^3)^\Delta})^{\Delta(\Delta-1)+2(\Delta-1)^2} (1 - \frac{1}{8\Delta^2})^{3\Delta-2}, \tag{2.1}$$

$$\exp(-16) \leq \frac{1}{8\Delta^2} (1 - \frac{2}{16^2\Delta^2})^{(2\Delta-1)(2\Delta-2)} (1 - \frac{1}{8\Delta^2})^{(2\Delta-2)\Delta+1} (1 - \frac{2}{(64\Delta^3)^\Delta})^{(2\Delta-1)\Delta(\Delta-1)}, \tag{2.2}$$

$$\begin{aligned}
 & \frac{1}{(64\Delta^3)^\Delta} \leq \frac{2}{(64\Delta^3)^\Delta} (1 - \frac{2}{16^2\Delta^2})^{2\Delta(2\Delta-1)+1+2(\Delta-2)} (1 - \frac{1}{8\Delta^2})^{2\Delta^2+\Delta-2} \\
 & (1 - \frac{2}{(64\Delta^3)^\Delta})^{2(\Delta-1)^3+\Delta(\Delta-1)}. \tag{2.3}
 \end{aligned}$$

Now since $(1 - \frac{1}{z})^z \geq \frac{1}{4}$, to prove (2.1), it suffices to prove the following inequality:

$$\frac{1}{2} \leq \left(\frac{1}{2}\right)^{(4\Delta-5) + \frac{3\Delta-2}{4\Delta^2} + \frac{4[\Delta(\Delta-1)+2(\Delta-1)^2]}{(64\Delta^3)^\Delta}}. \quad (2.1*)$$

In order to proof (2.1*), it is only to prove

$$1 \geq \frac{(4\Delta-5)}{64\Delta^2} + \frac{3\Delta-2}{4\Delta^2} + \frac{4(\Delta-1)(3\Delta-2)}{(64\Delta^3)^\Delta}, \quad (2.1**)$$

when $\Delta > 2$, inequality (2.1**) is true, thus the inequality is hold (2.1), in order to prove (2.2), it is only need to prove

$$\begin{aligned} \exp(-16) &\leq \exp(-\ln(8\Delta^2)) \exp\left(\frac{-4(2\Delta-1)(2\Delta-2)}{16^2\Delta^2}\right) \\ &\exp\left(\frac{-2[(2\Delta-2)\Delta+1]}{8\Delta^2}\right) \exp\left(\frac{-4(2\Delta-1)\Delta(\Delta-1)}{(64\Delta^3)^\Delta}\right), \end{aligned} \quad (2.2*)$$

in order to prove (2*), it is only need to prove

$$-16 \leq -\ln(8\Delta^2) - \frac{-4(2\Delta-1)(2\Delta-2)}{16^2\Delta^2} - \frac{-2[(2\Delta-2)\Delta+1]}{8\Delta^2} - \frac{-4(2\Delta-1)\Delta(\Delta-1)}{(64\Delta^3)^\Delta}, \quad (2.2**)$$

when $\Delta \leq 730$, inequality (2.2**) is true, thus inequality (2.2) is hold.

Now since $(1 - \frac{1}{z})^z \geq \frac{1}{4}$, to prove (2.3), it suffices to prove the following inequality.

In order to prove (2.3), it is only need to prove

$$\frac{1}{2} \leq \left(\frac{1}{2}\right)^{\frac{(4\Delta^2-3)}{16^2\Delta^2} + \frac{2\Delta^2+\Delta-2}{4\Delta^2} + \frac{4(\Delta-1)(2\Delta^2-3\Delta+2)}{(64\Delta^3)^\Delta}}. \quad (2.3*)$$

In order to prove (2.3*), it is only need to prove

$$1 \geq \frac{(4\Delta^2-3)}{16^2\Delta^2} + \frac{2\Delta^2+\Delta-2}{4\Delta^2} + \frac{4(\Delta-1)(2\Delta^2-3\Delta+2)}{(64\Delta^3)^\Delta}, \quad (2.3**)$$

when $\Delta \geq 2$, inequality (2.3**) is true, thus inequality (2.3) is hold.

Above all, G has $2\Delta + 1 - \chi'_s(G, 2)$. This complete the proof

References

- [1] Mlynski D A. The graph theoretic approach to the circuit layout program[J]. The Mathematical Intelligencer, 1999, 5: 304–308.
- [2] Bertram E, Horak P. Some applications of graph theory to other parts of mathematics[J]. The Mathematical Intelligencer, 1999, 21(3): 6–10.
- [3] Jensen T, Toft B. Graph coloring problems[M]. New York: Wiley, 1995.
- [4] Zhang Zhongfu, Liu Linzhong, Wang Weifan. Adjacent strong edge coloring of graphs[J]. Applied Mathematics Letters, 2002, 5: 623–626.
- [5] Zhang Zhongfu, Woodall Douglas R, Li Jingwen. Adjacent strong edge coloring and total coloring of regular graphs[J]. Sci. China Ser. A, 2009, 52(2): 973–980.

- [6] Wang Weifan. Equitable total coloring of graphs with maximum degree 3[J]. Graphs and Combinatorics, 2002, 18: 677–685.
- [7] Bazgan C, Harkat-Benhamdine A, Li Hao. On the vertex-distinguishing proper edge-colorings of graphs[J]. J. Combin Theory (Ser. B), 1999, 75: 288–301.
- [8] Alon N, Sadakov B, Zaks A. Acyclic edge coloring of graphs[J]. Journal of Graph Theory, 2001, 37: 157–167.
- [9] Rahul Muthu, Narayanan N, Subramanian C R. Improved bounds on acyclic edge colouring[J]. Discrete Mathematic, 2007, 307: 3063–3069.
- [10] Hatami H. $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number of graphs[J]. Journal of Combinatorial Theory, 2003, 36(2): 135–139.
- [11] Zhang Zhongfu, Li Jinwen, Chen Xianguan. $D(\beta)$ -vertex edge coloring of graph[J]. Acta Mathematica Sinica, Chinese Series, 2006, 49(3): 703–708.
- [12] Akbari S, Bidkhor H, Nosrati N. r -strong edge coloring of graph[J]. Discrete Math., 2006, 306: 3005–3010.
- [13] Michael M, Bruce R. Graph coloring and the probabilistic method[M]. Springer, 2002.
- [14] Alon Noga, Spencer Joel H. The probabilistic method[M]. New York: Wiley, 1992.
- [15] Bondy J A, Murty U S R. Graph theory with applications[M]. Berlin: Springer, 2008.

图的2-强边色数的上界

田京京¹, 聂玉峰¹, 王力工¹, 常建²

(1. 西北工业大学应用数学系, 陕西 西安 710129)

(2. 内蒙古师范大学数学科学学院, 内蒙古 呼和浩特 010022)

摘要: 本文研究了图的2-强边色数的上界. 利用图染色的概率方法中的一般局部引理, 得到了 $3 \leq \Delta \leq 730$ 时, $\chi'_s(G, 2) \leq 2\Delta + 1$, 推广了参考文献[11, 12]中的结果

关键词: 2-强边染色; 2-强边色数; 一般局部引理.

MR(2010)主题分类号: 05C15 中图分类号: O157.5