THE CLASSIFICATION OF GRADIENT RICCI ALMOST SOLITONS

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Abstract: We study the classification of a gradient Ricci almost soliton. Using similar methods as in [11] for \( n \geq 5 \), we obtain that the Weyl curvature tensor is harmonic or Einstein under the assumption that the Bach tensor is flat.

Keywords: gradient Ricci almost solitons; Bach tensor; Weyl curvature tensor

2010 MR Subject Classification: 53C21; 53C25

1 Introduction

Let \( (M^n, g) \) be an \( n \)-dimensional Riemannian manifold. If there exist two smooth functions \( f, \lambda \) on \( (M^n, g) \) such that

\[
R_{ij} + f_{ij} = \lambda g_{ij},
\]

then \( (M^n, g) \) is called a gradient Ricci almost soliton which was introduced by Pigola, Rigoli, Rimoldi and Setti in [1], where \( R_{ij} \) denotes the Ricci curvature of \( (M^n, g) \). Clearly, the above gradient Ricci almost solitons generalize the concept of gradient Ricci solitons which play a very important role in Hamilton’s Ricci flow as it corresponds to the self-similar solutions and often arises as singularity models, for a survey in this subject we refer to the work due to Cao in [2]. When \( \lambda = \rho R + \mu \) in (1.1) with \( \rho, \mu \) two real constants, \( (M^n, g) \) is called the gradient \( \rho \)-Einstein soliton (see [3]) which is a special case of \( (m, \rho) \)-quasi-Einstein manifolds defined in [4], where \( R \) is the scalar curvature of \( (M^n, g) \). For the recent research on this direction, see [5–10] and the references therein.

In this paper, using a similar idea used in [11–13], we derive some formulas, and establish a link between the Cotton tensor \( C_{ijk} \) and the 3-tensor \( D_{ijk} \), that is, \( C_{ijk} = D_{ijk} - W_{ijkf}f^f \), where \( W_{ijkf} \) is the Weyl curvature tensor. By virtue of this relationship we give some classifications of gradient Ricci almost solitons.

2 Preliminaries

Received date: 2013-06-27
Accepted date: 2013-07-30
Foundation item: Supported by National Natural Science Foundation of China (11371018; 11171368).
Biography: Zeng Fanqi (1987–), male, born at Xinyang, Henan, master, major in differential geometry.
We use moving frames in all calculations and adopt the following index convention:

\[ 1 \leq i, j, k, \cdots \leq n, \quad 2 \leq a, b, c, \cdots \leq n \]

throughout this paper.

**Lemma 2.1** Let \((M^n, g)\) be a gradient Ricci almost soliton satisfying (1.1). Then we have

\[
\Delta f = n\lambda - R, \tag{2.1}
\]

\[
(|\nabla f|^2)_i = 2\lambda f_i - 2R_{ij}f^j, \tag{2.2}
\]

\[
\frac{1}{2}R_{,i} = (n-1)\lambda_i + R_{ij}f^j, \tag{2.3}
\]

where \(f^j = g^{jk}f_k\).

**Proof** Equations (2.1) and (2.2) are direct consequences of (1.1) and the fact

\[
(|\nabla f|^2)_i = 2f_jf_{ij} = 2f_j(\lambda g_{ij} - R_{ij}) = 2\lambda f_i - 2R_{ij}f^j.
\]

By the second Bianchi identity, we get

\[
\frac{1}{2}R_{,i} = R_{ij,j} = (\lambda g_{ij} - f_{ij},_j)
\]

\[= \lambda_i - f_{ijj}
\]

\[= \lambda_i - (\Delta f)_i - R_{ij}f^j,
\]

where we used the Ricci identity \(f_{ijj} = (\Delta f)_i + R_{ij}f^j\). Inserting (2.1) and (2.2) into the above equation gives (2.3). We complete the proof of Lemma 2.1.

For \(n \geq 3\), the Weyl curvature tensor and the Cotton tensor are defined by

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2} \left( A_{ik}g_{jl} - A_{il}g_{jk} + A_{jl}g_{ik} - A_{jk}g_{il} \right)
\]

\[= R_{ijkl} - \frac{1}{n-2} \left( R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il} \right)
\]

\[+ \frac{R}{(n-1)(n-2)} (g_{ik}g_{jl} - g_{il}g_{jk}) \tag{2.4}
\]

and

\[
C_{ijk} = A_{kji} - A_{kj,i}, \tag{2.5}
\]

where \(A_{ij}\) is called the Schouten tensor given by

\[
A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij},
\]

From the definition of the Cotton tensor, we have that \(C_{ijk}\) is skew-symmetric in the first two indices and trace-free in any two indices:

\[
C_{ijk} = -C_{jik}, \quad g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0.
\]
The divergence of the Weyl curvature tensor is related to the Cotton tensor by

$$-\frac{n-3}{n-2}C_{ijkl} = W_{ijkl}.$$  \hspace{1cm} (2.6)

For $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}W_{ikjl,k} + \frac{1}{n-2}W_{ikjl}R^{kl}.$$  \hspace{1cm} (2.7)

Using (2.6), we may extend the definition of Bach tensor in dimensions including 3 as follows:

$$B_{ij} = \frac{1}{n-2}(C_{kij} + W_{ikjl}R^{kl}).$$  \hspace{1cm} (2.8)

As in [11], see also [8, 12, 13], we define the following 3-tensor $D$ by

$$D_{ijk} = \frac{1}{n-2}(R_{kj}f_i - R_{ki}f_j) + \frac{1}{(n-1)(n-2)}(R_{kl}g_{jk}f^l - R_{jl}g_{ki}f^l)$$
$$- \frac{R}{(n-1)(n-2)}(g_{kj}f_i - g_{ki}f_j).$$  \hspace{1cm} (2.9)

Then we have that $D_{ijk}$ is skew-symmetric in the first two indices and trace-free in any two indices:

$$D_{ijk} = -D_{jik}, \quad g^{ij}D_{ijk} = g^{ik}D_{ij} = 0.$$

**Lemma 2.2** Let $(M^n, g)$ be a gradient Ricci almost soliton satisfying (1.1). Then the Cotton tensor, D-tensor and the Weyl curvature tensor are related by

$$C_{ijk} = D_{ijk} - W_{ijkl}f^l.$$  \hspace{1cm} (2.10)

**proof** Using formula (1.1), we have

$$R_{kj,i} - R_{ki,j} = (\lambda g_{kj} - f_{kj},i) - (\lambda g_{ki} - f_{ki},j)$$
$$= \lambda_i g_{kj} - \lambda_j g_{ki} + f_{kij} - f_{ki}$$
$$= \lambda_i g_{kj} - \lambda_j g_{ki} - R_{ijkl}f^l.$$  

Therefore,

$$C_{ijk} = A_{kij} - A_{ki,j}$$
$$= \lambda_i g_{kj} - \lambda_j g_{ki} - R_{ijkl}f^l$$
$$- \frac{1}{2(n-1)}(R_{ij}g_{jk} - R_{ij}g_{ik})$$
$$= -\frac{1}{(n-1)}(R_{ij}g_{kj}f_i - R_{ij}g_{ki}f_i) - R_{ijkl}f^l$$
$$= D_{ijk} - W_{ijkl}f^l,$$

where the third equality used equation (2.3). It completes the proof of Lemma 2.2.
The next lemma links the norm of $D_{ijk}$ to the geometry of the level surfaces of the function $f$ on $(M^n, g)$. The proof can be found in [15, Proposition 2.3] and [11, Proposition 3.1].

**Lemma 2.3** Let $(M^n, g)$ be a Riemannian manifold and let $\Sigma_c = \{ x | f(x) = c \}$ be the level surface with respect to regular value $c$ of $f$. Choose local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ on $(M^n, g)$ such that $e_1 = \nabla f / |\nabla f|$ and $\{e_2, \cdots, e_n\}$ tangent to $\Sigma$. Denote by $|D_{ijk}|$ the norm of the 3-tensor $D$, and by $g_{ab}$ the induced metric on $\Sigma$. We have

$$|D_{ijk}|^2 = \frac{2|\nabla f|^2}{(n-1)(n-2)^2} \left( (n-2) \sum_{a=2}^n R_{1a}^2 + (n-1)|R_{ab} - \frac{R - R_{11}}{n-1} g_{ab}|^2 \right),$$

where $R_{ij} = \text{Ric}(e_i, e_j)$ are the components of the Ricci curvature on $(M^n, g)$, $R$ is the scalar curvature of $(M^n, g)$. Note that the indices $2 \leq a, b, c, \cdots \leq n$, then $R_{ab}$ denotes the Ricci tensor of $(M^n, g)$ restricted to the tangent space of $\Sigma$ and $g^{ab}R_{ab} = R - R_{11}$.  

### 3 Some Results

With the help of Lemma 2.3, we can obtain the following result.

**Proposition 3.1** Let $(M^n, g)$ be a gradient Ricci almost soliton satisfying (1.1) with $D_{ijk} = 0$. Let $\Sigma_c = \{ x | f(x) = c \}$ be the level surface with respect to regular value $c$ of $f$. Then for any local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ with $e_1 = \nabla f / |\nabla f|$ and $\{e_2, \cdots, e_n\}$ tangent to $\Sigma$, we have

1. $|\nabla f|, \Delta f, \lambda$ and the scalar curvature $R$ of $(M^n, g)$ are all constant on $\Sigma$;
2. $R_{1a} = 0$ and $e_1 = \nabla f / |\nabla f|$ is an eigenvector of the Ricci operator;
3. the second fundamental form $h_{ab}$ of $\Sigma$ is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$;
4. the mean curvature $H = \frac{(n-1)\lambda - (R - R_{11})}{n-1}$ is constant on $\Sigma$;
5. on $\Sigma$, the Ricci tensor of $(M^n, g)$ either has a unique eigenvalue $\nu$, or has two distinct eigenvalues $\nu$ and $\sigma$ of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \nabla f / |\nabla f|$ is an eigenvector of $\nu$. Moreover, both $\nu$ and $\sigma$ are constant on $\Sigma$.

**Proof** Under this chosen orthonormal frame, we have $f_1 = |\nabla f|$ and $f_2 = f_3 = \cdots = f_n = 0$. When $D_{ijk} = 0$, we have from Lemma 2.3 that

$$R_{1a} = 0 \quad \text{(3.1)}$$

and

$$R_{ab} = \frac{R - R_{11}}{n-1} g_{ab}. \quad \text{(3.2)}$$

Therefore, we obtain from (2.2) and (2.3)

$$\left( |\nabla f|^2 \right)_a = 0, \quad \forall a,$$

which show that $|\nabla f|$ is constant on $\Sigma$. We derive form (2.2) and (2.3)

$$R_{ia} = 2(n-1) \lambda_i + 2 \lambda f_i - (|\nabla f|^2)_i$$
which means that

\[ dR = 2(n - 1)d\lambda + 2\lambda df - d(|\nabla f|^2). \]  

Taking exterior differential of both sides of (3.3), we obtain \( d\lambda \wedge df = 0 \). Therefore, according to the well-known Cartan’s lemma, there exists a smooth function \( \varphi \) such that

\[ d\lambda = \varphi df, \]

which shows that \( \lambda \) is also constant on \( \Sigma_c \). Hence, (1) is proved.

In particular, (2) can be obtained from (3.1) directly.

By the definition of \( h_{ab} \), we have

\[ h_{ab} = \langle \nabla e_a (\nabla f), e_b \rangle = \frac{1}{|\nabla f|} f_{ab} = \frac{1}{|\nabla f|} \left( \lambda - \frac{R - R_{11}}{n - 1} \right) g_{ab}, \]

where the last equality used (3.2). Hence,

\[ H = g^{ab} h_{ab} = \frac{(n - 1)\lambda - (R - R_{11})}{|\nabla f|} \]

and (3) is proved.

By the Codazzi equation

\[ R_{1bc} = \nabla^\Sigma_a h_{ac} - \nabla^\Sigma_b h_{ac}, \]

we get from tracing over \( b \) and \( c \)

\[ R_{1a} = \nabla^\Sigma_a H - \nabla^\Sigma_b h_{ab} = \frac{n - 2}{n - 1} H_a \]

and (4) follows from \( R_{1a} = 0 \).

Since \( H \) is constant on \( \Sigma_c \), we have from (3.5)

\[ R_{11.a} = 0. \]

Applying

\[ R_{11.a} = e_a(R_{11}) - 2R(\nabla e_a, e_1, e_1) = e_a(R_{11}) - 2h_{ab} R_{1b} = e_a(R_{11}) \]

yields \( e_a(R_{11}) = 0 \), which shows that \( \nu = R_{11} \) is constant on \( \Sigma_c \). By (3.2) we know that for distinct \( a \), the eigenvalues of \( R_{aa} \) are the same. Hence, we have the eigenvalue \( \sigma \) is also constant. We obtain (5) and complete the proof of Proposition 3.1.

**Theorem 3.2** Let \((M^n, g)\) be a gradient Ricci almost soliton satisfying (1.1). Then

\[ (n - 2)B_{ij} = D_{kij} + \frac{n - 3}{n - 2} C_{kji} f^k. \]  

If \((M^n, g)\) is compact, then for \( p \geq 0 \),

\[ \int_M f^p B_{ij} f^i f^j dv_g = -\frac{1}{2} \int_M f^p |D|^2 dv_g. \]
In particular, if $B_{ij} = 0$, we obtain from (3.7) the 3-tensor $D_{ijk} = 0$.

**Proof** By virtue of (2.8) and (2.10), we have

$$(n-2)B_{ij} = C_{ki,j,k} + W_{ikjl}R_{kl}$$

$$= (D_{ki,j} - W_{ki,j,l}f_l)_k + W_{ikjl}R_{kl}$$

$$= D_{ki,j,k} - W_{ki,j,l}f_l - W_{ki,j}f_{kl} + W_{ikjl}R_{kl}$$

$$= D_{ki,j,k} + \frac{n-3}{n-2} C_{lji}f_l.$$

If $(M^n, g)$ is compact, we obtain using integrating by parts

$$(n-2)\int_{M^n} f^p B_{ij} f_i f_j \, dv_g$$

$$= \int_{M^n} f^p \left( D_{ki,j,k} + \frac{n-3}{n-2} C_{kji}f^k \right) f_i f_j \, dv_g$$

$$= \int_{M^n} f^p D_{ki,j} f_i f_j \, dv_g$$

$$= -\int_{M^n} f^p D_{ki,j} f^i f^{kj} \, dv_g$$

$$= -\frac{n-2}{2} \int_{M^n} f^p |D|^2 \, dv_g.$$

Therefore, we obtain (3.7) and complete the proof of Theorem 3.2.

**Proposition 3.3** Let $(M^n, g)$ be a compact gradient Ricci almost soliton satisfying (1.1) with $B_{ij} = 0$. If $n \geq 4$, then the Cotton tensor $C_{ijk} = 0$ at all points where $\nabla f \neq 0$.

**Proof** From Lemma 2.2 and Theorem 3.2, we conclude that $C_{ijk} = -W_{ijkl}f_l$. Under the orthonormal frame as in Lemma 2.3, we have

$$C_{ijk} = -W_{ijkl}|\nabla f|. \tag{3.8}$$

In particular, we obtain from (3.8)

$$C_{ij1} = 0. \tag{3.9}$$

From Theorem 3.2, we get

$$\frac{n-3}{n-2} C_{ij1} |\nabla f| = 0.$$

Hence, if $n \geq 4$, then

$$C_{ij1} = C_{j1i} = 0. \tag{3.10}$$

Moreover, from (3.8) we also have that $C_{abc} = -W_{abc}|\nabla f|$. Using (2.4) and Proposition 3.1, we obtain

$$W_{abc1} = R_{abc1} = R_{1cba} = \nabla^\Sigma_{ea} h_{ac} - \nabla^\Sigma_{ea} h_{bc} = 0.$$
Therefore, we obtain

\[ C_{abc} = 0. \] (3.11)

Combining (3.9) with (3.10) and (3.11), we arrive at the conclusion of Proposition 3.3.

**Proposition 3.4** Let \((M^4, g)\) be a compact gradient Ricci almost soliton satisfying (1.1). If \(B_{ij} = 0\), then the Weyl curvature tensor \(W_{ijkl} = 0\) at all points where \(\nabla f \neq 0\).

**Proof** Since \(B_{ij} = 0\), we have \(D_{ijk} = C_{ijk} = 0\). Hence, Lemma 2.2 shows that \(W_{ijkl} = 0\) for \(1 \leq i, j, k \leq 4\). It remains to show that \(W_{abcd} = 0\) for \(2 \leq a, b, c, d \leq 4\). This essentially reduces to show the Weyl curvature tensor is equal to zero in 3 dimensions (see [14, p.276–277] or [11, p.13]). Therefore, we have \(W_{ijkl} = 0\).

**Theorem 3.5** Let \((M^n, g)\) be a compact gradient Ricci almost soliton satisfying (1.1) with \(B_{ij} = 0\).

1. If \(n \geq 5\), then the Weyl curvature tensor is harmonic or Einstein.
2. If \(n = 4\) and it has positive sectional curvature, then \((M^4, g)\) is rotational symmetric or Einstein.

**Proof** (1) If \((M^n, g)\) is not Einstein, then from the set \(\{p|\nabla f(p) = 0\}\) is of measure zero we have \(C_{ijk} = 0\) on \(\Omega = \{x|\nabla f \neq 0\}\) everywhere according to Proposition 3.3 and the continuity. Hence, the Weyl curvature tensor is harmonic.

(2) Under the assumption of Theorem 3.1, Proposition 3.4 shows that \((M^4, g)\) has vanishing Weyl curvature tensor at all points where \(\nabla f \neq 0\). So if the set \(\Omega = \{x|\nabla f \neq 0\}\) is dense, by continuity of the Weyl curvature tensor we have \(W_{ijkl} = 0\) everywhere and \((M^4, g)\) is locally conformally flat. Recall that in any neighborhood of the level surface \(\Sigma_c\), where \(\nabla f \neq 0\), we can express the metric \(ds^2\) by

\[ ds^2 = \frac{1}{|\nabla f|^2}(f, \theta)d\theta^a d\theta^a, \] (3.12)

where \(\theta = (\theta^2, \cdots, \theta^n)\) denote the intrinsic coordinates on \(\Sigma_c\). Since \((M^4, g)\) has vanishing Weyl curvature tensor and positive sectional curvature, the Gauss equation

\[ R^\Sigma_{abcd} = R_{abcd} + h_{aa}h_{bb} - h^2_{ab} \]

and Proposition 3.1 tells us that \((\Sigma_c, g_{ab})\) is a space form with constant positive sectional curvature and \(\frac{1}{|\nabla f|^2}(f, \theta) = \frac{1}{|\nabla f|^2}(f)\). Hence on \(\Omega\) we have

\[ ds^2 = \frac{1}{|\nabla f|^2}(f)df^2 + \varphi^2(f)g_{n-1}, \] (3.13)

where \(g_{n-1}\) denotes the standard metric on unit sphere \(S^{n-1}\). We conclude that \((M^4, g)\) is rotationally symmetric.

**References**


近黎奇梯度孤立子的分类

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摘要：本文研究黎奇梯度孤立子的分类问题，利用文献[11]类似的方法，在Bach张量等于零的条件下，对于$n \geq 5$，证明了流形是Einstein的或者Weyl曲率张量是调和的。

关键词：黎奇梯度孤立子；Bach张量；Weyl曲率张量