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THE HERMITIAN R-SYMMETRIC EXTREMAL RANK SOLUTIONS OF A MATRIX EQUATION

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Abstract: The Hermitian R-symmetric maximal and minimal rank solutions to the matrix equation $AX = B$ and their optimal approximation are considered. By applying the matrix rank method, the necessary and sufficient conditions for the existence of the maximal and minimal rank solutions with hermitian R-symmetric to the equation is obtained . The expressions of such solutions to this equation are also given when the solvability conditions are satisfied. In addition, corresponding minimal rank solution set to the equation and the explicit expression of the nearest matrix to a given matrix in the Frobenius norm are provided.

Keywords: matrix equation; Hermitian R-symmetric matrix; maximal rank; minimal rank; optimal approximate solution

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1 Introduction

Throughout this paper, let $\mathbb{C}^{n\times m}$ be the set of all $n\times m$ complex matrices, $\mathbb{H}^{n\times n}$ denote the class of $n\times n$ Hermitian matrices, $\mathbb{U}^{n\times n}$ be the set of all $n\times n$ unitary matrices. Denote by I_n the identity matrix with order n. Let $J = (e_n, e_{n-1}, \cdots, e_1)$, where e_i is the ith column of I_n. For matrix A, A^*, A^+, $\|A\|_F$ and $r(A)$ represent its conjugate transpose, Moore-Penrose inverse, Frobenius norm and rank, respectively. For a matrix A, the two matrices L_A and R_A stand for the two orthogonal projectors $L_A = I - A^+A$, $R_A = I - AA^+$ induced by A.

Definition 1 Let $R \in \mathbb{C}^{n\times n}$ be a nontrivial unitary involution, i.e., $R = R^* = R^{-1} \neq I_n$. We say that $A \in \mathbb{C}^{n\times n}$ is a Hermitian R-symmetric matrix, if $A^* = A, RAR = A$. We denoted by $\mathbb{HRS}^{n\times n}$ the set of all $n \times n$ Hermitian R-symmetric matrices.

In matrix theory and applications, many problems are closely related to the ranks of some matrix expressions with variable entries, and so it is necessary to explicitly characterize the possible ranks of the matrix expressions concerned. The study on the possible ranks of matrix equations can be traced back to the late 1970s (see, e.g. [1–3]). Recently, the extremal ranks, i.e., maximal and minimal ranks, of some matrix expressions have found many applications in control theory [4, 5], statistics, and economics (see, e.g. [6, 7]).

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In this paper, we consider the Hermitian R-symmetric extremal rank solutions of the matrix equation

$$AX = B, \tag{1.1}$$

where X and B are given matrices in $\mathbb{C}^{n\times m}$.

We also consider the matrix nearness problem

$$\min_{A\in S_m} \left\|A-\tilde{A}\right\|_F, \tag{1.2}$$

where $\tilde{A}$ is a given matrix in $\mathbb{C}^{n\times m}$ and S_m is the minimal rank solution set of eq. (1.1).

2 Some Lemmas

We first know that Hermitian R-symmetric matrices have the following properties.

Since $R = R^* = R^{-1} \neq I_n$, the only possible eigenvalues of R are $+1$ and -1. Let r and s be respectively the dimensions of the eigenspaces of R associated with the eigenvalues $\lambda = 1$, and $\lambda = -1$; thus $r, s > 1$ and $r + s = n$. Let

$$P = [p_1 \quad \cdots \quad p_r] \quad \text{and} \quad Q = [q_1 \quad \cdots \quad q_s], \tag{2.1}$$

where $\{p_1, \ldots, p_r\}$ and $\{q_1, \ldots, q_s\}$ are orthonormal bases for the eigenspaces. P and Q can be found by applying the Gram-Schmidt process to the columns of $I + R$ and $I - R$, respectively.

Lemma 1 [8] $A \in \mathbb{C}^{n\times n}$ is Hermitian and R-symmetric if and only if

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_P & 0 \\ 0 & A_Q \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix} \tag{2.2}$$

with $A_P = P^*AP \in \mathbb{H}^{r\times r}$, $A_Q = Q^*AQ \in \mathbb{H}^{s\times s}$.

Given matrix $X_1, B_1 \in \mathbb{C}^{n\times m}$, the singular value decomposition of X_1 be

$$X_1 = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^* = U_{11}\Sigma_1 V_{11}^*, \tag{2.3}$$

where $U_1 = [U_{11}, U_{12}] \in \mathbb{U}^{n\times n}$, $U_{11} \in \mathbb{C}^{n\times r_1}$, $V_1 = [V_{11}, V_{12}] \in \mathbb{U}^{m\times m}$, $V_{11} \in \mathbb{C}^{m\times r_1}$, $r_1 = r(X_1)$, $\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_{r_1})$, $\sigma_1 \geq \cdots \geq \sigma_{r_1} > 0$.

Let $A_{11} = U_{11}^* B_1 V_{11}\Sigma_1^{-1}$, $A_{12} = U_{12}^* B_1 V_{11}\Sigma_1^{-1}$, $G_1 = A_{12}L_{A_{11}}$, the singular value decomposition of G_1 be

$$G_1 = P_1 \begin{bmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{bmatrix} Q_1^* = P_{11}\Gamma_1 Q_{11}^*, \tag{2.4}$$

where $P_1 = [P_{11}, P_{12}] \in \mathbb{U}^{(n-r_1)\times(n-r_1)}$, $P_{11} \in \mathbb{C}^{(n-r_1)\times s_1}$, $Q_1 = [Q_{11}, Q_{12}] \in \mathbb{U}^{r_1\times r_1}$, $Q_{11} \in \mathbb{C}^{r_1\times s_1}$, $s_1 = r(G_1)$, $\Gamma_1 = \operatorname{diag}(\gamma_1, \ldots, \gamma_{s_1})$, $\gamma_1 \geq \cdots \geq \gamma_{s_1} > 0$.

Lemma 2 [9] Given matrices $X_1, B_1 \in \mathbb{C}^{n\times m}$. Let the singular value decompositions of X_1 and G_1 be (2.3), (2.4), respectively. Then the matrix equation $A_1X_1 = B_1$ has a solution A_1 in $\mathbb{H}^{n\times n}$ if and only if

$$X_1^*B_1 = B_1^*X_1, \quad B_1X_1^+X_1 = B_1. \tag{2.5}$$

In this case, let Ω_1 be the set of all Hermitian solutions of equation $A_1X_1 = B_1$, then the extreme ranks of A_1 are as follows:

(1) The maximal rank of A_1 is

$$\max_{A_1\in\Omega_1} r(A_1) = n + r(B_1) - r(X_1). \tag{2.6}$$

The general expression of A_1 satisfying (2.6) is

$$A_1 = A_0 + U_{12}N_1U_{12}^* \tag{2.7}$$

where $A_0 = B_1X_1^+ + (B_1X_1^+)^+R_{X_1} + R_{X_1}B_1X_1^+(X_1X_1^+B_1X_1^+)^+(B_1X_1^+)^*R_{X_1}$ and $N_1 \in \mathbb{H}^{(n-r_1)\times(n-r_1)}$ is chosen such that $r(R_{G_1}N_1R_{G_1}) = n + r(X_1^*B_1) - r(B_1) - r(X_1)$.

(2) The minimal rank of A_1 is

$$\min_{A_1\in\Omega_1} r(A_1) = 2r(B_1) - r(X_1^*B_1). \tag{2.8}$$

The general expression of A_1 satisfying (2.8) is

$$A_1 = A_0 + U_{12}P_{11}P_{11}^*M_1P_{11}P_{11}^*U_{12}^* \tag{2.9}$$

where $A_0 = B_1X_1^+ + (B_1X_1^+)^+R_{X_1} + R_{X_1}B_1X_1^+(X_1X_1^+B_1X_1^+)^+(B_1X_1^+)^*R_{X_1}$ and $M_1 \in \mathbb{H}^{(n-r_1)\times(n-r_1)}$ is arbitrary.

3 Hermitian and R-Symmetric Extremal Rank Solutions to $AX = B$

Assume P, Q with the forms of (2.1). Let

$$\begin{bmatrix} P^* \\ Q^* \end{bmatrix} X = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix}, \quad \begin{bmatrix} P^* \\ Q^* \end{bmatrix} B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \tag{3.1}$$

where $X_2 \in \mathbb{C}^{r\times m}$, $X_3 \in \mathbb{C}^{s\times m}$, $B_2 \in \mathbb{C}^{r\times m}$, $B_3 \in \mathbb{C}^{s\times m}$, and the singular value decomposition of matrices X_2, X_3 are, respectively,

$$X_2 = U_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} V_2^* = U_{21}\Sigma_2V_{21}^*, \tag{3.2}$$

where $U_2 = [U_{21}, U_{22}] \in \mathbb{U}^{r\times r}$, $U_{21} \in \mathbb{C}^{r\times r_2}$, $V_2 = [V_{21}, V_{22}] \in \mathbb{U}^{m\times m}$, $V_{21} \in \mathbb{C}^{m\times r_2}$, $r_2 = r(X_2)$, $\Sigma_2 = \mathrm{diag}(\alpha_1, \ldots, \alpha_{r_2})$, $\alpha_1 \geq \cdots \geq \alpha_{r_2} > 0$,

$$X_3 = U_3 \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} V_3^T = U_{31}\Sigma_3V_{31}^T, \tag{3.3}$$

where $U_3=[U_{31},U_{32}]\in\mathbb{U}^{s\times s}$, $U_{31}\in\mathbb{C}^{s\times r_3}$, $V_3=[V_{31},V_{32}]\in\mathbb{U}^{m\times m}$, $V_{31}\in\mathbb{C}^{m\times r_3}$, $r_3=r(X_3)$, $\Sigma_3=\mathrm{diag}(\beta_1,\cdots,\beta_{r_3})$, $\beta_1\geq\cdots\geq\beta_{r_3}>0$.

Let $A_{21}=U_{21}^*B_2V_{21}\Sigma_2^{-1}$, $A_{22}=U_{22}^*B_2V_{21}\Sigma_2^{-1}$, $G_2=A_{22}L_{A_{21}}$, $A_{31}=U_{31}^*B_3V_{31}\Sigma_3^{-1}$, $A_{32}=U_{32}^*B_3V_{31}\Sigma_3^{-1}$, $G_3=A_{32}L_{A_{31}}$, the singular value decomposition of matrices G_2, G_3 are, respectively,

$$G_2=P_2\begin{bmatrix}\Gamma_2 & 0\\ 0 & 0\end{bmatrix}Q_2^*=P_{21}\Gamma_2Q_{21}^*, \tag{3.4}$$

where $P_2=[P_{21},P_{22}]\in\mathbb{U}^{(r-r_2)\times(r-r_2)}$, $P_{21}\in\mathbb{C}^{(r-r_2)\times s_2}$, $Q_2=[Q_{21},Q_{22}]\in\mathbb{U}^{r_2\times r_2}$, $Q_{21}\in\mathbb{C}^{r_2\times s_2}$, $s_2=r(G_2)$, $\Gamma_2=\mathrm{diag}(\zeta_1,\ldots,\zeta_{s_2})$, $\zeta_1\geq\cdots\geq\zeta_{s_2}>0$.

$$G_3=P_3\begin{bmatrix}\Gamma_3 & 0\\ 0 & 0\end{bmatrix}Q_3^*=P_{31}\Gamma_3Q_{31}^*, \tag{3.5}$$

where $P_3=[P_{31},P_{32}]\in\mathbb{U}^{(s-r_3)\times(s-r_3)}$, $P_{31}\in\mathbb{C}^{(s-r_3)\times s_3}$, $Q_3=[Q_{31},Q_{32}]\in\mathbb{U}^{r_3\times r_3}$, $Q_{31}\in\mathbb{C}^{r_3\times s_3}$, $s_3=r(G_3)$, $\Gamma_3=\mathrm{diag}(\xi_1,\cdots,\xi_{s_3})$, $\xi_1\geq\cdots\geq\xi_{s_3}>0$.

Now we can establish the existence theorems as follows.

Theorem 1 Let $X,B\in\mathbb{C}^{n\times m}$ be known. Suppose P, Q with the forms of (2.1), $\begin{bmatrix}P^*\\ Q^*\end{bmatrix}X$, $\begin{bmatrix}P^*\\ Q^*\end{bmatrix}B$ have the partition forms of (3.1), and the singular value decompositions of the matrices X_2, X_3 and G_2, G_3 are given by (3.2), (3.3) and (3.4), (3.5), respectively. Then equation (1.1) has a solution $A\in\mathbb{HRS}^{n\times n}$ if and only if

$$X_2^*B_2=B_2^*X_2,\quad B_2X_2^+X_2=B_2,\quad X_3^*B_3=B_3^*X_3,\quad B_3X_3^+X_3=B_3. \tag{3.6}$$

In this case, let Ω be the set of all Hermitian R-symmetric solutions of equation (1.1), then the extreme ranks of A are as follows:

(1) The maximal rank of A is

$$\max_{A\in\Omega}r(A)=n+r(B_2)+r(B_3)-r(X_2)-r(X_3). \tag{3.7}$$

The general expression of A satisfying (3.7) is

$$A=\begin{bmatrix}P & Q\end{bmatrix}\begin{bmatrix}A_2+U_{22}N_2U_{22}^* & 0\\ 0 & A_3+U_{32}N_3U_{32}^*\end{bmatrix}\begin{bmatrix}P^*\\ Q^*\end{bmatrix}, \tag{3.8}$$

where

$$A_i=B_iX_i^++(B_iX_i^+)^+R_{X_i}+R_{X_i}B_iX_i^+(X_iX_i^+B_iX_i^+)^+(B_iX_i^+)^*R_{X_i},$$

$i=2,3$ and $N_2\in\mathbb{H}^{(r-r_2)\times(r-r_2)}$, $N_3\in\mathbb{H}^{(s-r_3)\times(s-r_3)}$ are chosen such that

$$r(R_{G_2}N_2R_{G_2})=r+r(X_2^*B_2)-r(B_2)-r(X_2),$$
$$r(R_{G_3}N_3R_{G_3})=s+r(X_3^*B_3)-r(B_3)-r(X_3).$$

(2) The minimal rank of A is

$$\min_{A\in\Omega} r(A) = 2r(B_2) + 2r(B_3) - r(X_2^* B_2) - r(X_3^* B_3). \tag{3.9}$$

The general expression of A satisfying (3.9) is

$$A = [P \quad Q] \begin{bmatrix} A_2 + U_{22}P_{21}P_{21}^T M_2 P_{21}P_{21}^T U_{22}^T & 0 \\ 0 & A_3 + U_{32}P_{31}P_{31}^T M_3 P_{31}P_{31}^T U_{32}^T \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix}, \tag{3.10}$$

where $A_i = B_iX_i^+ + (B_iX_i^+)^+ R_{X_i} + R_{X_i}B_iX_i^+(X_iX_i^+B_iX_i^+)^+(B_iX_i^+)^* R_{X_i}, i = 2,3,$ and $M_2 \in \mathbb{H}^{(r-r_2)\times(r-r_2)}$, $M_3 \in \mathbb{H}^{(s-r_3)\times(s-r_3)}$ are arbitrary.

Proof Suppose the matrix equation (1.1) has a solution A which is Hermitian R-symmetric, then it follows from Lemma 1 that there exist $A_P \in \mathbb{H}^{r\times r}$, $A_Q \in \mathbb{H}^{s\times s}$ satisfying

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_P & 0 \\ 0 & A_Q \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix} \quad \text{and} \quad AX = B. \tag{3.11}$$

By (3.1), that is

$$\begin{bmatrix} A_P & 0 \\ 0 & A_Q \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \tag{3.12}$$

i.e.,

$$A_P X_2 = B_2, A_Q X_3 = B_3. \tag{3.13}$$

Therefore by Lemma 2, (3.6) hold, and in this case, let Ω be the set of all Hermitian R-symmetric solutions of equation (1.1), we have

(1) By (3.1),

$$\max_{A\in\Omega} r(A) = \max_{\substack{A_P X_2 = B_2 \\ A_P^* = A_2}} r(A_P) + \max_{\substack{A_Q X_3 = B_3 \\ A_Q^* = A_3}} r(A_Q). \tag{3.14}$$

By Lemma 2,

$$\max_{\substack{A_P X_2 = B_2 \\ A_P^* = A_2}} r(A_P) = r + r(B_2) - r(X_2) \tag{3.15}$$

and

$$\max_{\substack{A_Q X_3 = B_3 \\ A_Q^* = A_3}} r(A_Q) = s + r(B_3) - r(X_3). \tag{3.16}$$

Taking (3.15) and (3.16) into (3.14) yields (3.7).

According to the general expression of the solution in Lemma 2, it is easy to verify the rest of part in (1).

(2) The proof is very similar to that of (1) By (3.1) and Lemma 2, so we omit it.

4 The Expression of the Optimal Approximation Solution to the Set of the Minimal Rank Solution

From (3.10), when the solution set $S_m = \{A \mid AX = B, A \in \mathbb{HRS}^{n\times n}, r(A) = \min\limits_{Y\in\Omega} r(Y)\}$ is nonempty, it is easy to verify that S_m is a closed convex set, therefore there exists a unique solution $\hat{A}$ to the matrix nearness problem (1.2).

Theorem 2 Given matrix $\tilde{A}$, and the other given notations and conditions are the same as in Theorem 1. Let

$$\begin{bmatrix} P^* \\ Q^* \end{bmatrix} \tilde{A} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \tilde{A}_{11} \in \mathbb{C}^{r\times r}, \tilde{A}_{22} \in \mathbb{C}^{s\times s}, \tag{3.1}$$

and we denote

$$U_2^*(\tilde{A}_{11} - A_2)U_2 = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}, \tilde{B}_{11} \in \mathbb{C}^{r_2\times r_2}, \tilde{B}_{22} \in \mathbb{C}^{(r-r_2)\times(r-r_2)}, \tag{3.2}$$

$$U_3^*(\tilde{A_{22}} - A_3)U_3 = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix}, \tilde{C}_{11} \in \mathbb{C}^{r_3\times r_3}, \tilde{C}_{22} \in \mathbb{C}^{(s-r_3)\times(s-r_3)}. \tag{3.3}$$

If S_m is nonempty, then problem (1.2) has a unique $\hat{A}$ which can be represented as

$$\hat{A} = [P \;\; Q] \begin{bmatrix} A_2 + U_{22}P_{21}P_{21}^*\tilde{B}_{22}P_{21}P_{21}^*U_{22}^* & 0 \\ 0 & A_3 + U_{32}P_{31}P_{31}^*\tilde{C}_{22}P_{31}P_{31}^*U_{32}^* \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix}, \tag{3.4}$$

where $\tilde{B}_{22}$, $\tilde{C}_{22}$ are the same as in (3.2), (3.3).

Proof When S_m is nonempty, it is easy to verify from (3.10) that S_m is a closed convex set. Problem (1.2) has a unique solution $\hat{A}$. By Theorem 1, for any $A \in S_m$, A can be expressed as

$$A = [P \;\; Q] \begin{bmatrix} A_2 + U_{22}P_{21}P_{21}^*M_2P_{21}P_{21}^*U_{22}^* & 0 \\ 0 & A_3 + U_{32}P_{31}P_{31}^*M_3P_{31}P_{31}^*U_{32}^* \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix}, \tag{3.5}$$

where

$$A_i = B_iX_i^+ + (B_iX_i^+)^+R_{X_i} + R_{X_i}B_iX_i^+(X_iX_i^+B_iX_i^+)^+(B_iX_i^+)^*R_{X_i},$$

$i = 2, 3$, and $M_2 \in \mathbb{H}^{(k-r_2)\times(r-r_2)}$, $M_3 \in \mathbb{H}^{(s-r_3)\times(s-r_3)}$ are arbitrary.

Using the invariance of the Frobenius norm under unitary transformations, and

$$P_{21}P_{21}^* + P_{22}P_{22}^* = I, P_{31}P_{31}^* + P_{32}P_{32}^* = I,$$

where $P_{21}P_{21}^*$, $P_{22}P_{22}^*$, $P_{31}P_{31}^*$, $P_{32}P_{32}^*$ are unitary projection matrices, and

$$P_{21}P_{21}^*P_{22}P_{22}^* = 0, P_{31}P_{31}^*P_{32}P_{32}^* = 0,$$

we have

$$\begin{aligned}
\|\tilde{A} - A\|_F^2 &= \left\|\tilde{A}_{11} - A_2 - U_{22}P_{21}P_{21}^*M_2P_{21}P_{21}^*U_{22}^T\right\|_F^2 + \left\|\tilde{A}_{12}\right\|_F^2 \\
&\quad + \left\|\tilde{A}_{22} - A_3 - U_{32}P_{31}P_{31}^*M_3P_{31}P_{31}^*U_{32}^*\right\|_F^2 + \left\|\tilde{A}_{21}\right\|_F^2 \\
&= \left\|U_2^*(\tilde{A}_{11} - A_2)U_2 - \begin{bmatrix} 0 & 0 \\ 0 & P_{21}P_{21}^*M_2P_{21}P_{21}^* \end{bmatrix}\right\|_F^2 + \left\|\tilde{A}_{12}\right\|_F^2 \\
&\quad + \left\|U_3^*(\tilde{A}_{22} - A_3)U_3 - \begin{bmatrix} 0 & 0 \\ 0 & P_{31}P_{31}^*M_3P_{31}P_{31}^* \end{bmatrix}\right\|_F^2 + \left\|\tilde{A}_{21}\right\|_F^2 \\
&= \left\|\tilde{A}_{12}\right\|_F^2 + \left\|\tilde{A}_{21}\right\|_F^2 + \left\|\tilde{B}_{11}\right\|_F^2 + \left\|\tilde{B}_{12}\right\|_F^2 + \left\|\tilde{B}_{21}\right\|_F^2 + \left\|\tilde{C}_{11}\right\|_F^2 \\
&\quad + \left\|\tilde{C}_{12}\right\|_F^2 + \left\|\tilde{B}_{22}P_{22}P_{22}^*\right\|_F^2 + \left\|P_{22}P_{22}^*\tilde{B}_{22}P_{21}P_{21}^*\right\|_F^2 \\
&\quad + \left\|P_{21}P_{21}^*\tilde{B}_{22}P_{21}P_{21}^* - P_{21}P_{21}^*M_2P_{21}P_{21}^*\right\|_F^2 \\
&\quad + \left\|\tilde{C}_{21}\right\|_F^2 + \left\|\tilde{C}_{22}P_{32}P_{32}^*\right\|_F^2 + \left\|P_{32}P_{32}^*\tilde{C}_{22}P_{31}P_{31}^*\right\|_F^2 \\
&\quad + \left\|P_{31}P_{31}^*\tilde{C}_{22}P_{31}P_{31}^* - P_{31}P_{31}^*M_3P_{31}P_{31}^*\right\|_F^2.
\end{aligned}$$

Therefore, $\min\limits_{A\in S_m} \|\tilde{A} - A\|_F$ is equivalent to

$$\min_{M_2\in\mathbb{H}^{(r-r_2)\times(r-r_2)}} \left\|P_{21}P_{21}^T\tilde{B}_{22}P_{21}P_{21}^* - P_{21}P_{21}^*M_2P_{21}P_{21}^*\right\|_F, \tag{3.6}$$

$$\min_{M_3\in\mathbb{H}^{(s-r_3)\times(s-r_3)}} \left\|P_{31}P_{31}^*\tilde{C}_{22}P_{31}P_{31}^* - P_{31}P_{31}^*M_3P_{31}P_{31}^*\right\|_F. \tag{3.7}$$

Obviously, the solutions of (3.6), (3.7) can be written as

$$M_2 = \tilde{B}_{22} + P_{22}P_{22}^*\tilde{M}_2P_{22}P_{22}^*, \quad \forall\tilde{M}_2 \in \mathbb{H}^{(r-r_2)\times(r-r_2)}, \tag{3.8}$$

$$M_3 = \tilde{C}_{22} + P_{32}P_{32}^*\tilde{M}_3P_{32}P_{32}^*, \quad \forall\tilde{M}_3 \in \mathbb{H}^{(s-r_3)\times(s-r_3)}. \tag{3.9}$$

Substituting (3.8), (3.9) into (3.5), then we get that the unique solution to problem (1.2) can be expressed in (3.4). The proof is completed.

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一类矩阵方程的Hermitian R-对称定秩解

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摘要: 本文研究了矩阵方程$AX = B$ 的Hermitian R-对称最大秩和最小秩解问题. 利用矩阵秩的方法, 获得了矩阵方程$AX = B$有最大秩和最小秩解的充分必要条件以及解的表达式, 同时对于最小秩解的解集合, 得到了最佳逼近解.

关键词: 矩阵方程; Hermitian R-对称矩阵; 最大秩; 最小秩; 最佳逼近解

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